Some relations between countable Cohen-Macaulay representation type and super-stretched

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October 14, 2011

Finite Cohen-Macaulay Type

Definition

A local Cohen-Macaulay ring has finite (resp. countably) Cohen-Macaulay type provided there are, up to isomorphism, only finitely (resp. countably) many indecomposable maximal Cohen-Macaulay modules.

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Examples of finite type:

- Regular local rings
- (Herzog 1978)
  0-dimensional hypersurface rings;

ADE Singularities (Knörrer 1987, Buchweitz-Greuel-Schreyer 1987)

If $k = \mathbb{C}$, then the complete ADE plane curve singularities over $\mathbb{C}$ are

$$k[\![x, y, z_1, \ldots, z_r]\!]/(f),$$

where f is one of the following polynomials:

$$\begin{aligned}
(A_n) &: x^{n+1} + y^2 + z_1^2 + \cdots + z_r^2, \;\; n \geqslant 1; \\
(D_n) &: x^{n-1} + xy^2 + z_1^2 + \cdots + z_r^2, \;\; n \geqslant 4; \\
(E_6) &: x^4 + y^3 + z_1^2 + \cdots + z_r^2; \\
(E_7) &: x^3y + y^3 + z_1^2 + \cdots + z_r^2; \\
(E_8) &: x^5 + y^3 + z_1^2 + \cdots + z_r^2.
\end{aligned}$$

Countable Cohen-Macaulay Type

Example (Buchweitz-Greuel-Schreyer 1987)

A complete hypersurface singularity over an algebraically closed uncountable field k has (infinite) countable Cohen-Macaulay type iff it is isomorphic to one of the following:

$$A_\infty : \ k[\![x, y, z_2, \dots, z_r]\!]/(y^2 + z_2^2 + \cdots + z_r^2);$$
$$D_\infty : \ k[\![x, y, z_2, \dots, z_r]\!]/(xy^2 + z_2^2 + \cdots + z_r^2).$$

Motivating Question

Question (Huneke-Leuschke)

Let R be a complete local Cohen-Macaulay ring of countable Cohen-Macaulay representation type, and assume that R has an isolated singularity. Is R then necessarily of finite Cohen-Macaulay representation type?

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- (Knörrer 1987, Buchweitz-Greuel-Schreyer 1987)
  True for hypersurfaces;
- (Karr-Wiegand 2010)
  True for one dimensional case;

Stretched

In 1988, D. Eisenbud and J. Herzog completely classified the graded Cohen-Macaulay rings of finite type. To do this they showed that such rings are stretched in the sense of J. Sally (1979).

Definition

A standard graded Cohen-Macaulay ring R of dimension d is said to be stretched if there exists a regular sequence $x_1, \ldots, x_d$ of degree 1 elements such that

$$\dim_k \left(\frac{R}{(x_1, \ldots, x_d)} \right)_i \leqslant 1$$

for all $i \geqslant 2$.

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for all $i \geqslant 2$. $(1, n, 1, 1, \ldots, 1)$

Super-Stretched

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A standard graded ring R of dimension d is said to be super-stretched if for all system of parameters $x_1, \ldots, x_d$, we have that

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Modulo y gives $(1, 1, 1, 1)$

Module y^2 gives $(1, 2, 2, 2, 1)$

More Examples of super-stretched

- The complete ADE plane curve singularities over $\mathbb{C}$;
- Any ring of finite type;
- $A_\infty : \quad k[\![x, y, z_2, \ldots, z_r]\!]/(y^2 + z_2^2 + \cdots + z_r^2)$;
- $D_\infty : \quad k[\![x, y, z_2, \ldots, z_r]\!]/(xy^2 + z_2^2 + \cdots + z_r^2)$;
- $k[\![x, y, a, b, z]\!]/(xa, xb, ya, yb, xz - y^n, az - b^m),\ n, m \geqslant 0$ (Burban-Drozd 2010)

Main Theorem

Theorem (Stone 2011)

A graded, noetherian, Cohen-Macaulay ring of countable Cohen-Macaulay type and uncountable residue field is super-stretched.

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The main tool in the proof is my ability to recover an ideal from its d^{th} syzygy. That is, given a free resolution of an $\mathfrak{m}$-primary ideal J, I am able to regain the ideal from the d^{th} syzygy of the resolution.

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A Gorenstein ring of countable Cohen-Macaulay representation type is an hypersurface.

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Proposition (Stone 2011)

Let R be a graded complete intersection with uncountable residue field and of countable Cohen-Macaulay representation type. Then R is a hypersurface with multiplicity at most three.

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Theorem (Stone 2011)

Let R be a graded Gorenstein ring of dimension one and uncountable residue field. If R is of countable Cohen-Macaulay representation type, then R is a hypersurface.

Thank You