Local cohomology and G-modules

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However, they are often very large (not finitely generated) and can be difficult to describe.

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- $\mathbb{C}[X]$ is the polynomial ring in the entries of X,
- $\Delta = (\Delta_1, \ldots, \Delta_n)$ the ideal generated by the *maximal* minors of X

Goal.

Describe $H^i_\Delta(\mathbb{C}[X])$.

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- Note that $N = \dim \mathbb{C}[\Delta_1, \ldots, \Delta_n]$.

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the injective hull of $\mathbb{C}$ as a $\mathbb{C}[X]$-module.

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Goal: Show $\alpha = 1$.

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Critical: G is *linearly reductive.*

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Sketch of proof that $H^N_\Delta(\mathbb{C}[X]) \cong E_{\mathbb{C}[X]}(\mathbb{C})$.

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- We may study $H_I^i(R)$.