The weak Lefschetz property for monomial complete intersections in positive characteristic

Adela Vraciu

University of South Carolina

joint work with Andy Kustin

Definitions

Let $A = \oplus A_i$ be a standard graded algebra over an algebraically closed field k.

We say that A has the *weak Lefschetz property* (WLP) if there exists a linear form $L \in A_1$ such that the map

$$\times L : A_i \to A_{i+1}$$

has maximal rank (i.e. it is injective or surjective) for all i.

Such a linear form L is called a Lefschetz element.
The set of Lefschetz elements forms a (possibly empty) Zarisky subset of A_1.

A key fact

Migliore, Miro-Roig, and Nagel (2011) show that if $A = k[x_1, \ldots, x_n]/I$ is a standard graded Artinian Gorenstein algebra, then A has WLP if and only if $L = x_1 + \ldots + x_n$ is a Lefschetz element, and this is if and only if the map

$$\times L : A_{\lfloor \frac{e-1}{2} \rfloor} \to A_{\lfloor \frac{e+1}{2} \rfloor}$$

is injective, where e is the degree of the socle generator of A.

In the case when $I = (x_1^{d_1}, \ldots, x_n^{d_n})$, we translate this into a condition on the degrees of the non-Koszul relations on $x_1^{d_1}, \ldots, x_{n-1}^{d_{n-1}}, (x_1 + \ldots + x_{n-1})^{d_n} \in k[x_1, \ldots, x_{n-1}]$:

Proposition

$A = \dfrac{k[x_1, \ldots, x_n]}{(x_1^{d_1}, \ldots, x_n^{d_n})}$ has WLP if and only if

the smallest total degree of a non-Koszul relation on $x_1^{d_1}, \ldots, x_{n-1}^{d_{n-1}}, (x_1 + \ldots + x_{n-1})^{d_n}$ is $\lfloor \dfrac{\Sigma_{i=1}^n d_i - n + 3}{2} \rfloor$.

We use the convention that the total degree of a relation $a_1 x_1^{d_1} + \cdots + a_n (x_1 + \ldots x_{n-1})^{d_n}$ is $\deg(a_1) + d_1$.

The role of the characteristic

Theorem [Stanley - J. Watanabe]:
If $\mathrm{char}(k) = 0$, then $A = \dfrac{k[x_1, \dots, x_n]}{(x_1^{d_1}, \dots, x_n^{d_n})}$ has WLP for every $d_1, \dots, d_n \geq 1$.

THIS IS NO LONGER TRUE IN POSITIVE CHARACTERISTIC!

Li-Zanello (2010) found a surprising connection between the monomial complete intersections in three variables that have WLP (as a function of the characteristic) and enumerations of plane partitions.

Brenner-Kaid (2011) gave an explicit description of the values of d (in terms of p) such that $\dfrac{k[x, y, z]}{(x^d, y^d, z^d)}$ has WLP.

WLP in three variables and finite projective dimension

Theorem [Kustin - Rahmati -V.]
Let $R = \dfrac{k[x,y,z]}{(x^n + y^n + z^n)}$ where $n \geq 2$, and let $N \geq n$ be an integer not divisible by n. Then

$$R/(x^N, y^N, z^N)R$$

has finite projective dimension as an R-module if and only if

$$A = \frac{k[x,y,z]}{(x^a, y^a, z^a)}$$

DOES NOT have WLP for at least one of the values $a = \lfloor \frac{N}{n} \rfloor$ or $a = \lceil \frac{N}{n} \rceil$.

Our main results

Theorem 1 [Kustin - V.]
Assume $\mathrm{char}(k) = p \geq 3$, and let $d = lp^e + d'$, where $l \leq p-1$ and $d' < p^e$.

Then $\dfrac{k[x,y,z,w]}{(x^d, y^d, z^d, w^d)}$ has WLP if and only if $l \leq \dfrac{p-1}{2}$ and

$d' \in \{\dfrac{p^e-1}{2}, \dfrac{p^e+1}{2}\}$.

Main results - continued

Theorem 2 [Kustin - V.]
Let $n \geq 5$ and $\mathrm{char}(k) = p > 0$. Then

$$A = \frac{k[x_1, \dots, x_n]}{(x_1^d, \dots, x_n^d)}$$

has WLP if and only if

$$\lfloor \frac{n(d-1)+3}{2} \rfloor \leq p$$

In particular, A does not have WLP for any $d \geq p$.

Necessary conditions for WLP

We use the Frobenius endomorphism to create relations of small degree. The following is the key observation:

Write $d = lp^e + d'$ and fix $1 \leq i \leq n$.
If

$$a_1x_1^l + \ldots + a_ix_i^l + b_{i+1}x_i^{l+1} + \ldots + b_{n-1}x_{n-1}^{l+1} + b_n(x_1 + \ldots + x_{n-1})^{l+1} = 0$$

is a relation on $x_1^l, \ldots, x_i^l, x_{i+1}^{l+1}, \ldots, x_{n-1}^{l+1}(x_1 + \ldots + x_{n-1})^{l+1}$, then

$$(x_1 \cdots x_i)^{d'} \left(a_1^{p^e} x_1^{lp^e} + \ldots + b_{n-1}^{p^e} x_{n-1}^{(l+1)p^e} + b_n^{p^e} (x_1 + \ldots + x_{n-1})^{(l+1)p^e} \right)$$

is a relation on $x_1^d, \ldots, x_{n-1}^d, (x_1 + \ldots, x_{n-1})^d$.

Necessary conditions - continued

If $\underline{l} = (l_1, \ldots, l_n)$, we use $N(\underline{l})$ to denote the smallest total degree of a non-Koszul relation on $x_1^{l_1}, \ldots, x_{n-1}^{l_{n-1}}, (x_1 + \ldots + x_{n-1})^{l_n}$.

We have shown that

$$N(\underline{d}) \leq p^e N(\underline{l}) + id'$$

where $\underline{d} = (d, \ldots, d)$, $\underline{l} = (l, \ldots, l, l+1, \ldots, l+1)$ (i l's and $n-i$ $l+1$'s), and $d = lp^e + d'$.

Necessary conditions - continued

Lemma:

Let $\underline{l} = (l, \ldots, l, l+1, \ldots, l+1)$ be as above. Then we have

$$N(\underline{l}) \leq \lfloor \frac{nl - i + 3}{2} \rfloor.$$

Sketch of Proof:

Let $A_{\underline{l}} = \dfrac{k[x_1, \ldots, x_n]}{(x_1^l, \ldots, x_i^l, x_{i+1}^{l+1}, \ldots, x_n^{l+1})}$. The socle generator of $A_{\underline{l}}$ has degree $e = nl - i$ and the Hilbert function of $A_{\underline{l}}$ becomes strictly decreasing after step $\lfloor \dfrac{e+1}{2} \rfloor$, which implies that the map

$$\times L : [A_{\underline{l}}]_{\lfloor \frac{e+1}{2} \rfloor} \to [A_{\underline{l}}]_{\lfloor \frac{e+3}{2} \rfloor}$$

is not injective. This gives rise to a non-Koszul relation of degree $\lfloor \dfrac{e+3}{2} \rfloor$.

Necessary conditions - conclusion

Corollary:
Assume that $d = lp^e + d'$ (where $p = \mathrm{char}(k)$) and $1 \leq i \leq n$. If

$$A = \frac{k[x_1, \ldots, x_n]}{(x_1^d, \ldots, x_n^d)}$$

has WLP, then

$$\lfloor \frac{n(d-1)+3}{2} \rfloor \leq N(\underline{d}) \leq p^e \lfloor \frac{nl - i + 3}{2} \rfloor + id'.$$

Sufficient conditions for WLP

Lemma:
Let $c \leq d$. The ideal $(x^d, y^d) : (x+y)^{2c}$ in $k[x, y]$ is generated in degrees $\geq d - c$ if and only if $\Delta_c(d) \neq 0$ in k, where

$$\Delta_c(d) = \begin{vmatrix} \binom{d}{1} & \binom{d}{2} & \cdots & \binom{d}{c} \\ \binom{d}{2} & \binom{d}{3} & \cdots & \binom{d}{c+1} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{d}{c} & \binom{d}{c+1} & \cdots & \binom{d}{2c-1} \end{vmatrix}$$

Proof of Lemma

The statement is equivalent to $(x^d, y^d) \cap (x+y)^{2c}$ has no non-zero elements of degree $\leq d-c-1$ if and only if $\Delta_c(d) \neq 0$. Write a general element of (x^d, y^d) of degree $d-c-1$ as a polynomial in the variables x and $x+y$:

$$H = \left[a_1 x^{c-1} + a_2 x^{c-2}(x+y) + \ldots + a_c (x+y)^{c-1}\right] x^d +$$

$$\left[b_1 x^{c-1} + b_2 x^{c-2}(x+y) + \ldots + b_c (x+y)^{c-1}\right] y^d$$

Write $y = (x+y) - x$ and use the binomial expansion for y^d; the condition that $H \in (x+y)^{2c}$ amounts to saying that the coefficients for $(x+y)^{2c-1}, (x+y)^{2c-2}, \ldots, (x+y), 1$ that are obtained when the expression for H is expanded are equal to zero. This gives rise to a homogeneous system of $2c$ equations in the unknowns $a_1, \ldots, a_c, b_1, \ldots, b_c$. The first c equations tell us that the a_i's can be expressed as linear combinations of the b_i's. The last c equations (in the unknowns $b_1, \ldots, b_c$) have determinant $\Delta_c(d)$, so $\Delta_c(d) \neq 0 \Leftrightarrow$ there is no non-trivial solution.

Sufficient conditions for WLP in four variables

Theorem:
Assume that $\Delta_c(d) \neq 0$ in k for every $c \in \{1, \ldots, d\}$.
Then

$$A = \frac{k[x, y, z, w]}{(x^d, y^d, z^d, w^d)}$$

has WLP.

Sketch of Proof:
WLP is equivalent to the statement that $(x^d, y^d, z^d) : (x + y + z)^d$ has no non-zero elements of degree $d - 2$.
Let $u = u_{d-2} + u_{d-3}z + \ldots + u_0 z^{d-2} \in (x^d, y^d, z^d) : (x + y + z)^d$, with $u_i \in k[x, y]$ homogeneous of degree i. We want to show $u = 0$. Expand $(x + y + z)^d = \Sigma_{i=0}^{d} \left(\begin{array}{c} d \\ i \end{array} \right) (x + y)^i z^{d-i}$ and multiply $u(x + y + z)^d$; the condition is that the coefficients of $1, z, \ldots, z^{d-1}$ in the resulting expression are in (x^d, y^d).

The coefficient of z^{d-1} is

$$d(x+y)u_{d-2} + \begin{pmatrix} d \\ 2 \end{pmatrix} (x+y)^2 u_{d-3} + \ldots + d(x+y)^{d-1} u_0.$$

Since it has degree $d-1$, the only way it can be in (x^d, y^d) is if it is zero; this implies $u_{d-2} \in (x+y)$.
Now the coefficient of z^{d-2}

$$\begin{pmatrix} d \\ 2 \end{pmatrix} (x+y)^2 u_{d-2} + \begin{pmatrix} d \\ 3 \end{pmatrix} (x+y)^3 u_{d-3} + \ldots + u_0 (x+y)^d$$

must be in $(x^d, y^d) \cap (x+y)^2$.
According to the lemma, it has insufficient degree, so it must be zero; this implies $u_{d-2} \in (x+y)^2, u_{d-3} \in (x+y)$, etc.

The determinants $\Delta_c(d)$

It is known that

$$\Delta_c(d) = \frac{\binom{d}{c}\binom{d+1}{c}\cdots\binom{d+c-1}{c}}{\binom{c}{c}\binom{c+1}{c}\cdots\binom{2c-1}{c}}$$

If $d = lp^e + d'$, with $l \leq \frac{p-1}{2}$ and $d' \in \{\frac{p^e-1}{2}, \frac{p^e+1}{2}\}$, we show that $\Delta_c(d) \neq 0$ in k (where $\mathrm{char}(k) = p$) for all $c = 1, \ldots, d$ by counting the powers of p in the numerator and in the denominator.