Connected Sums of Simplicial Complexes

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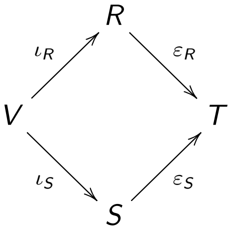
Let R,S,T be commutative rings, and let V be a T-module. A connected sum diagram is a commutative diagram



We will let $I = \ker \varepsilon_R$ and $J = \ker \varepsilon_S$.
The connected sum of the above diagram is the ring

$$R \#_T S = R \times_T S / \{(\iota_R(v), \iota_S(v)) \mid v \in V\}.$$

Theorem (AAM)

Let R and S be Gorenstein local rings of dimension d and T a Cohen-Macaulay local ring of dim d. Let V be a canonical module for T, and choose isomorphisms of V with ideals of R and S, respectively, via

$$\iota_R(V) = (0 : I) \qquad \iota_S(V) = (0 : J).$$

If I or J is nonzero, then $R\#_T S$ is a Gorenstein local ring of dimension d.

Example

$R = k[a, b]/(a^2, b^2),\ S = k[c, d]/(c^2, d^2),\ T = k$

Here, I, J are the socles of R and S, so

$$R\#_k S = k[a, b, c, d]/(a^2, b^2, c^2, d^2, ac, ad, bc, bd, ab - cd)$$

Topologically, this corresponds to the example: $X = S^2 \times S^2 = Y$

Then $\mathrm{H}^*(X) \cong R$ and $\mathrm{H}^*(Y) \cong S$, and

$$\mathrm{H}^*(X \# Y) \cong R \#_k S.$$

Question

Is there a similar topological construction that realizes the connected sum construction for higher dimensional rings?

Let Δ_1 and Δ_2 be simplicial complexes on a vertex set $[m]$.

Let $Z \subset \Delta_1 \cap \Delta_2$ be a subset such that $O_{\Delta_1 \cup \Delta_2}(Z) \subseteq \Delta_1 \cap \Delta_2$

The connected sum of Δ_1 and Δ_2 along Z is:

$$\Delta_1 \#^Z \Delta_2 := \mathrm{Del}_Z(\Delta_1 \cup \Delta_2)$$

This matches the definition of 'connected sum along a facet' that appears in Buchstaber-Panov.

Example

$\mathcal{F}(\Delta_1) = \{abc, bcd\}$, $\mathcal{F}(\Delta_2) = \{abc, ace\}$, $\Delta = \Delta_1 \cup \Delta_2$.
Let $\mathcal{F}(Z) = \{abc\} = O_\Delta(Z)$.

Example

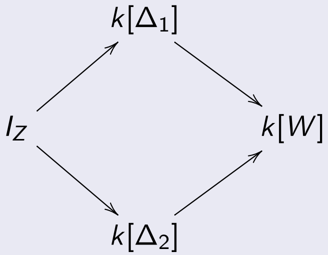
$\mathcal{F}(\Delta_1) = \{12, 25, 53, 34, 14\}$, $\mathcal{F}(\Delta_2) = \{25, 53, 23\}$.
$\mathcal{F}(Z) = \{25, 53\}$.

Recall that given a simplicial complex Δ on a vertex set $[m]$, the Stanley-Reisner ring of Δ is the k-algebra

$$k[\Delta] = k[x_1, \ldots, x_m]/\langle \{x_{i_1} \cdots x_{i_l} \mid \{i_1, \ldots, i_l\} \notin \Delta \rangle.$$

Proposition

Let Δ_1 and Δ_2 be simplicial complexes on $[m]$, $W = \Delta_1 \cap \Delta_2$, and $Z \subset W$ so that $O(Z) \subset W$. Then there is a commutative diagram, with $I_Z = \{x_\sigma \mid \sigma \in Z\}$:



whose connected sum satisfies:

$$k[\Delta_1] \#_{k[W]}^{I_Z} k[\Delta_2] \cong k[\Delta_1 \#^Z \Delta_2].$$

A connected sum $\Delta_1 \#^Z \Delta_2$ is called *strong* provided Δ_1, Δ_2 and $W = \Delta_1 \cap \Delta_2$ are pure of the same dimension, and

$$Z = W \setminus \overline{(\Delta_1 \setminus W)} = W \setminus \overline{(\Delta_2 \setminus W)}$$

Proposition

Assume that $\Delta_1 \#^Z \Delta_2$ is a strong connected sum, and that Δ_1, Δ_2 and $W = \Delta_1 \cap \Delta_2$ are pure of the same dimension.

If Δ_1 and Δ_2 are Gorenstein, and W is Cohen-Macaulay, then $k[\Delta_1 \#^Z \Delta_2]$ is Gorenstein.

Let $T^m \cong \mathbb{C}^m$, and let X be a space with a T-action.

Then one may define the T-equivariant cohomology of X to be the cohomology of the space $(ET^m \times X)/\sim$ where $(e,x) \sim (eg, g^{-1}x)$, for all $e \in ET^m$, $g \in T$.

The T-equivariant cohomology of X is denoted $\mathrm{H}^*_T(X)$.

An important example is that of $X = pt$. Then:

$$\begin{aligned} \mathrm{H}^*_T(pt) &= \mathrm{H}^*((ET^m \times pt)/\sim) \\ &= \mathrm{H}^*(BT^m) \\ &= \mathrm{H}^*((\mathbb{C}P^\infty)^m) \\ &= k[x_1, \dots, x_m] \quad \text{(generated in degree two)} \end{aligned}$$

Therefore, given any T-space X, the equivariant projection $X \to pt$ induces

$$\mathrm{H}^*_T(pt) \to \mathrm{H}^*_T(X)$$

therefore making $\mathrm{H}^*_T(X)$ into an algebra over $k[x_1, \dots, x_m]$.

Theorem (Davis-Januszkiewicz)

Given a simplicial complex Δ on the vertex set $[m]$, there is a topological space $\mathcal{Z}_\Delta \subseteq (D^2)^m$ with an action of T such that

$$\mathrm{H}^*_T(\mathcal{Z}_\Delta) \cong k(\Delta).$$

If $\Delta = \Delta_P$ for some simple polytope P, then one may define $\mathcal{Z}_P := \mathcal{Z}_{\Delta_P}$.

For a simple polytope P, $\mathcal{Z}_P$ is a manifold, and hence $\mathrm{H}^*_T(\mathcal{Z}_P)$ is a Gorenstein ring.

Let P be a polytope, and P_- and P_+ the polytopes obtained by cutting P by a general hyperplane. Let $P_0 = P_- \cap P_+$, and let o be the vertex of Δ_{P_-} and Δ_{P_+} corresponding to the hyperplane. Then one has

$$k[\Delta_P] \cong k[\Delta_{P_-} \#^Z \Delta_{P_+}]$$

where $Z = O_{\Delta_1 \cup \Delta_2}(o)$

In particular, one has

$$\mathrm{H}^*_T(\mathcal{Z}_P) \cong \mathrm{H}^*_T(\mathcal{Z}_{P_-}) \#^{\mathrm{H}^*_T(\mathcal{Z}_{P_0})}_{\mathrm{H}^*_T(\mathcal{Z}_{P_0})} \mathrm{H}^*_T(\mathcal{Z}_{P_+})$$