DG Ext and Yoneda Ext for DG modules

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Assumption

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1. The homothety map $\chi^R_C\colon R \to \mathrm{Hom}_R(C, C)$ given by $\chi^R_C(r)(c) = rc$ is an isomorphism, and
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3. in Wakamatsu's work on tilting theory

Conjecture (W. V. Vasconcelos, 1974)

The set of isomorphism classes of semidualizing modules over a Cohen-Macaulay local ring is finite.

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If R is Cohen-Macaulay and equicharacteristic, then the set of isomorphism classes of semidualizing modules is finite.

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3. $\mathrm{GL}_n(R)$ acts on this scheme so that orbits are exactly the isomorphism classes
4. Ext vanishing implies that every semidualizing R-module has an open orbit
5. There can only be finitely many open orbits, so there are only finitely many isomorphism classes of semidualizing R-modules

Definition

A commutative differential graded algebra over R (*DG R-algebra* for short) is an R-complex A with $A_i = 0$ for $i < 0$ equipped with a chain map $\mu^A \colon A \otimes_R A \to A$ denoted $\mu^A(a \otimes b) = ab$ (which is called the product) that is

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1. associative: for all $a, b, c \in A$ we have $(ab)c = a(bc)$;
2. unital: there is an element $1 \in A_0$ such that for all $a \in A$ we have $1a = a$;
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Examples

1. R is a DG R-algebra.
2. The Koszul complex $K^R(x_1, \cdots, x_n)$ with $x_1, \cdots, x_n \in R$ is a DG R-algebra.

Definition

Let A be a DG R-algebra. A differential graded module over A (DG A-module for short) is an R-complex M with a chain map $\mu^M \colon A \otimes_R M \to M$ with $am := \mu^M(a \otimes m)$ that is unitary and associative. The map μ^M is the scalar product on M.

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Example

If we consider R as a DG R-algebra, then the DG R-modules are exactly the R-complexes.

Definition

Let A be a DG R-algebra, and let P be a DG A-module. P is called semiprojective if $\mathrm{Hom}_A(P, -)$ preserves surjective quasiisomorphisms.

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Every homologically finite DG A-module M has a semiprojective resolution $P \xrightarrow{\simeq} M$.

Definition

Let A be a DG R-algebra, and let M, N be DG A-modules. Given a semiprojective resolution $P \xrightarrow{\simeq} M$, we set $\mathrm{Ext}^i_A(M, N) = \mathrm{H}_{-i}(\mathrm{Hom}_A(P, N))$ for each integer i.

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Let $R = k[[x]]$ where k is a field. Then we have the following exact sequence of R-complexes and chain maps:

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$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \xrightarrow{x\cdot} & R & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & R & \xrightarrow{x\cdot} & R & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
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\end{array}$$

Now we have $\mathrm{Ext}^1_R(\underline{k}, \underline{R}) = 0$, but $\mathrm{YExt}^1_R(\underline{k}, \underline{R}) \neq 0$.

Theorem

Let A be a DG R-algebra, and let P be a semiprojective DG A-module. Then for each DG A-module N we have

$$\mathrm{YExt}^1_A(P,N) \cong \mathrm{Ext}^1_A(P,N).$$

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Sketch of Proof. Let $\alpha \in \mathrm{YExt}^1_A(P,N)$ be represented by the exact sequence $0 \longrightarrow N \longrightarrow X \longrightarrow P \longrightarrow 0$. This gives a graded split exact sequence of $A^\natural$-modules:

$$0 \longrightarrow N^\natural \longrightarrow X^\natural \longrightarrow P^\natural \longrightarrow 0.$$

Hence, α is isomorphic to a degreewise split sequence of the form

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_i & \longrightarrow & N_i \oplus P_i & \longrightarrow & P_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_i^N} & & \downarrow{\scriptstyle \begin{bmatrix} \partial_i^N & \lambda_i \\ 0 & \partial_i^P \end{bmatrix}} & & \downarrow{\scriptstyle \partial_i^P} & & \\
0 & \longrightarrow & N_{i-1} & \longrightarrow & N_{i-1} \oplus P_{i-1} & \longrightarrow & P_{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
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\end{array}
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Define the isomorphism by $[\alpha] \longmapsto \left[(\lambda_i)_{i\in\mathbb{N}}\right]$.

Theorem

Let A be a DG R-algebra, and let M, N be DG A-modules where $H_i(M) = 0$ for all $i > n$ and

$$N = \quad 0 \longrightarrow N_n \longrightarrow N_{n-1} \longrightarrow N_{n-2} \longrightarrow \cdots$$

for an integer n. Set

$$t_n M = \quad 0 \longrightarrow \frac{M_n}{\operatorname{Im} \partial^M_{n+1}} \xrightarrow{\overline{\partial}^M_n} M_{n-1} \xrightarrow{\partial^M_{n-1}} M_{n-2} \longrightarrow \cdots .$$

Then the natural map

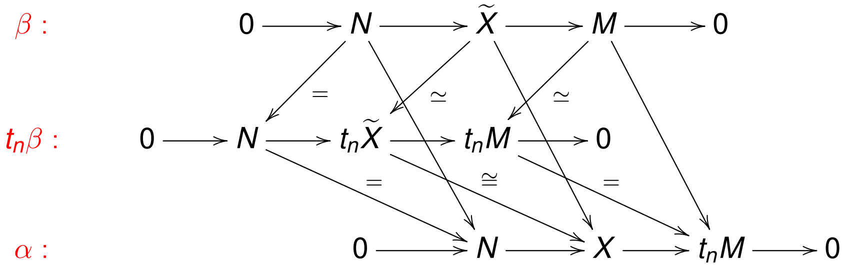
$$\mathrm{YExt}^1_A(t_n M, N) \xrightarrow{\Psi} \mathrm{YExt}^1_A(M, N)$$

is one-to-one.

Sketch of Proof. Let $\alpha \in \mathrm{YExt}^1_A(t_nM, N)$ be represented by the exact sequence $0 \longrightarrow N \longrightarrow X \longrightarrow t_nM \longrightarrow 0$. Consider the pull-back diagram

$$\begin{array}{llllllllll}
\beta: & 0 & \longrightarrow & N & \longrightarrow & \widetilde{X} & \longrightarrow & M & \longrightarrow & 0\,. \\
 & & & \downarrow{=} & & \downarrow{\simeq} & & \downarrow{\simeq} & & \\
\alpha: & 0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & t_nM & \longrightarrow & 0
\end{array}$$

The Ψ is defined by the formula $\Psi([\alpha]) = [\beta]$. It can be shown that $t_n\widetilde{X} \cong X$ and we have the following commutative diagram:



Hence, if β is split, then α is split. Therefore, Ψ is 1-1.

Corollary

Let A be a DG R-algebra, and let C be a semiprojective semidualizing DG A-module. Then $\mathrm{YExt}^1_A(t_nC, t_nC) = 0$.