Notes on Line Integrals

Suppose $\vec{\mathbf{F}} = \langle F_1, F_2, F_3 \rangle$ is a vector field and C is an oriented curve given by a position vector $\vec{\mathbf{r}}$. We can think of the vector field as "pushing" something along the curve. Examples are a force field, in which case the total amount of "push" is called *work*, and a velocity field with a closed curve, in which the total amount of "push" is called *circulation*. In these notes, we'll use the generic term *push* and the symbol P; while not a recognized term in either mathematics or physics, the term *push* has the advantage of indicating meaning, without the baggage of using a physical term (work, flow, or circulation) to mean something different from what it means in standard English.

In the simplest case, the vector field is constant and the curve is a straight line in the x direction. The total push is simply the product of the horizontal component F_1 of the vector field and the change in position Δx: $P = F_1 \Delta x$. For easy generalization, we can write this as a dot product:

$$P = F_1 \Delta x = \langle F_1, F_2, F_3 \rangle \cdot \langle \Delta x, 0, 0 \rangle = \vec{\mathbf{F}} \cdot \Delta \vec{\mathbf{r}}.$$

This dot product formula holds for any straight line, regardless of the direction, but only when the vector field is the same everywhere on the curve.

Now suppose the vector field is different at different points along the curve and the curve is not necessarily a straight line. To calculate the total amount of push, it is necessary to calculate the amount of push for a little bit of the curve and then add the totals for all the little bits. For an arbitrarily small bit of curve, we can take the vector field to be the constant value $\vec{\mathbf{F}}(x, y, z)$ and the change in position to be $d\vec{\mathbf{r}}$. Since we are dealing with a vector force and displacement, it is only the component of the field that is directed along the curve that counts toward the push; hence we need the dot product rather than some other type of multiplication. Hence, the little bit of push at this point of the curve is

$$dP = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

Summing the infinitesimal contributions at each point, the total push is given by

$$P = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}.$$

Evaluating Line Integrals

The evaluation of definite integrals has a common mathematical structure that is independent of the geometry. The idea is to set up an iterated integral using as many integration variables as the dimension of the region; that is, we need one parameter for a curve, two for a surface, and three for a solid region. Integrals along lines that can be described using only one spatial coordinate are trivial in that they almost immediately reduce to a simple one-variable integral. Other curves must be parameterized, and the parameter used for that parameterization serves as the variable for the integration. Hence, we have the following general plan for direct calculation of any definite integral:

1. Choose the integration variable(s). An n-dimensional region requires n variables. Any spatial variable other than an integration variable needs to be written in terms of integration variables.

2. Write the differential in terms of the integration variable(s).

3. Write the integrand in terms of the integration variables.
4. Identify the ranges of the integration variable(s) using a nested set of inequalities, and use these inequalities to set the integral up as an iterated integral.
5. Compute the integral from the inside out.

In the case of line integrals, there are also indirect methods that can sometimes be used instead of the direct method.

Method 1 — Direct evaluation by parameterization

Consider first the special case where the curve is parallel to a coordinate axis. As an example, suppose the curve is the portion of the line $x = 2$ that runs from $y = 0$ to $y = 4$ and suppose the vector field is $\vec{\mathbf{F}} = \langle x, 2x + y\rangle$. In this case, any little bit of the curve runs only in the y direction, so we simply have $d\vec{\mathbf{r}} = \vec{\mathbf{j}}\,dy$. Similarly, every point on the curve has $x = 2$, so $\vec{\mathbf{F}} = \langle 2, 4 + y\rangle$. Hence,

$$P = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{y=0}^{4} (\vec{\mathbf{F}} \cdot \vec{\mathbf{j}})\,dy = \int_0^4 (4 + y)\,dy = \cdots = 24.$$

In the more general case, we can compute $d\vec{\mathbf{r}}$ for a curve given as a $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$ using a simple multivariable substitution formula:

$$d\vec{\mathbf{r}} = \frac{d\vec{\mathbf{r}}}{dt}\,dt.$$

With this formula, line integrals become ordinary one-variable integrals in the parameter t:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \left(\vec{\mathbf{F}} \cdot \frac{d\vec{\mathbf{r}}}{dt}\right)\,dt. \tag{1}$$

As an example, suppose $\vec{\mathbf{F}} = \langle x, 2x + y\rangle$ as before, but C is the top half of the unit circle, traveled clockwise. We can use the curve $-C$, parameterized by

$$x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le \pi,$$

instead of the original curve C. Using this parameterization, we have

$$\vec{\mathbf{F}} = \langle \cos t, 2\cos t + \sin t\rangle, \qquad \frac{d\vec{\mathbf{r}}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle -\sin t, \cos t\rangle,$$

so

$$\vec{\mathbf{F}} \cdot \frac{d\vec{\mathbf{r}}}{dt} = (\cos t)(-\sin t) + (2\cos t + \sin t)(\cos t) = 2\cos^2 t = 1 + \cos 2t,$$

and

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_{-C} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\int_0^{\pi} \left(\vec{\mathbf{F}} \cdot \frac{d\vec{\mathbf{r}}}{dt}\right)\,dt = -\int_0^{\pi} (1 + \cos 2t)\,dt = \cdots = -(\pi + 0) = -\pi.$$

Note that the direct method makes sense in either two or three dimensions.

Method 2 — The fundamental theorem for line integrals

The line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ can sometimes be evaluated with a potential function.[1] The following conditions must hold:

1. The vector field must be a gradient field: $\vec{\mathbf{F}} = \vec{\nabla} f$.
2. The curve C must lie in a domain in which the vector field is continuous.
3. The curve C must be piecewise smooth.

When these conditions are met, the fundamental theorem of line integrals says

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(B) - f(A), \tag{2}$$

where A and B are the initial and terminal points, respectively, of the curve C. The fundamental theorem for line integrals is analogous to the fundamental theorem of calculus, but it is applicable in a much narrower range of problems. Whereas any continuous function of one variable is the derivative of some function, an arbitrarily chosen continuous function of several variables is almost certainly not the gradient of some function.

Method 2.1 – Reparameterization for line integrals

If $\vec{\mathbf{F}}$ is known to be a gradient field, then the line integral from A to B of $\vec{\mathbf{F}}$ is independent of path. In addition to allowing for computation using the fundamental theorem for line integrals, this property allows the integral to be computed by the direct method applied to *any* curve from A to B. This may be convenient in cases where it is difficult to compute a potential function.

Method 3 — Green's theorem

Green's theorem can be used for line integrals in the plane, provided the curve and vector field satisfy several conditions:

1. The curve must be the boundary of a plane region R, traveled counterclockwise.
2. The components of the vector field $\vec{\mathbf{F}} = \langle F_1, F_2 \rangle$ must have continuous first partial derivatives in a connected open region that contains the curve.

If these conditions are met, then Green's theorem asserts the equivalence of integrals of F_1 and F_2 over the curve with integrals of derivatives of F_1 and F_2 over the region inside the curve:

$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA. \tag{3}$$

[1] See the appendix.

Green's theorem can also be written in vector form as

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R \left(\vec{\nabla} \times \vec{\mathbf{F}} \cdot \vec{\mathbf{k}}\right) dA. \tag{4}$$

Before using Green's theorem, it is important to check the necessary conditions for the curve and vector field, as they are not always met.

Method 4 — Stokes' theorem

Stokes' Theorem is the three-dimensional generalization of Green's Theorem. The requirements are a little more general:

1. The curve must be the boundary of a surface S, traveled counterclockwise as viewed from above, with "above" defined by the particular normal vector chosen to represent S.
2. The components of the vector field $\vec{\mathbf{F}} = \langle F_1, F_2, F_3\rangle$ must have continuous first partial derivatives in a connected open region that contains the curve.

If these conditions are met, then Stokes's theorem asserts the equivalence of the line integral around the closed curve C with an integral taken over the enclosed surface S:[2]

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \vec{\nabla} \times \vec{\mathbf{F}}\; d\vec{\mathbf{A}}. \tag{5}$$

[2]Surface integrals are discussed later in the course. Indeed, even the notation for the integral on the right side will likely be unfamiliar at this point. Stokes' Theorem is mentioned here for completeness only.

Appendix – Finding Potential Functions

Suppose we want to use the fundamental theorem for line integrals (method 2) to evaluate

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}, \qquad \vec{\mathbf{F}} = \langle 2x, 1 + 2yz^3, 3y^2z^2 \rangle \tag{6}$$

for some complicated curve C. This will only work if there is a potential function f for which

$$\vec{\nabla} f = \vec{\mathbf{F}}.$$

If there is such a function (if not, we'll be able to tell), it has to satisfy three requirements:

$$f_x = 2x, \qquad f_y = 1 + 2yz^3, \qquad f_z = 3x^2z^2.$$

The first requirement seems to mean $f = x^2$, but actually the most general possibility is that f is x^2 plus additional terms that might include y and z, but not x. In other words,

$$f = x^2 + f_1(y, z).$$

The same analysis can be applied to the other requirements, giving us

$$f = y + y^2z^3 + f_2(x, z)$$

and

$$f = y^2z^3 + f_3(x, y).$$

We now need to determine if there is a function f that satisfies all three of these formulas. Such a function would have to include the terms x^2, y and y^2z^3. Careful checking shows that

$$f = x^2 + y + y^2z^3$$

works with $f_1(y, z) = y + y^2z^3$, $f_2(x, z) = x^2$, $f_3(x, y) = x^2 + y$.

Note that this procedure fails if there is no potential function. For example, suppose

$$\vec{\mathbf{F}} = \langle 2xy + y, x^2 \rangle.$$

From $\vec{\nabla} f = \vec{\mathbf{F}}$, we get

$$f_x = 2xy + y, \qquad f_y = x^2$$

and then

$$f = x^2y + xy + f_1(y), \qquad f = x^2y + f_2(x).$$

These results cannot be reconciled because the term xy in the first formula cannot be $f_2(x)$ in the second.