Dimensions of derived categories and singularity categories

Ryo Takahashi

Shinshu University/University of Nebraska-Lincoln

October 16, 2011

Joint work with Takuma Aihara

Joint work with Takuma Aihara

Definition

Joint work with Takuma Aihara

Definition

Let $\mathcal{T}$ be a triangulated category

Joint work with Takuma Aihara

Definition

Let $\mathcal{T}$ be a triangulated category
(e.g. $\mathsf{D}^{\mathsf{b}}(\mathsf{R})$, $\mathsf{D}_{\mathsf{sg}}(\mathsf{R}) = \mathsf{D}^{\mathsf{b}}(\mathsf{R})/\mathsf{perf}(\mathsf{R})$, $\underline{\mathsf{CM}}(\mathsf{R})$, $\dots$).

Joint work with Takuma Aihara

Definition

Let $\mathcal{T}$ be a triangulated category
(e.g. $D^b(R)$, $D_{sg}(R) = D^b(R)/\text{perf}(R)$, $\underline{CM}(R)$, $\ldots$).
Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{T}$ and $n \geq 1$.

Joint work with Takuma Aihara

Definition

Let $\mathcal{T}$ be a triangulated category
(e.g. $\mathrm{D^b(R)}$, $\mathrm{D_{sg}(R) = D^b(R)/perf(R)}$, $\underline{\mathrm{CM}}(\mathrm{R})$, $\dots$).
Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{T}$ and $\mathrm{n} \geq 1$.

1. $\mathcal{X} * \mathcal{Y}$ consists of $\mathrm{C} \in \mathcal{T}$ admitting exact triangles $\mathrm{M} \to \mathrm{C} \to \mathrm{N} \to$ with $\mathrm{M} \in \mathcal{X}$ and $\mathrm{N} \in \mathcal{Y}$.

Joint work with Takuma Aihara

Definition

Let $\mathcal{T}$ be a triangulated category
(e.g. $\mathsf{D^b(R)}$, $\mathsf{D_{sg}(R) = D^b(R)/perf(R)}$, $\underline{\mathsf{CM}}\mathsf{(R)}$, $\dots$).
Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{T}$ and $\mathsf{n} \geq 1$.

1. $\mathcal{X} * \mathcal{Y}$ consists of $\mathsf{C} \in \mathcal{T}$ admitting exact triangles $\mathsf{M} \to \mathsf{C} \to \mathsf{N} \to$ with $\mathsf{M} \in \mathcal{X}$ and $\mathsf{N} \in \mathcal{Y}$.
2. $\langle \mathcal{X} \rangle$ is the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{X}$ and closed under finite direct sums, direct summands and shifts.

Joint work with Takuma Aihara

Definition

Let $\mathcal{T}$ be a triangulated category
(e.g. $\mathrm{D^b(R)}$, $\mathrm{D_{sg}(R) = D^b(R)/perf(R)}$, $\underline{\mathrm{CM}}(\mathrm{R})$, $\dots$).
Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{T}$ and $\mathrm{n} \geq 1$.

1. $\mathcal{X} * \mathcal{Y}$ consists of $\mathrm{C} \in \mathcal{T}$ admitting exact triangles $\mathrm{M} \to \mathrm{C} \to \mathrm{N} \to$ with $\mathrm{M} \in \mathcal{X}$ and $\mathrm{N} \in \mathcal{Y}$.
2. $\langle \mathcal{X} \rangle$ is the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{X}$ and closed under finite direct sums, direct summands and shifts.

$$\langle \mathcal{X} \rangle_{\mathrm{n}} = \langle \overbrace{\langle \mathcal{X} \rangle * \cdots * \langle \mathcal{X} \rangle}^{\mathrm{n}} \rangle,$$

Joint work with Takuma Aihara

Definition

Let $\mathcal{T}$ be a triangulated category (e.g. $D^b(R)$, $D_{sg}(R) = D^b(R)/\mathrm{perf}(R)$, $\underline{\mathrm{CM}}(R)$, ...).
Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{T}$ and $n \geq 1$.

1. $\mathcal{X} * \mathcal{Y}$ consists of $C \in \mathcal{T}$ admitting exact triangles $M \to C \to N \to$ with $M \in \mathcal{X}$ and $N \in \mathcal{Y}$.
2. $\langle \mathcal{X} \rangle$ is the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{X}$ and closed under finite direct sums, direct summands and shifts.

$$\langle \mathcal{X} \rangle_n = \langle \overbrace{\langle \mathcal{X} \rangle * \cdots * \langle \mathcal{X} \rangle}^{n} \rangle, \ \langle G \rangle_n := \langle \{G\} \rangle_n \text{ for } G \in \mathcal{T}.$$

Joint work with Takuma Aihara

Definition

Let $\mathcal{T}$ be a triangulated category (e.g. $D^b(R)$, $D_{sg}(R) = D^b(R)/\mathrm{perf}(R)$, $\underline{CM}(R)$, $\dots$).
Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{T}$ and $n \geq 1$.

1. $\mathcal{X} * \mathcal{Y}$ consists of $C \in \mathcal{T}$ admitting exact triangles $M \to C \to N \to$ with $M \in \mathcal{X}$ and $N \in \mathcal{Y}$.
2. $\langle \mathcal{X} \rangle$ is the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{X}$ and closed under finite direct sums, direct summands and shifts.
$$\langle \mathcal{X} \rangle_n = \langle \overbrace{\langle \mathcal{X} \rangle * \cdots * \langle \mathcal{X} \rangle}^{n} \rangle, \ \langle G \rangle_n := \langle \{G\} \rangle_n \text{ for } G \in \mathcal{T}.$$
3. The dimension $\dim \mathcal{T}$ of $\mathcal{T}$ is the infimum of $r \geq 0$ such that $\mathcal{T} = \langle G \rangle_{r+1}$ for some $G \in \mathcal{T}$.

Let R be a commutative Noetherian ring.

Let R be a commutative Noetherian ring.

Fact

Let R be a commutative Noetherian ring.

Fact

1. $\dim D^b(R) \geq \dim D_{sg}(R)$.

Let **R** be a commutative Noetherian ring.

Fact

1. $\dim \mathsf{D}^b(R) \geq \dim \mathsf{D}_{sg}(R)$.
2. [Rouquier 2008]
   If R is a reduced affine algebra, then $\dim \mathsf{D}^b(R) \geq \dim R$.

Let R be a commutative Noetherian ring.

Fact

1. $\dim D^b(R) \geq \dim D_{sg}(R)$.
2. [Rouquier 2008]
   If R is a reduced affine algebra, then $\dim D^b(R) \geq \dim R$.
3. [Avramov-Iyengar]
   If R is local, then $\dim D_{sg}(R) \geq \operatorname{cfrank} R - 1$.

Question

Question

When $\dim D^b(R) < \infty$?

Question

When $\dim D^b(R) < \infty$?

Fact

Question

When $\dim D^b(R) < \infty$?

Fact

1. If R is Artinian, then $D^b(R) = \langle R/\operatorname{rad} R\rangle_{\ell\ell(R)}$, and $\dim D^b(R) \le \ell\ell(R) - 1 < \infty$.

Question

When $\dim D^b(R) < \infty$?

Fact

1. If R is Artinian, then $D^b(R) = \langle R/\operatorname{rad} R\rangle_{\ell\ell(R)}$, and $\dim D^b(R) \leq \ell\ell(R) - 1 < \infty$.
2. [D. Christensen 1998, Krause-Kussin 2006]
   If R is regular with $\dim R < \infty$, then $D^b(R) = \langle R\rangle_{\dim R+1}$, and $\dim D^b(R) \leq \dim R < \infty$.

Question

When $\dim D^b(R) < \infty$?

Fact

1. If R is Artinian, then $D^b(R) = \langle R/\operatorname{rad} R\rangle_{\ell\ell(R)}$, and $\dim D^b(R) \leq \ell\ell(R) - 1 < \infty$.
2. [D. Christensen 1998, Krause-Kussin 2006] If R is regular with $\dim R < \infty$, then $D^b(R) = \langle R\rangle_{\dim R+1}$, and $\dim D^b(R) \leq \dim R < \infty$.
3. [Rouquier 2008, Keller-Van den Bergh] If R is essentially of finite type, then $\dim D^b(R) < \infty$.

Question

When $\dim D^b(R) < \infty$?

Fact

1. If R is Artinian, then $D^b(R) = \langle R/\operatorname{rad} R\rangle_{\ell\ell(R)}$, and $\dim D^b(R) \leq \ell\ell(R) - 1 < \infty$.
2. [D. Christensen 1998, Krause-Kussin 2006]
   If R is regular with $\dim R < \infty$, then $D^b(R) = \langle R\rangle_{\dim R+1}$, and $\dim D^b(R) \leq \dim R < \infty$.
3. [Rouquier 2008, Keller-Van den Bergh]
   If R is essentially of finite type, then $\dim D^b(R) < \infty$.

Question

Question

When $\dim D^b(R) < \infty$?

Fact

1. If R is Artinian, then $D^b(R) = \langle R/\operatorname{rad} R\rangle_{\ell\ell(R)}$, and $\dim D^b(R) \le \ell\ell(R) - 1 < \infty$.
2. [D. Christensen 1998, Krause-Kussin 2006]
   If R is regular with $\dim R < \infty$, then $D^b(R) = \langle R\rangle_{\dim R+1}$, and $\dim D^b(R) \le \dim R < \infty$.
3. [Rouquier 2008, Keller-Van den Bergh]
   If R is essentially of finite type, then $\dim D^b(R) < \infty$.

Question

If R is complete local, then $\dim D^b(R) < \infty$?

Example

Example

Let $R = k[[t^4, t^5, t^6]] \subseteq k[[t]] = S$.

Example

Let $R = k[[t^4, t^5, t^6]] \subseteq k[[t]] = S$. We have $C_R = t^8 S \subseteq R$.

Example

Let $R = k[[t^4, t^5, t^6]] \subseteq k[[t]] = S$. We have $C_R = t^8 S \subseteq R$.
There is an exact sequence $0 \to R \xrightarrow{f} S \to S/R \to 0$,

Example

Let $R = k[[t^4, t^5, t^6]] \subseteq k[[t]] = S$. We have $C_R = t^8 S \subseteq R$.
There is an exact sequence $0 \to R \xrightarrow{f} S \to S/R \to 0$, where S/R is an R/C_R-module.

Example

Let $R = k[[t^4, t^5, t^6]] \subseteq k[[t]] = S$. We have $C_R = t^8 S \subseteq R$. There is an exact sequence $0 \to R \xrightarrow{f} S \to S/R \to 0$, where S/R is an R/C_R-module. Fix $X \in D^b(R)$.

Example

Let $R = k[[t^4, t^5, t^6]] \subseteq k[[t]] = S$. We have $C_R = t^8S \subseteq R$. There is an exact sequence $0 \to R \xrightarrow{f} S \to S/R \to 0$, where S/R is an R/C_R-module. Fix $X \in D^b(R)$. There is an exact sequence of R-complexes

$$0 \to M \to X \xrightarrow{X \otimes_R f} X \otimes_R S \to N \to 0,$$

Example

Let $R = k[[t^4, t^5, t^6]] \subseteq k[[t]] = S$. We have $C_R = t^8S \subseteq R$. There is an exact sequence $0 \to R \xrightarrow{f} S \to S/R \to 0$, where S/R is an R/C_R-module. Fix $X \in D^b(R)$. There is an exact sequence of R-complexes

$$0 \to M \to X \xrightarrow{X \otimes_R f} X \otimes_R S \to N \to 0,$$

which gives exact triangles

$$M \to X \to L \to \quad \text{and} \quad L \to X \otimes_R S \to N \to .$$

Example

Let $R = k[[t^4, t^5, t^6]] \subseteq k[[t]] = S$. We have $C_R = t^8 S \subseteq R$. There is an exact sequence $0 \to R \xrightarrow{f} S \to S/R \to 0$, where S/R is an R/C_R-module. Fix $X \in D^b(R)$. There is an exact sequence of R-complexes

$$0 \to M \to X \xrightarrow{X \otimes_R f} X \otimes_R S \to N \to 0,$$

which gives exact triangles

$$M \to X \to L \to \quad \text{and} \quad L \to X \otimes_R S \to N \to .$$

We have $X \otimes_R S \in D^b(S) = \langle S \rangle_2$

Example

Let $R = k[[t^4, t^5, t^6]] \subseteq k[[t]] = S$. We have $C_R = t^8 S \subseteq R$. There is an exact sequence $0 \to R \xrightarrow{f} S \to S/R \to 0$, where S/R is an R/C_R-module. Fix $X \in D^b(R)$. There is an exact sequence of R-complexes

$$0 \to M \to X \xrightarrow{X \otimes_R f} X \otimes_R S \to N \to 0,$$

which gives exact triangles

$$M \to X \to L \to \quad \text{and} \quad L \to X \otimes_R S \to N \to .$$

We have $X \otimes_R S \in D^b(S) = \langle S \rangle_2$ and $M, N \in D^b(R/C_R) = \langle k \rangle_2$,

Example

Let $R = k[[t^4, t^5, t^6]] \subseteq k[[t]] = S$. We have $C_R = t^8 S \subseteq R$. There is an exact sequence $0 \to R \xrightarrow{f} S \to S/R \to 0$, where S/R is an R/C_R-module. Fix $X \in D^b(R)$. There is an exact sequence of R-complexes

$$0 \to M \to X \xrightarrow{X \otimes_R f} X \otimes_R S \to N \to 0,$$

which gives exact triangles

$$M \to X \to L \to \quad \text{and} \quad L \to X \otimes_R S \to N \to .$$

We have $X \otimes_R S \in D^b(S) = \langle S \rangle_2$ and $M, N \in D^b(R/C_R) = \langle k \rangle_2$, since $R/C_R \cong k[[x, y, z]]/(x, y, z)^2$ and $\ell\ell(R) = 2$.

Example

Let $R = k[[t^4, t^5, t^6]] \subseteq k[[t]] = S$. We have $C_R = t^8 S \subseteq R$. There is an exact sequence $0 \to R \xrightarrow{f} S \to S/R \to 0$, where S/R is an R/C_R-module. Fix $X \in D^b(R)$. There is an exact sequence of R-complexes

$$0 \to M \to X \xrightarrow{X \otimes_R f} X \otimes_R S \to N \to 0,$$

which gives exact triangles

$$M \to X \to L \to \quad \text{and} \quad L \to X \otimes_R S \to N \to .$$

We have $X \otimes_R S \in D^b(S) = \langle S \rangle_2$ and $M, N \in D^b(R/C_R) = \langle k \rangle_2$, since $R/C_R \cong k[[x, y, z]]/(x, y, z)^2$ and $\ell\ell(R) = 2$. Hence $L \in \langle S \oplus k \rangle_4$,

Example

Let $R = k[[t^4, t^5, t^6]] \subseteq k[[t]] = S$. We have $C_R = t^8 S \subseteq R$. There is an exact sequence $0 \to R \xrightarrow{f} S \to S/R \to 0$, where S/R is an R/C_R-module. Fix $X \in D^b(R)$. There is an exact sequence of R-complexes

$$0 \to M \to X \xrightarrow{X \otimes_R f} X \otimes_R S \to N \to 0,$$

which gives exact triangles

$$M \to X \to L \to \quad \text{and} \quad L \to X \otimes_R S \to N \to .$$

We have $X \otimes_R S \in D^b(S) = \langle S \rangle_2$ and $M, N \in D^b(R/C_R) = \langle k \rangle_2$, since $R/C_R \cong k[[x, y, z]]/(x, y, z)^2$ and $\ell\ell(R) = 2$. Hence $L \in \langle S \oplus k \rangle_4$, and $X \in \langle S \oplus k \rangle_6$.

Example

Let $R = k[[t^4, t^5, t^6]] \subseteq k[[t]] = S$. We have $C_R = t^8 S \subseteq R$. There is an exact sequence $0 \to R \xrightarrow{f} S \to S/R \to 0$, where S/R is an R/C_R-module. Fix $X \in D^b(R)$. There is an exact sequence of R-complexes

$$0 \to M \to X \xrightarrow{X \otimes_R f} X \otimes_R S \to N \to 0,$$

which gives exact triangles

$$M \to X \to L \to \quad \text{and} \quad L \to X \otimes_R S \to N \to .$$

We have $X \otimes_R S \in D^b(S) = \langle S \rangle_2$ and $M, N \in D^b(R/C_R) = \langle k \rangle_2$, since $R/C_R \cong k[[x, y, z]]/(x, y, z)^2$ and $\ell\ell(R) = 2$. Hence $L \in \langle S \oplus k \rangle_4$, and $X \in \langle S \oplus k \rangle_6$. This shows $D^b(R) = \langle S \oplus k \rangle_6$,

Example

Let $R = k[[t^4, t^5, t^6]] \subseteq k[[t]] = S$. We have $C_R = t^8 S \subseteq R$. There is an exact sequence $0 \to R \xrightarrow{f} S \to S/R \to 0$, where S/R is an R/C_R-module. Fix $X \in D^b(R)$. There is an exact sequence of R-complexes

$$0 \to M \to X \xrightarrow{X \otimes_R f} X \otimes_R S \to N \to 0,$$

which gives exact triangles

$$M \to X \to L \to \quad \text{and} \quad L \to X \otimes_R S \to N \to .$$

We have $X \otimes_R S \in D^b(S) = \langle S \rangle_2$ and $M, N \in D^b(R/C_R) = \langle k \rangle_2$, since $R/C_R \cong k[[x, y, z]]/(x, y, z)^2$ and $\ell\ell(R) = 2$. Hence $L \in \langle S \oplus k \rangle_4$, and $X \in \langle S \oplus k \rangle_6$. This shows $D^b(R) = \langle S \oplus k \rangle_6$, and $\dim D^b(R) \leq 5 < \infty$.

Theorem (Aihara-T)

Theorem (Aihara-T)

Let R be a complete equicharacteristic local ring with perfect residue field. Then $\dim D^b(R) < \infty$.