Some relations between countable Cohen-Macaulay representation type and super-stretched

Branden Stone
University of Kansas

October 14, 2011
Finite Cohen-Macaulay Type

Definition
A local Cohen-Macaulay ring has finite (resp. countably) Cohen-Macaulay type provided there are, up to isomorphism, only finitely (resp. countably) many indecomposable maximal Cohen-Macaulay modules.
Finite Cohen-Macaulay Type

Definition

A local Cohen-Macaulay ring has finite (resp. countably) Cohen-Macaulay type provided there are, up to isomorphism, only finitely (resp. countably) many indecomposable maximal Cohen-Macaulay modules.

Examples of finite type:
Definition

A local Cohen-Macaulay ring has finite (resp. countably) Cohen-Macaulay type provided there are, up to isomorphism, only finitely (resp. countably) many indecomposable maximal Cohen-Macaulay modules.

Examples of finite type:
- Regular local rings
Finite Cohen-Macaulay Type

**Definition**
A local Cohen-Macaulay ring has finite (resp. countably) Cohen-Macaulay type provided there are, up to isomorphism, only finitely (resp. countably) many indecomposable maximal Cohen-Macaulay modules.

Examples of finite type:
- Regular local rings
- (Herzog 1978) 0-dimensional hypersurface rings;
ADE Singularities (Knörrer 1987, Buchweitz-Greuel-Schreyer 1987)

If $k = \mathbb{C}$, then the complete ADE plane curve singularities over $\mathbb{C}$ are

$$k[x, y, z_1, \ldots, z_r]/(f),$$

where $f$ is one of the following polynomials:

$(A_n)$: $x^{n+1} + y^2 + z_1^2 + \cdots + z_r^2$, $n \geq 1$;

$(D_n)$: $x^{n-1} + xy^2 + z_1^2 + \cdots + z_r^2$, $n \geq 4$;

$(E_6)$: $x^4 + y^3 + z_1^2 + \cdots + z_r^2$;

$(E_7)$: $x^3y + y^3 + z_1^2 + \cdots + z_r^2$;

$(E_8)$: $x^5 + y^3 + z_1^2 + \cdots + z_r^2$. 
Countable Cohen-Macaulay Type

Example (Buchweitz-Greuel-Schreyer 1987)

A complete hypersurface singularity over an algebraically closed uncountable field $k$ has (infinite) countable Cohen-Macaulay type iff it is isomorphic to one of the following:

\[
A_{\infty} : \ k[[x, y, z_2, \ldots, z_r]]/(y^2 + z_2^2 + \cdots + z_r^2);
\]

\[
D_{\infty} : \ k[[x, y, z_2, \ldots, z_r]]/(xy^2 + z_2^2 + \cdots + z_r^2).
\]
Motivating Question

Question (Huneke-Leuschke)

Let $R$ be a complete local Cohen-Macaulay ring of countable Cohen-Macaulay representation type, and assume that $R$ has an isolated singularity. Is $R$ then necessarily of finite Cohen-Macaulay representation type?
Motivating Question

<table>
<thead>
<tr>
<th>Question (Huneke-Leuschke)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $R$ be a complete local Cohen-Macaulay ring of countable Cohen-Macaulay representation type, and assume that $R$ has an isolated singularity. Is $R$ then necessarily of finite Cohen-Macaulay representation type?</td>
</tr>
</tbody>
</table>

- (Knörrer 1987, Buchweitz-Greuel-Schreyer 1987) True for hypersurfaces;
Motivating Question

Question (Huneke-Leuschke)

Let $R$ be a complete local Cohen-Macaulay ring of countable Cohen-Macaulay representation type, and assume that $R$ has an isolated singularity. Is $R$ then necessarily of finite Cohen-Macaulay representation type?

- (Knörrer 1987, Buchweitz-Greuel-Schreyer 1987) True for hypersurfaces;
- (Karr-Wiegand 2010) True for one dimensional case;
In 1988, D. Eisenbud and J. Herzog completely classified the graded Cohen-Macaulay rings of finite type. To do this they showed that such rings are stretched in the sense of J. Sally (1979).

**Definition**

A standard graded Cohen-Macaulay ring $R$ of dimension $d$ is said to be \textbf{stretched} if there exists a regular sequence $x_1, \ldots, x_d$ of degree 1 elements such that

$$\dim_k \left( \frac{R}{(x_1, \ldots, x_d)} \right)_i \leq 1$$

for all $i \geq 2$. 

B. Stone — October 14, 2011
In 1988, D. Eisenbud and J. Herzog completely classified the graded Cohen-Macaulay rings of finite type. To do this they showed that such rings are stretched in the sense of J. Sally (1979).

**Definition**

A standard graded Cohen-Macaulay ring $R$ of dimension $d$ is said to be **stretched** if there exists a regular sequence $x_1, \ldots, x_d$ of degree 1 elements such that

$$\dim_k \left( \frac{R}{(x_1, \ldots, x_d)} \right)_i \leq 1$$

for all $i \geq 2$. 

$(1, n, 1, 1, \ldots, 1)$
Super-Stretched

**Definition**

A standard graded ring $R$ of dimension $d$ is said to be **super-stretched** if for all system of parameters $x_1, \ldots, x_d$, we have that

$$\dim_k \left( \frac{R}{(x_1, \ldots, x_d)} \right)_i \leq 1$$

for all $i \geq \sum \deg(x_i) - d + 2$.
Super-Stretched

**Definition**

A standard graded ring $R$ of dimension $d$ is said to be **super-stretched** if for all system of parameters $x_1, \ldots, x_d$, we have that

$$\dim_k \left( \frac{R}{(x_1, \ldots, x_d)} \right)_i \leq 1$$

for all $i \geq \sum \text{deg}(x_i) - d + 2$.

Example: $k[\![x, y]\!] / x^4$ is stretched but not super-stretched.
Super-Stretched

**Definition**

A standard graded ring $R$ of dimension $d$ is said to be **super-stretched** if for all system of parameters $x_1, \ldots, x_d$, we have that

$$\dim_k \left( \frac{R}{(x_1, \ldots, x_d)} \right)_i \leq 1$$

for all $i \geq \sum \deg(x_i) - d + 2$.

Example: $k[[x, y]]/x^4$ is stretched but not super-stretched.

Modulo $y$ gives $(1, 1, 1, 1)$
Super-Stretched

**Definition**

A standard graded ring $R$ of dimension $d$ is said to be **super-stretched** if for all system of parameters $x_1, \ldots, x_d$, we have that

$$\dim_k \left( \frac{R}{(x_1, \ldots, x_d)} \right)_i \leq 1$$

for all $i \geq \sum \deg(x_i) - d + 2$.

Example: $k[x, y]/x^4$ is stretched but not super-stretched.

Modulo $y$ gives $(1, 1, 1, 1)$

Module $y^2$ gives $(1, 2, 2, 2, 1)$
More Examples of super-stretched

- The complete ADE plane curve singularities over $\mathbb{C}$;
- Any ring of finite type;
- $A_\infty : k\llbracket x, y, z_2, \ldots, z_r \rrbracket / (y^2 + z_2^2 + \cdots + z_r^2)$;
- $D_\infty : k\llbracket x, y, z_2, \ldots, z_r \rrbracket / (xy^2 + z_2^2 + \cdots + z_r^2)$;
- $k\llbracket x, y, a, b, z \rrbracket / (xa, xb, ya, yb, xz - y^n, az - b^m), n, m \geq 0$ (Burban-Drozd 2010)
Main Theorem

Theorem (Stone 2011)

A graded, noetherian, Cohen-Macaulay ring of countable Cohen-Macaulay type and uncountable residue field is super-stretched.
Main Theorem

**Theorem (Stone 2011)**

A graded, noetherian, Cohen-Macaulay ring of countable Cohen-Macaulay type and uncountable residue field is super-stretched.

The main tool in the proof is my ability to recover an ideal from its $d^{th}$ syzygy. That is, given a free resolution of an $\mathfrak{m}$-primary ideal $J$, I am able to regain the ideal from the $d^{th}$ syzygy of the resolution.
Conjecture

Conjecture (Burban ??)

A Gorenstein ring of countable Cohen-Macaulay representation type is an hypersurface.
Conjecture

Conjecture (Burban ??)

A Gorenstein ring of countable Cohen-Macaulay representation type is an hypersurface.

Proposition (Stone 2011)

Let $R$ be a graded complete intersection with uncountable residue field and of countable Cohen-Macaulay representation type. Then $R$ is a hypersurface with multiplicity at most three.
### Conjecture

**Conjecture (Burban ??)**

*A Gorenstein ring of countable Cohen-Macaulay representation type is an hypersurface.*

### Proposition (Stone 2011)

**Proposition (Stone 2011)**

*Let $R$ be a graded complete intersection with uncountable residue field and of countable Cohen-Macaulay representation type. Then $R$ is a hypersurface with multiplicity at most three.*

### Theorem (Stone 2011)

**Theorem (Stone 2011)**

*Let $R$ be a graded Gorenstein ring of dimension one and uncountable residue field. If $R$ is of countable Cohen-Macaulay representation type, then $R$ is a hypersurface.*
Thank You