MATH 221: DIFFERENTIAL EQUATIONS

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MAY, 2000

This portfolio is based on the way I taught the course in Spring 1999.

Math 221, Differential Equations, is the introductory course in ordinary differential equations at the University of Nebraska–Lincoln. Students in the course have completed the three-semester calculus sequence and have generally not taken a linear algebra course. Most of the students are engineering majors, although there are some math and science majors and an occasional student with a major not related to the course content. Nearly all students in the course are taking it as a required course.

Math 221 is taught in classes of fewer than 40 students. The enrollment in the course then dictates that there be several sections of the course each semester. Because the course plays a key service role and is also a prerequisite for other courses, the Department of Mathematics and Statistics has worked to make the course essentially the same across all sections in the same semester, with only minor changes from one semester to the next. This standardization inevitably requires that individual faculty members agree to compromises that they may not fully endorse. In particular, there is an official statement of course goals. I was the principal author of the current version of this statement.

In order to try to achieve the goal of uniformity among different sections of the course, all instructors are expected to use the official textbook chosen by a committee appointed from among faculty who have an interest in the course. In Spring 1999, the official textbook was *Elementary Differential Equations*, sixth edition, by Boyce and DiPrima. I was constrained to use this text although I did not support its adoption.

Instructors generally follow a syllabus prepared by a faculty member appointed by the Department to be the convener of the course. They devise their own policies for the course and may alter the syllabus. There is frequent informal communication among the instructors, and this tends to result in similar practices for the different instructors, especially in grading systems, assignments, and exams. It is not possible to offer common exams because each section has its own exam schedule; however, the adoption of common goals, use of a common textbook, and frequent communication generally result in midterm and final exams that differ little across the sections. In practice, variations between sections are generally minor and deal more with administrative policies and course delivery than with topics covered or methods of assessment.

Within the constraints imposed by the Department, I made my course design decisions individually. I was the convener in Spring 1999, so I prepared the official syllabus. I did not use the official syllabus in my own section. I used the textbook only as a secondary reference and a source of problems, prefering instead to organize the course around my own lecture notes. I chose my own methods of assessment, which differed in significant ways from those of my colleagues.

1 OBJECTIVES AND TOPICS

Math 221 concerns the study of ordinary differential equations. Motivation for the importance of ordinary differential equations comes from their use in mathematical modeling. Physical laws generally take the form of differential equations. In particular, physical laws describing quantities that vary in time, but not space, and physical laws describing quantities that vary in one spatial dimension, and are independent of time, are generally ordinary differential equations. Typical applications of ordinary differential equations include decay processes, one-dimensional motion under the influence of forces, changes in homogeneous populations, and modes of vibration in a string or beam.

The subject matter of differential equations includes a body of theory of differential equations and their solutions, a collection of techniques for analyzing various categories of equations, and mathematical modeling. Since explicit solution formulas can be obtained for only a limited class of differential equations, some of the important techniques are graphical (yielding qualitative information about solutions) or numerical (yielding a set of approximate solution values). Nevertheless, symbolic techniques remain a large part of the course, as many mathematical models in common use include equations that can be solved exactly by symbolic methods.

MATH 221 AND ITS PLACE IN THE CURRICULUM

Most of the students who take Math 221 are majors in engineering or science who are expected to take the course as part of the requirements for their degree. The design of the course must acknowledge the service role that the course takes for the bulk of its students. The course is also required for majors in the Department; thus, it is also important for the course to be designed so as to prepare students for courses in advanced ordinary differential equations, partial differential equations, and applied mathematics.

COURSE GOALS AND RATIONALE

The primary goals of Math 221 can be divided into those that deal with modeling, those that deal with concepts, and those that deal with techniques.

Modeling

Math 221 is a mathematics course rather than a course in science or engineering, but the development of differential equation models to represent physical situations is an appropriate mathematical activity. The emphasis in Math 221 should be on the modeling process rather than the specific applications of models. In their subsequent work, students will learn about mathematical models in their own field of interest. No specific mathematical model is common to all fields of science and engineering, but the skills of building, analyzing, and improving mathematical models is.

1. **Math 221 students should learn to formulate differential equation models for physical processes and use them to explore questions of scientific interest.**

Concepts

Math 221 is not primarily a course about the theory of differential equations; however, a certain amount of theoretical knowledge is needed in order to be able to solve differential equations and interpret the results. In particular, results obtained by a computer algebra system are not always correct, and it is conceptual knowledge that allows a student to tell when answers provided by technology are wrong.

2. **Math 221 students should understand the difference between a solution and a solution formula and know what conditions are necessary to guarantee that an initial value problem has a unique solution.**

3. **Math 221 students should understand the special properties of linear equations and be able to use the linearity principle to construct general solutions for linear equations.**

4. **Math 221 students should understand the relationships between solution curves, the slope field, and numerical approximations for first-order equations and the notions of equilibrium solutions, stability, and the phase space for autonomous problems.**

Techniques

Students in Math 221 study a variety of techniques, some of which are symbolic and some of which are graphical. Mastery of the fundamental techniques is the core of the course. The techniques listed in the goals are those that the students are most likely to encounter in subsequent courses. In particular, the Laplace transform is included primarily because it is a priority of the College of Engineering.

5. **Math 221 students should learn to recognize and explicitly solve first-order equations that are separable or linear.**

6. **Math 221 students should learn to solve second-order equations with constant coefficients, using undetermined coefficients, variation of parameters, and the Laplace transform.**

7. **Math 221 students should learn to find eigenvalues and straight-line solutions for autonomous linear 2×2 systems and use them to classify the equilibrium point at the origin by stability and shape.**

8. **Math 221 students should learn to use direction fields, nullclines, and computer-generated phase portraits to determine the stability of equilibrium points and discuss the qualitative features of the solution curves.**

Non-content goals

The Department has endorsed an additional goal for the course that is not directly related to the subject matter of the course.

9. **Math 221 students should acquire some experience using a computer algebra system.**

Standard practice in engineering and science now includes the use of computer algebra systems as tools for analyzing differential equations. In particular, computer algebra systems can provide graphical information for problems too complicated or detailed to be worked by pencil and paper. The Department does not currently use computer algebra systems for calculus courses. Current Department policy is to introduce Maple to students in Math 221.

To the official Department list of goals, I add several noncontent goals that I believe are common to most mathematics courses.

10. **Math 221 students should learn to think abstractly, analytically, and logically.**

Mathematics involves thinking about classes of problems (abstract thinking), dividing problems up into parts (analytical thinking), and making clear the connections between assumptions and conclusions (logical thinking).

11. **Math 221 students should learn to solve problems using knowledge, skills, intuition, and discipline.**

Mathematics is best viewed not as a collection of facts, but as a way of thinking. Learning and practicing mathematics therefore requires the development and use of problem-solving skills. Successful problem-solving requires an appropriate level of knowledge, facility in computational skills, an ability to judge how reasonable a result is based on experience (intuition), and sufficient self-discipline to work through difficulties rather than abandon the effort.

12. **Math 221 students should learn to communicate their work orally and in writing.**

In any field of endeavor in which the products are thought rather than objects, the value of one's work is only as good as one's ability to share the work with others. When asked what attributes they want new employees to have, employers of mathematicians, scientists, and engineers almost always list communication skills, both oral and written. An important goal of any mathematics course is to help students improve their ability to communicate their work.

TOPICS

The topics that I included in my syllabus were generally the same as those that appeared in the standard syllabus. However, I organized them as I saw fit rather than as they were organized in the textbook. My notes were divided into a total of 33 sections, each of which corresponded to one class period. The 33 topics were organized into 7 units.

Introduction

These topics occur in chapters 1 and 8 of the current textbook. The introduction consists of the basic definitions, slope fields, and Euler's method.

First-order equations

These topics occur in chapter 2 of the current textbook. We currently cover the existence and uniqueness theorems for the general case and the linear case, and we cover the methods of solution for separable equations and linear equations. Some emphasis is given to the interval of definition for the solution of an initial value problem. Applications of these symbolic techniques generally include radioactive decay, homogeneous cooling, and chemical mixing. There is also some material on graphical methods for autonomous nonlinear equations, including applications to population models and chemical reactions.

Second-order equations

These topics occur in chapters 3 and 5 of the current textbook. We cover the theory of second-order linear equations, emphasizing those aspects of the theory that are necessary to understand how and when one can construct general solutions from a particular solution and a 2-parameter complementary solution. The solution technique for homogeneous constant-coefficient equations is covered in detail along with the methods of undetermined coefficients and variation of parameters for particular solutions of nonhomogeneous problems. The main application of these equations is in linear oscillator problems, and we generally consider both mechanical models and electrical models. We also cover Euler equations.

Systems of equations

These topics occur in chapters 7 and 9 of the current textbook. We spend some time covering linear algebra in 2 dimensions, emphasizing the eigenvalue problem, because students who take Math 221 have generally not had a course in linear algebra. We cover the method for solving systems of two

homogeneous linear first-order equations with constant coefficients, with the emphasis on classifying the critical point at the origin as well as constructing a general solution from real-valued functions. We also cover basic graphical techniques, including direction fields and nullclines, and we discuss straight-line solutions and how to use all of our tools to sketch the phase portrait of a nonlinear system and determine the stability of equilibrium points.

The Laplace transform

These topics occur in chapter 6 of the current textbook. We present the definition of the Laplace transform and its use in solving initial value problems. Students are expected to use partial fraction decomposition and completing the square to invert the transform for simple cases. We also cover unit step functions and use them to solve initial value problems where the nonhomogeneous terms are only piecewise continuous.

SOME SPECIFIC OBJECTIVES

In addition to the general goals that drove the choice of topics, I also had a list of specific objectives based on the goals and topics. The list of objectives was divided into basic objectives, which I expected all students to accomplish, and advanced objectives, which I expected only of the better students.

Symbolic Techniques

I expected all my students to be able to use the integration technique for solving separable equations and the method of undetermined coefficients and the integrating factor method for linear equations of the appropriate subclass. I also expected all my students to be able to write a second-order differential equation as a system, determine equilibrium points for an autonomous nonlinear system, determine eigenvalues for a system of two linear equations, and use the eigenvalues to determine the stability of the origin in all cases and the solution formulas for the case of real and distinct eigenvalues. I expected the better students to also learn the method of variation of parameters and the Laplace transform method, to be able to construct solution formulas for autonomous linear systems with complex eigenvalues, and to be able to examine the stability of equilibrium points of a nonlinear system by linearization.

Graphical Techniques

I expected all my students to be able to sketch minitangents in the slope field of a first-order equation or in the direction field of an autonomous system of two equations, to sketch and interpret the phase line for a first-order equation and to interpret a phase portrait for an autonomous system, and to use eigenvalues and straight-line solutions to sketch the phase portrait for an autonomous system. I expected the better students to also be able to use nullclines as a tool for sketching phase portraits.

Concepts and Models

I expected all my students to be able to discuss the concepts of equilibrium solutions and their stability and interval of existence of solutions and to be able to answer questions based on these concepts. I also expected them to be able to identify whether or not a given method is appropriate for a given problem and to be able to solve word problems involving decay processes or unforced linear oscillators. The better students were expected to be able to discuss basic differential equation theory, derive mathematical models from a verbal description of a process, and to solve word problems involving unfamiliar mathematical models.

2 COURSE PRESENTATION

The term *course presentation* is used here to indicate ways the instructor and students work to achieve the course goals: activities to be done during class time, activities to be done outside of class, and grading policies. I place the greatest emphasis on the grading policies, because I believe that the grading policies have more effect on student learning than anything else an instructor does.

Most instructors in the Department of Mathematics and Statistics adopt similar grading policies for Math 221. Typically, grades are determined using a point-based system, with a 200-point final exam, three 100-point midterm exams, and an additional 150 to 200 points made up of some combination of group project reports, homework, and quizzes.

I have long been dissatisfied with the point-based grading systems. Data collected in the Department shows that students who get a C in a calculus or precalculus course seldom pass the next course. I see this as a problem because the prerequisite for these courses is a C or better in the previous course. We have a system in which many students meet the official prerequisite for a course but are not sufficiently prepared to succeed in the course. I believe that two features of the point-based grading system contribute greatly to this problem.

- **Students can get a C by doing D work on exams, C work on quizzes, and A work on projects and homework done primarily by a helpful classmate.**

- **Success in the next course depends on mastery of the basics, but students generally earn a C on an exam without mastering anything.**

SOME WORKING HYPOTHESES AND THEIR IMPLICATIONS

My course design was based on a pair of working hypotheses that I have gradually developed over my teaching career.

1. **The average student does not spend enough time outside of class to meet the course objectives.** (The peer review project at UNL has compiled an enormous body of evidence that the average student spends less than one hour outside of class for every hour of class time.)

2. **In a course whose purpose is to prepare students for further study, grades should measure what students *can do* at the end of the course, rather than what they *have done* during the semester.** Specifically, a grade of C should mean that the student has satisfied a set of basic objectives at a high level of competence.

The two working hypotheses have contrary implications on the question of what should and should not be graded. One way to get students to spend more time on task is to reward them with points for everything we want them to do. By this argument, we should use a point-based grading system that includes points for attendance, points for homework done, points for participation in project groups, points for rewriting problems missed on exams, and points for preparing good summaries of each chapter or study guides for exams. Unfortunately, these items measure what a student has done rather than what the student can do. The more points we assign for these activities, the more time on task we are likely to get, but the easier it will be for students to pass without becoming competent in the course material. The second hypothesis implies that grades should be based only on instruments that measure competence rather than effort, even if that leads to less time on task. The difficulty with this is that many students are unlikely to read the book, study their notes, or work exercises, unless they are directly compensated for these activities.

Perhaps it is best to reward time on task directly in the hope that more time will lead to greater competence. Or perhaps it is better to require competence and hope that students will spend a sufficient amount of time on task to achieve it. In my experience, it seems clear that rewarding time

on task does not lead to competence. I therefore resolved to require competence and count on that requirement to increase time on task.

AN EXPERIMENT IN MASTERY GRADING

In light of the above discussion, I resolved to make an experiment in mastery grading. This idea has been used in the Department of Psychology. The plan there involves a sequence of chapter quizzes. Students must achieve a specific score on each quiz in order to go on, otherwise they are expected to take another quiz from the same question bank. At the end of the semester, the grades are based on the number of quizzes passed by the student.

The plan outlined above rests on the idea that there is only one level of mastery, so the only way to excel is to study more topics. For a course based on concepts and facts, this may well be appropriate. For a course in mathematics, it seemed to me to be too naive. I believe students should be able to demonstrate excellence by showing that they can work more difficult problems or do routine work at a level higher than adequate, as well as by demonstrating competence on a larger list of topics.

As a compromise, I devised a two-tiered grading system. The basic tier was a pass-fail mastery-based system. In order to pass the course, students had to achieve predetermined minimum scores on 7 chapter-level progress tests and a basic final exam. The progress tests were half the length of a standard midterm and the basic final exam was half the length of a standard final exam. The material on these exams was limited to a subset (say about 70%) of the material actually covered and consisted of routine and moderate questions. The passing level was set at what would normally have been a minimal B+ on a point-based system. Students could take new versions of each progress test as needed, but were limited to one try each week. Owing to time constraints, the basic final exam could be taken only twice.

The upper tier of the grading system was a point-based system used to assign a letter grade to students who passed the basic tier. These points were accumulated during the semester, and consisted of points earned on advanced tests (given in conjunction with the progress tests), the comprehensive final exam, and individual reports on group projects.

The experimental grading scheme attempted to resolve all of the dilemmas outlined above. By requiring B+ level work on the basic material, once when the material was first learned and a second time at the end of the course, I hoped to have a system by which grades would to a large extent measure what a student can do rather than what he/she has done. I hoped that time on task would be enhanced in two ways. Some students would need a lot of time on task in order to pass the progress tests. I would not give points for their study activities, but without these they could not pass the course. Other students would need time on task to do the group projects. The problem of students getting grade boosts from the accomplishments of others would be limited by the division of the grading into two tiers. Students could still get a higher passing grade than they deserved with project points, but they would not be able to pass the course on the strength of project points. Students could not get by on projects without demonstrating good writing skills because each student had to write an individual report of the group work.

CLASS TIME AND OUTSIDE ACTIVITIES

I began a typical 50-minute class period with from 10 to 15 minutes spent answering student questions. I was generally dissatisfied with the quantity of student questions; I suspected that few students had attempted the reading or exercises. Most of the remaining time in class was spent in the traditional lecture format, with a combination of presentation of ideas, presentation of examples, and summarizing.

Outside activities included daily reading assignments, daily exercise sets that were not collected, and miniprojects for five of the seven units. None of these activities had a direct effect on the issue of passing the course. Reading and exercises affected passing indirectly by improving test scores, while

project work affected the grades of only those students who met the basic requirements to pass the course.

Projects serve two purposes in Math 221. The principal purpose is that it is hard to find any other way to meet goal 1, that of learning to formulate differential equations and use them to explore questions of scientific interest. The goal is not for the students to learn about any one particular scientific application, but rather for them to gain some experience in using mathematics outside of a math problem. The idea is that differential equations are primarily of interest as a tool for science and engineering, just as grammar is primarily of interest as a tool for writing. A course in pure grammar with no application to writing would clearly be missing something, and I feel that a differential equations course with no projects is analogous. Projects also have the advantage of getting students to work in teams, as is the rule in the workplace.

At the same time, there are some drawbacks to projects. Most problems occur when students in a particular project group differ greatly in motivation or ability. I try to group students by motivation, so that dedicated students are not burdened by slackers. I don't favor grouping by ability, as the same project that is too hard for the weaker groups might be too easy for the stronger groups. However, the amount of work put into a project and the amount of experience gained can be vastly different for the students in a group; finding ways to give a fair project grade to each student is problematic. I hoped to solve this problem by having the students write individual reports. Group members were allowed to help each other with calculations and to share computer-generated graphs, but they were not allowed to share writing.

3 COURSE OUTCOMES

Twenty-nine students were in my section at the beginning of the semester. The table compares the grade distribution of my class with my typical grade distribution. In a typical semester, I use a point-based system in which the boundaries between consecutive grades are drawn separately for each instrument. The boundaries are determined subjectively, but with an ideal objective standard in mind. Thus, I do not grade on the curve; nevertheless, grade distributions tend to be fairly similar from one semester to another. I count as members of the class anyone who stays in it long enough to take the first exam. Those who drop the class after that time are considered as withdrawals and classified with those who receive an F.

Grade	Current Semester		Typical Semester, 1996-99
	Number of Students	Cumulative Percentage	Cumulative Percentage
A	4	14	13
B$^+$	4	28	23
B	4	41	36
C$^+$	4	55	55
C	9	86	70
D	0	86	77
F / W	4	100	100

The most significant result from this class was the large increase in the percentage of students who passed the course. I have taught this course using a point-based system seven times, with pass rates of 76%, 84%, 46%, 76%, 71%, 69%, and 66%. My class this semester achieved a pass rate slightly higher than the best of my previous classes. By my subjective analysis, this semester's class was a little better than my average class, so I would have expected the pass rate to be about 75%. This class included three students who needed a heroic effort to pass and who probably would not have passed without the repetition of a mastery system. These three students comprised about 10% of the class, and so the conclusion seems fairly clear.

- **More students pass under a mastery system because of a small but significant number of marginal students who succeed by perseverance.**

The principal goal of the mastery system was to improve the *quality* of the C students. The anecdotal evidence shows that the mastery system achieved this goal. The key evidence is in the final exam results of the three marginal students. One of these students was unable to complete the course on time. I gave him an incomplete, which I changed to a C after he passed the makeup final in the week after the end of the semester. A second of the marginal students scored 59% on his first attempt at the basic final exam. Although this was not a passing mark, it was probably better than that student would have got on the final in a standard course. On the second try at the basic final, this student did earn his C.

The third marginal student, whom I judged to be the weakest of the three, was for me the real test case. This student had needed a total of 21 attempts to pass the 7 progress tests, including 5 attempts for test 3, which included the most important topics in the course. The fourth attempt was close enough that I offered him an alternative format for the fifth attempt. He was given the same questions as on the fourth attempt, but he had to do the problems on my blackboard and answer my oral questions. In spite of the difficulty he had throughout the semester, this student passed the final on the first try,[1] with a paper that is far better than what is typically achieved on a final exam by a student who gets a C in the course. My colleagues all felt on the basis of this final paper that this

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[1]This final exam paper appears in the appendix, along with the second, third, and fourth attempts at test 3.

student earned a passing mark for the course. This contrasts with my usual observation that I give at least one C every semester to a student who clearly did not learn enough to merit a passing grade.

A total of 25 students of my original 29 managed to pass all 7 progress tests. All 25 of these passed the course, with 80% of them passing the final on the first try. In a typical semester, about 10 to 20% of the students show a marked decline in performance on the final exam. This semester, there were 4 students (14%) whose final exams showed a marked decline; however, only 1 of these failed to pass the final on the first try. Although the total amount of evidence is limited, I am confidant of the following conclusion.

- **A grade of C in my Spring 1999 class represents a higher level of competence than is normally represented by a C in a Math 221 class.**

While the results of my experiment in mastery grading were encouraging, they were not uniformly positive. One difficulty was that some students made less effort in the course than they would have in a standard class. In particular, 5 capable students opted to work only for a C grade. These students did nothing beyond the progress tests and the basic final, even though they were clearly capable of more work. Equally disappointing were the results of the surveys conducted for the peer review project. On the average, students in my class spent barely an hour outside of class for every hour in class. This low amount of time on task is consistent with that of my regular classes and with the average amount of time on task my students reported for their other classes. It is, however, only about half of what I believe to be an appropriate amount of time to be spending on a college course.

4 REFLECTIONS ON THE COURSE

The biggest drawback of the experimental course was that it required too much instructor effort to be practical. I put in a great deal of time writing makeup tests and administering them. I made one change in the middle of the semester that made the process more efficient; I began to require that students must work out missed test problems at home before being permitted to take a repeat test. I had noticed that many students repeated their mistakes on repeat tests, so it was clear that they had not learned anything more since failing the first test. The new policy required them to better prepare for make-up tests, and the result was a marked improvement in the success rate on make-up tests.

Some changes are clearly needed to make a practical mastery grading plan, in addition to requiring students to work out problems they missed before repeating a test. The most important of these is that the basic tests should be given two class periods prior to the corrsponding advanced test, with success on the basic test a necessary condition for permission to take the advanced test. I also plan to try computerized testing for the basic tests. The computerized test would have to be proctored and would require a data base of test questions. It remains to be seen whether it is possible to make an adequate computerized test. One difficulty with computerized tests is that it is not possible to ask open-ended questions, such as questions where the answer is a sketch. A second difficulty is that computers cannot award partial credit for solutions that include only one minor error. A compromise idea would be to use a computerized system to create the tests, but to have the students work the tests with pencil and paper. In order for this to work, the grading must be streamlined with a simple rubric.

The course drew mixed reviews from the students. On the whole, the student evaluations of the course were within my normal range. I also asked the students to comment on the grading system I used. Two of these comments were particularly revealing.

- The grading system, I felt wasn't very good. I liked having basic and advanced portions and being able to retake the basic part if you didn't pass (helped to make sure that I learned the material) but I felt like it took too many advanced points to be able to get above a C. With all of the time and effort needed to prepare for the basic parts, it was hard to find time to do the projects. Also, if you get behind and can't take the exam on time, you don't have a chance to get advanced points on that one.

- I disliked the grading system at first but now that I am done I would have to say it is in fact quite effective in teaching the material.

The first of these comments was intended to be critical, but seems to me to make a positive case for the grading system. The student objected to the amount of work required for the course and resented having to master basic material before moving on. These were two of the goals of the grading system. Note that the student did not address the issue of how the grading system affected his/her learning, but only the issue of whether the system made the course more difficult.