j-stretched ideals and Sally's Conjecture

Paolo Mantero

Purdue University

Joint work(s) with Yu Xie (U. of Notre Dame)

October 15, 2011

Based on the following papers:

P. Mantero and Y. Xie, *On the Cohen-Macaulayness of the conormal module of an ideal* (2010), 24 pages, submitted. Available at `arxiv:1103.5518`.

P. Mantero and Y. Xie, *j-stretched ideals and Sally's Conjecture*, 22 pages, preprint.

Based on the following papers:

P. Mantero and Y. Xie, *On the Cohen-Macaulayness of the conormal module of an ideal* (2010), 24 pages, submitted. Available at `arxiv:1103.5518`.

P. Mantero and Y. Xie, *j-stretched ideals and Sally's Conjecture*, 22 pages, preprint.

Cohen-Macaulayness of the conormal module

Question 1 (Vasconcelos 1987, 1994)

Let R be a RLR and I be a perfect ideal that is generically a complete intersection (i.e., $I_{\mathfrak{p}}$ is a complete intersection $\forall\, \mathfrak{p} \in \mathrm{Ass}_R(R/I)$).
If I/I^2 (equivalently, R/I^2) is CM $\overset{?}{\Rightarrow}$ R/I is Gorenstein?

Answer is YES for:

- perfect *prime* ideals of height 2 (Herzog, 1978);
- licci ideals (Huneke and Ulrich, 1989);
- squarefree monomial ideals (Rinaldo, Terai and Yoshida, 2011).

In particular, it is true for all perfect ideals of height 2.

Cohen-Macaulayness of the conormal module

Question 1 (Vasconcelos 1987, 1994)

Let R be a RLR and I be a perfect ideal that is generically a complete intersection (i.e., $I_{\mathfrak{p}}$ is a complete intersection $\forall\, \mathfrak{p} \in \operatorname{Ass}_R(R/I)$).
If I/I^2 (equivalently, R/I^2) is CM $\overset{?}{\Rightarrow}$ R/I is Gorenstein?

Answer is YES for:

- perfect *prime* ideals of height 2 (Herzog, 1978);
- licci ideals (Huneke and Ulrich, 1989);
- squarefree monomial ideals (Rinaldo, Terai and Yoshida, 2011).

In particular, it is true for all perfect ideals of height 2.

Cohen-Macaulayness of the conormal module

Question 1 (Vasconcelos 1987, 1994)

Let R be a RLR and I be a perfect ideal that is generically a complete intersection (i.e., $I_{\mathfrak{p}}$ is a complete intersection $\forall\, \mathfrak{p} \in \operatorname{Ass}_R(R/I)$).
If I/I^2 (equivalently, R/I^2) is CM $\overset{?}{\Rightarrow}$ R/I is Gorenstein?

Answer is YES for:

- perfect *prime* ideals of height 2 (Herzog, 1978);
- licci ideals (Huneke and Ulrich, 1989);
- squarefree monomial ideals (Rinaldo, Terai and Yoshida, 2011).

In particular, it is true for all perfect ideals of height 2.

Cohen-Macaulayness of the conormal module

Question 1 (Vasconcelos 1987, 1994)

Let R be a RLR and I be a perfect ideal that is generically a complete intersection (i.e., $I_{\mathfrak{p}}$ is a complete intersection $\forall\, \mathfrak{p} \in \mathrm{Ass}_R(R/I)$).
If I/I^2 (equivalently, R/I^2) is CM $\overset{?}{\Rightarrow}$ R/I is Gorenstein?

Answer is YES for:

- perfect *prime* ideals of height 2 (Herzog, 1978);
- licci ideals (Huneke and Ulrich, 1989);
- squarefree monomial ideals (Rinaldo, Terai and Yoshida, 2011).

In particular, it is true for all perfect ideals of height 2.

Cohen-Macaulayness of the conormal module

Question 1 (Vasconcelos 1987, 1994)

Let R be a RLR and I be a perfect ideal that is generically a complete intersection (i.e., $I_{\mathfrak{p}}$ is a complete intersection $\forall\, \mathfrak{p} \in \mathrm{Ass}_R(R/I)$).
If I/I^2 (equivalently, R/I^2) is CM $\overset{?}{\Rightarrow}$ R/I is Gorenstein?

Answer is YES for:

- perfect *prime* ideals of height 2 (Herzog, 1978);
- licci ideals (Huneke and Ulrich, 1989);
- squarefree monomial ideals (Rinaldo, Terai and Yoshida, 2011).

In particular, it is true for all perfect ideals of height 2.

Cohen-Macaulayness of the conormal module, cont'd

Using tools from linkage theory, we proved the following

Proposition 2 (M-Xie 2010)

Question 1 can be reduced to the case of prime ideals.

Theorem(s) 3 (M-Xie 2010)

Question 1 holds true for:

(a) *any monomial ideal I;*

(b) *almost every ideal I defining a short algebra;*

(c) *any ideal I such that R/I has multiplicity $\leq \mathrm{ecodim} R/I + 4$;*

(d) *any ideal I such that R/I is a stretched algebra.*

We also provide examples of a prime ideal $\mathfrak{p}$ such that $e(R/\mathfrak{p}) = \mathrm{ecodim} R/I + 5$ and answer to Vasconcelos' Question is NO.

Cohen-Macaulayness of the conormal module, cont'd

Using tools from linkage theory, we proved the following

Proposition 2 (M-Xie 2010)

Question 1 can be reduced to the case of prime ideals.

Theorem(s) 3 (M-Xie 2010)

Question 1 holds true for:

(a) *any monomial ideal I;*

(b) *almost every ideal I defining a short algebra;*

(c) *any ideal I such that R/I has multiplicity $\leq \mathrm{ecodim} R/I + 4$;*

(d) *any ideal I such that R/I is a stretched algebra.*

We also provide examples of a prime ideal $\mathfrak{p}$ such that $e(R/\mathfrak{p}) = \mathrm{ecodim} R/I + 5$ and answer to Vasconcelos' Question is NO.

Cohen-Macaulayness of the conormal module, cont'd

Using tools from linkage theory, we proved the following

Proposition 2 (M-Xie 2010)

Question 1 can be reduced to the case of prime ideals.

Theorem(s) 3 (M-Xie 2010)

Question 1 holds true for:

(a) *any monomial ideal I;*

(b) *almost every ideal I defining a short algebra;*

(c) *any ideal I such that R/I has multiplicity $\leq \mathrm{ecodim} R/I + 4$;*

(d) *any ideal I such that R/I is a stretched algebra.*

We also provide examples of a prime ideal $\mathfrak{p}$ such that $e(R/\mathfrak{p}) = \mathrm{ecodim} R/I + 5$ and answer to Vasconcelos' Question is NO.

Cohen-Macaulayness of the conormal module, cont'd

Using tools from linkage theory, we proved the following

Proposition 2 (M-Xie 2010)

Question 1 can be reduced to the case of prime ideals.

Theorem(s) 3 (M-Xie 2010)

Question 1 holds true for:

(a) *any monomial ideal I;*

(b) *almost every ideal I defining a short algebra;*

(c) *any ideal I such that R/I has multiplicity $\leq \mathrm{ecodim} R/I + 4$;*

(d) *any ideal I such that R/I is a stretched algebra.*

We also provide examples of a prime ideal $\mathfrak{p}$ such that $e(R/\mathfrak{p}) = \mathrm{ecodim} R/I + 5$ and answer to Vasconcelos' Question is NO.

Cohen-Macaulayness of the conormal module, cont'd

Using tools from linkage theory, we proved the following

Proposition 2 (M-Xie 2010)

Question 1 can be reduced to the case of prime ideals.

Theorem(s) 3 (M-Xie 2010)

Question 1 holds true for:

(a) *any monomial ideal I;*

(b) *almost every ideal I defining a short algebra;*

(c) *any ideal I such that R/I has multiplicity $\leq \mathrm{ecodim} R/I + 4$;*

(d) *any ideal I such that R/I is a stretched algebra.*

We also provide examples of a prime ideal $\mathfrak{p}$ such that $e(R/\mathfrak{p}) = \mathrm{ecodim} R/I + 5$ and answer to Vasconcelos' Question is NO.

Cohen-Macaulayness of the conormal module, cont'd

Using tools from linkage theory, we proved the following

Proposition 2 (M-Xie 2010)

Question 1 can be reduced to the case of prime ideals.

Theorem(s) 3 (M-Xie 2010)

Question 1 holds true for:

(a) *any monomial ideal I;*

(b) *almost every ideal I defining a short algebra;*

(c) *any ideal I such that R/I has multiplicity $\leq \mathrm{ecodim} R/I + 4$;*

(d) *any ideal I such that R/I is a stretched algebra.*

We also provide examples of a prime ideal $\mathfrak{p}$ such that $e(R/\mathfrak{p}) = \mathrm{ecodim} R/I + 5$ and answer to Vasconcelos' Question is NO.

Cohen-Macaulayness of the conormal module, cont'd

Using tools from linkage theory, we proved the following

Proposition 2 (M-Xie 2010)

Question 1 can be reduced to the case of prime ideals.

Theorem(s) 3 (M-Xie 2010)

Question 1 holds true for:

(a) *any monomial ideal I;*

(b) *almost every ideal I defining a short algebra;*

(c) *any ideal I such that R/I has multiplicity $\leq \mathrm{ecodim} R/I + 4$;*

(d) *any ideal I such that R/I is a stretched algebra.*

We also provide examples of a prime ideal $\mathfrak{p}$ such that $e(R/\mathfrak{p}) = \mathrm{ecodim} R/I + 5$ and answer to Vasconcelos' Question is NO.

Stretched algebras

- An Artinian local ring $(A, \mathfrak{n})$ is **stretched** if $\mathfrak{n}^2$ is a principal ideal.

Example

Set $A_n = k[\![X, Y, Z]\!]/(X^2, XY, XZ, YZ, Z^n - Y^2)$ *with* $n \geq 2 \Rightarrow$ A_n *is a stretched algebra.*

- An Artinian algebra is stretched iff its Hilbert function has the shape

$$1 \quad c \quad 1 \quad \dots \quad 1 \quad 0 \longrightarrow$$

Stretched algebras

- An Artinian local ring $(A, \mathfrak{n})$ is **stretched** if $\mathfrak{n}^2$ is a principal ideal.

Example

Set $A_n = k[\![X, Y, Z]\!]/(X^2, XY, XZ, YZ, Z^n - Y^2)$ *with* $n \geq 2 \Rightarrow$ A_n *is a stretched algebra.*

- An Artinian algebra is stretched iff its Hilbert function has the shape

$$1 \quad c \quad 1 \quad \dots \quad 1 \quad 0 \longrightarrow$$

Stretched algebras

- An Artinian local ring $(A, \mathfrak{n})$ is **stretched** if $\mathfrak{n}^2$ is a principal ideal.

Example

Set $A_n = k[\![X, Y, Z]\!]/(X^2, XY, XZ, YZ, Z^n - Y^2)$ *with* $n \geq 2 \Rightarrow$ A_n *is a stretched algebra.*

- An Artinian algebra is stretched iff its Hilbert function has the shape

$$1 \quad c \quad 1 \quad \dots \quad 1 \quad 0_{\longrightarrow}$$

Structure of Artinian stretched algebras

Theorem 4 (Sally 1981, Elias-Valla 2008, M-Xie 2010)

Let $(R, \mathfrak{m})$ be a RLR of dimension c with $\operatorname{char} R/\mathfrak{m} \neq 2$. Let $I \subseteq \mathfrak{m}^2$ be an $\mathfrak{m}$-primary ideal with R/I stretched with $\mathfrak{m}^2_{R/I} \neq 0$. Write $\tau(R/I) = r + 1$ for some non negative integer r.
$\Rightarrow \exists$ minimal generators $x_1, \ldots, x_c$ for $\mathfrak{m}$, and units $u_{r+1}, \ldots, u_{c-1}$ in R with

$$I = (x_1\mathfrak{m}, \ldots, x_r\mathfrak{m}) + J$$

where
$$J = (x_{r+i}x_{r+j} \,|\, 1 \leq i < j \leq c-r) + (x_c^s - u_{r+i}x_{r+i}^2 \,|\, 1 \leq i \leq c-r-1).$$

As a consequence, we have a complete description of I solely based on the Hilbert function and the type of R/I.

Structure of Artinian stretched algebras

Theorem 4 (Sally 1981, Elias-Valla 2008, M-Xie 2010)

Let $(R, \mathfrak{m})$ be a RLR of dimension c with $\operatorname{char} R/\mathfrak{m} \neq 2$. Let $I \subseteq \mathfrak{m}^2$ be an $\mathfrak{m}$-primary ideal with R/I stretched with $\mathfrak{m}^2_{R/I} \neq 0$. Write $\tau(R/I) = r + 1$ for some non negative integer r.
$\Rightarrow \exists$ minimal generators $x_1, \ldots, x_c$ for $\mathfrak{m}$, and units $u_{r+1}, \ldots, u_{c-1}$ in R with

$$I = (x_1\mathfrak{m}, \ldots, x_r\mathfrak{m}) + J$$

where
$$J = (x_{r+i}x_{r+j} \,|\, 1 \leq i < j \leq c-r) + (x_c^s - u_{r+i}x_{r+i}^2 \,|\, 1 \leq i \leq c-r-1).$$

As a consequence, we have a complete description of I solely based on the Hilbert function and the type of R/I.

An example

Example

If R/I is Artinian algebra with Hilbert function

$$1 \qquad 3 \qquad 1 \qquad 0_{\longrightarrow}$$

and type 2 $\Rightarrow \exists$ a regular system of parameters, x, y, z, for R, and a unit u of R with

$$I = (x^2, xy, xz, yz, x^3 - uy^2).$$

Other examples

- A Cohen-Macaulay local ring $(R, \mathfrak{m})$ is **stretched** if there exists a minimal reduction J of $\mathfrak{m}$ ($J\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n) so that R/J is Artinian stretched.

If R is a Cohen-Macaulay local ring, Abhyankar proved that

$$e(R) \geq \operatorname{ecodim} R + 1.$$

- If $e(R) = \operatorname{ecodim} R + 1$, then R has *minimal multiplicity*;
- If $e(R) = \operatorname{ecodim} R + 2$, then R has *almost minimal multiplicity*.

Example

Let R be a Cohen-Macaulay local algebra with minimal or almost minimal multiplicity $\Rightarrow$ R is stretched.

Other examples

- A Cohen-Macaulay local ring $(R, \mathfrak{m})$ is **stretched** if there exists a minimal reduction J of $\mathfrak{m}$ ($J\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n) so that R/J is Artinian stretched.

If R is a Cohen-Macaulay local ring, Abhyankar proved that

$$e(R) \geq \operatorname{ecodim} R + 1.$$

- If $e(R) = \operatorname{ecodim} R + 1$, then R has *minimal multiplicity*;
- If $e(R) = \operatorname{ecodim} R + 2$, then R has *almost minimal multiplicity*.

Example

Let R be a Cohen-Macaulay local algebra with minimal or almost minimal multiplicity $\Rightarrow$ R is stretched.

Other examples

- A Cohen-Macaulay local ring $(R, \mathfrak{m})$ is **stretched** if there exists a minimal reduction J of $\mathfrak{m}$ ($J\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n) so that R/J is Artinian stretched.

If R is a Cohen-Macaulay local ring, Abhyankar proved that

$$e(R) \geq \operatorname{ecodim} R + 1.$$

- If $e(R) = \operatorname{ecodim} R + 1$, then R has *minimal multiplicity*;
- If $e(R) = \operatorname{ecodim} R + 2$, then R has *almost minimal multiplicity*.

Example

Let R be a Cohen-Macaulay local algebra with minimal or almost minimal multiplicity $\Rightarrow$ R is stretched.

Other examples

- A Cohen-Macaulay local ring $(R, \mathfrak{m})$ is **stretched** if there exists a minimal reduction J of $\mathfrak{m}$ ($J\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n) so that R/J is Artinian stretched.

If R is a Cohen-Macaulay local ring, Abhyankar proved that

$$e(R) \geq \operatorname{ecodim} R + 1.$$

- If $e(R) = \operatorname{ecodim} R + 1$, then R has *minimal multiplicity*;
- If $e(R) = \operatorname{ecodim} R + 2$, then R has *almost minimal multiplicity*.

Example

Let R be a Cohen-Macaulay local algebra with minimal or almost minimal multiplicity $\Rightarrow$ R is stretched.

Sally's Conjecture

Theorem 5

Let $(R, \mathfrak{m})$ be Cohen-Macaulay local ring.

(a) *(Sally 1979) If R has minimal multiplicity $\Rightarrow gr_{\mathfrak{m}}(R)$ is Cohen-Macaulay;*

(b) *(Sally 1981, Rossi-Valla 1994, Wang 1994) If R has almost minimal multiplicity $\Rightarrow gr_{\mathfrak{m}}(R)$ is almost Cohen-Macaulay ($i.e.$, $\operatorname{depth} gr_{\mathfrak{m}}(R) \geq \dim R - 1$).*

Part (b) is known as **Sally's Conjecture**.

Sally's Conjecture

Theorem 5

Let $(R, \mathfrak{m})$ be Cohen-Macaulay local ring.

(a) *(Sally 1979) If R has minimal multiplicity $\Rightarrow$ $gr_{\mathfrak{m}}(R)$ is Cohen-Macaulay;*

(b) *(Sally 1981, Rossi-Valla 1994, Wang 1994) If R has almost minimal multiplicity $\Rightarrow$ $gr_{\mathfrak{m}}(R)$ is almost Cohen-Macaulay ($i.e.$, $\operatorname{depth} gr_{\mathfrak{m}}(R) \geq \dim R - 1$).*

Part (b) is known as **Sally's Conjecture**.

Sally's Conjecture

Theorem 5

Let $(R, \mathfrak{m})$ *be Cohen-Macaulay local ring.*

(a) *(Sally 1979) If* R *has minimal multiplicity* $\Rightarrow gr_{\mathfrak{m}}(R)$ *is Cohen-Macaulay;*

(b) *(Sally 1981, Rossi-Valla 1994, Wang 1994) If* R *has almost minimal multiplicity* $\Rightarrow gr_{\mathfrak{m}}(R)$ *is almost Cohen-Macaulay* (*i.e.*, $\operatorname{depth} gr_{\mathfrak{m}}(R) \geq \dim R - 1$).

Part (b) is known as **Sally's Conjecture**.

Stretched $\mathfrak{m}$-primary ideals

Let $(R, \mathfrak{m})$ be Cohen-Macaulay, I be an $\mathfrak{m}$-primary ideal, J be a minimal reduction of I ($JI^n = I^{n+1}$ for some n). Then, I is **stretched** if

(i) $HF_{I/J}(2) \leq 1$, and

(ii) $I^2 \cap J = JI$.

When $I = \mathfrak{m}$, this definition is equivalent to say that R/J is a stretched algebra.

- Rossi and Valla (2001) proved the $\mathfrak{m}$-primary analogue of Sally's Conjecture for stretched $\mathfrak{m}$-primary ideals, under some additional assumptions on the ideal.
- **Problematic Remark:** $\mathfrak{m}$-primary stretched ideals do not generalize ideals defining algebras with almost minimal multiplicity.

Stretched $\mathfrak{m}$-primary ideals

Let $(R, \mathfrak{m})$ be Cohen-Macaulay, I be an $\mathfrak{m}$-primary ideal, J be a minimal reduction of I ($JI^n = I^{n+1}$ for some n). Then, I is **stretched** if

(i) $HF_{I/J}(2) \leq 1$, and

(ii) $I^2 \cap J = JI$.

When $I = \mathfrak{m}$, this definition is equivalent to say that R/J is a stretched algebra.

- Rossi and Valla (2001) proved the $\mathfrak{m}$-primary analogue of Sally's Conjecture for stretched $\mathfrak{m}$-primary ideals, under some additional assumptions on the ideal.
- **Problematic Remark:** $\mathfrak{m}$-primary stretched ideals do not generalize ideals defining algebras with almost minimal multiplicity.

Stretched $\mathfrak{m}$-primary ideals

Let $(R, \mathfrak{m})$ be Cohen-Macaulay, I be an $\mathfrak{m}$-primary ideal, J be a minimal reduction of I ($JI^n = I^{n+1}$ for some n). Then, I is **stretched** if

(i) $HF_{I/J}(2) \leq 1$, and
(ii) $I^2 \cap J = JI$.

When $I = \mathfrak{m}$, this definition is equivalent to say that R/J is a stretched algebra.

- Rossi and Valla (2001) proved the $\mathfrak{m}$-primary analogue of Sally's Conjecture for stretched $\mathfrak{m}$-primary ideals, under some additional assumptions on the ideal.
- **Problematic Remark:** $\mathfrak{m}$-primary stretched ideals do not generalize ideals defining algebras with almost minimal multiplicity.

Stretched $\mathfrak{m}$-primary ideals

Let $(R, \mathfrak{m})$ be Cohen-Macaulay, I be an $\mathfrak{m}$-primary ideal, J be a minimal reduction of I ($JI^n = I^{n+1}$ for some n). Then, I is **stretched** if

(i) $HF_{I/J}(2) \leq 1$, and

(ii) $I^2 \cap J = JI$.

When $I = \mathfrak{m}$, this definition is equivalent to say that R/J is a stretched algebra.

- Rossi and Valla (2001) proved the $\mathfrak{m}$-primary analogue of Sally's Conjecture for stretched $\mathfrak{m}$-primary ideals, under some additional assumptions on the ideal.
- **Problematic Remark:** $\mathfrak{m}$-primary stretched ideals do not generalize ideals defining algebras with almost minimal multiplicity.

Stretched $\mathfrak{m}$-primary ideals

Let $(R, \mathfrak{m})$ be Cohen-Macaulay, I be an $\mathfrak{m}$-primary ideal, J be a minimal reduction of I ($JI^n = I^{n+1}$ for some n). Then, I is **stretched** if

(i) $HF_{I/J}(2) \leq 1$, and

(ii) $I^2 \cap J = JI$.

When $I = \mathfrak{m}$, this definition is equivalent to say that R/J is a stretched algebra.

- Rossi and Valla (2001) proved the $\mathfrak{m}$-primary analogue of Sally's Conjecture for stretched $\mathfrak{m}$-primary ideals, under some additional assumptions on the ideal.
- **Problematic Remark:** $\mathfrak{m}$-primary stretched ideals do not generalize ideals defining algebras with almost minimal multiplicity.

Goals

The goals we achieve in our paper with Y. Xie are:

- provide a generalized notion of stretched ('j-stretched') such that
  1. it is well-defined even when $\dim R/I > 0$;
  2. it removes the intersection property.
  3. it generalizes the 'higher dimensional version' of minimal and almost minimal multiplicity.
- Characterize the CM-ness of $gr_I(R)$ for these ideals.
- Prove Sally's Conjecture for this class of ideals, under some (somewhat expected) assumptions.

Our tools come from residual intersection theory and j-multiplicity theory (=the higher-dimensional version of Hilbert-Samuel multiplicity).

Goals

The goals we achieve in our paper with Y. Xie are:

- provide a generalized notion of stretched ('j-stretched') such that
  1. (1) it is well-defined even when $\dim R/I > 0$;
  2. (2) it removes the intersection property.
  3. (3) it generalizes the 'higher dimensional version' of minimal and almost minimal multiplicity.
- Characterize the CM-ness of $gr_I(R)$ for these ideals.
- Prove Sally's Conjecture for this class of ideals, under some (somewhat expected) assumptions.

Our tools come from residual intersection theory and j-multiplicity theory (=the higher-dimensional version of Hilbert-Samuel multiplicity).

Goals

The goals we achieve in our paper with Y. Xie are:

- provide a generalized notion of stretched ('j-stretched') such that
  (1) it is well-defined even when $\dim R/I > 0$;
  (2) it removes the intersection property.
  (3) it generalizes the 'higher dimensional version' of minimal and almost minimal multiplicity.
- Characterize the CM-ness of $gr_I(R)$ for these ideals.
- Prove Sally's Conjecture for this class of ideals, under some (somewhat expected) assumptions.

Our tools come from residual intersection theory and j-multiplicity theory (=the higher-dimensional version of Hilbert-Samuel multiplicity).

Goals

The goals we achieve in our paper with Y. Xie are:

- provide a generalized notion of stretched ('j-stretched') such that
  1. it is well-defined even when $\dim R/I > 0$;
  2. it removes the intersection property.
  3. it generalizes the 'higher dimensional version' of minimal and almost minimal multiplicity.
- Characterize the CM-ness of $gr_I(R)$ for these ideals.
- Prove Sally's Conjecture for this class of ideals, under some (somewhat expected) assumptions.

Our tools come from residual intersection theory and j-multiplicity theory (=the higher-dimensional version of Hilbert-Samuel multiplicity).

Goals

The goals we achieve in our paper with Y. Xie are:

- provide a generalized notion of stretched ('j-stretched') such that
  1. it is well-defined even when $\dim R/I > 0$;
  2. it removes the intersection property.
  3. it generalizes the 'higher dimensional version' of minimal and almost minimal multiplicity.
- Characterize the CM-ness of $gr_I(R)$ for these ideals.
- Prove Sally's Conjecture for this class of ideals, under some (somewhat expected) assumptions.

Our tools come from residual intersection theory and j-multiplicity theory (=the higher-dimensional version of Hilbert-Samuel multiplicity).

Goals

The goals we achieve in our paper with Y. Xie are:

- provide a generalized notion of stretched ('j-stretched') such that
  1. it is well-defined even when $\dim R/I > 0$;
  2. it removes the intersection property.
  3. it generalizes the 'higher dimensional version' of minimal and almost minimal multiplicity.
- Characterize the CM-ness of $gr_I(R)$ for these ideals.
- Prove Sally's Conjecture for this class of ideals, under some (somewhat expected) assumptions.

Our tools come from residual intersection theory and j-multiplicity theory (=the higher-dimensional version of Hilbert-Samuel multiplicity).

Goals

The goals we achieve in our paper with Y. Xie are:

- provide a generalized notion of stretched ('j-stretched') such that
  1. it is well-defined even when $\dim R/I > 0$;
  2. it removes the intersection property.
  3. it generalizes the 'higher dimensional version' of minimal and almost minimal multiplicity.
- Characterize the CM-ness of $gr_I(R)$ for these ideals.
- Prove Sally's Conjecture for this class of ideals, under some (somewhat expected) assumptions.

Our tools come from residual intersection theory and j-multiplicity theory (=the higher-dimensional version of Hilbert-Samuel multiplicity).

j-stretched ideals

1-dimensional definition

Let R be a 1-dimensional Cohen-Macaulay local domain, I be a non zero ideal of R, and let J' be a general principal reduction of I. Then,

$$I \text{ is } j\text{-stretched} \iff \lambda(I^2/J'I + I^3) \leq 1.$$

Definition 6

*Let R be a Noetherian local ring and I be an ideal with analytic spread $\ell(I) = \dim R = d$. I is j-**stretched** if, for a general minimal reduction $J = (x_1, \ldots, x_d)$ of I, one has*

$$\lambda(I^2\overline{R}/x_d I\overline{R} + I^3\overline{R}) \leq 1$$

where $\overline{R} = R/J_{d-1}$ and $J_{d-1} = (x_1, \ldots, x_{d-1}) :_R I^\infty$.

j-stretched ideals

1-dimensional definition

Let R be a 1-dimensional Cohen-Macaulay local domain, I be a non zero ideal of R, and let J' be a general principal reduction of I. Then,

$$I \text{ is } j\text{-stretched} \iff \lambda(I^2/J'I + I^3) \leq 1.$$

Definition 6

*Let R be a Noetherian local ring and I be an ideal with analytic spread $\ell(I) = \dim R = d$. I is j-**stretched** if, for a general minimal reduction $J = (x_1, \ldots, x_d)$ of I, one has*

$$\lambda(I^2\overline{R}/x_d I\overline{R} + I^3\overline{R}) \leq 1$$

where $\overline{R} = R/J_{d-1}$ and $J_{d-1} = (x_1, \ldots, x_{d-1}) :_R I^\infty$.

Observations and Facts

Recall that j-multiplicity is the higher-dimensional version of Hilbert-Samuel multiplicity.

Remark. I has minimal/almost minimal j-multiplicity $\Rightarrow$ I is j-stretched
(while I with almost minimal multiplicity $\not\Rightarrow$ I stretched!)

Proposition 7

If I has the corresponding length property with respect to <u>one</u> minimal reduction $\Rightarrow$ I is j-stretched.

Comment. Proposition 7 is useful from the computational perspective.

Observations and Facts

Recall that j-multiplicity is the higher-dimensional version of Hilbert-Samuel multiplicity.

Remark. I has minimal/almost minimal j-multiplicity $\Rightarrow$ I is j-stretched
(while I with almost minimal multiplicity $\not\Rightarrow$ I stretched!)

Proposition 7

If I has the corresponding length property with respect to one minimal reduction $\Rightarrow$ I is j-stretched.

Comment. Proposition 7 is useful from the computational perspective.

Observations and Facts

Recall that j-multiplicity is the higher-dimensional version of Hilbert-Samuel multiplicity.

Remark. I has minimal/almost minimal j-multiplicity $\Rightarrow$ I is j-stretched
(while I with almost minimal multiplicity $\not\Rightarrow$ I stretched!)

Proposition 7

If I has the corresponding length property with respect to one minimal reduction $\Rightarrow$ I is j-stretched.

Comment. Proposition 7 is useful from the computational perspective.

Observations and Facts

Recall that j-multiplicity is the higher-dimensional version of Hilbert-Samuel multiplicity.

Remark. I has minimal/almost minimal j-multiplicity $\Rightarrow I$ is j-stretched
(while I with almost minimal multiplicity $\not\Rightarrow I$ stretched!)

Proposition 7

If I has the corresponding length property with respect to one minimal reduction $\Rightarrow I$ is j-stretched.

Comment. Proposition 7 is useful from the computational perspective.

j-stretched ideals vs. stretched ideals

Theorem 8 (M-Xie)

Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring, and I be an $\mathfrak{m}$-primary ideal. If I is stretched $\Rightarrow$ I is j-stretched.

Therefore, j-stretched ideals generalize simultaneously ideals having minimal/almost minimal j-multiplicity, and $\mathfrak{m}$-primary stretched ideals.

j-stretched ideals vs. stretched ideals

Theorem 8 (M-Xie)

Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring, and I be an $\mathfrak{m}$-primary ideal. If I is stretched $\Rightarrow$ I is j-stretched.

Therefore, j-stretched ideals generalize simultaneously ideals having minimal/almost minimal j-multiplicity, and $\mathfrak{m}$-primary stretched ideals.

CM-ness of the associated graded ring

Under some residual assumptions, we can characterize the j-stretched ideals for which $\mathrm{gr}_I(R)$ is CM.

Theorem 9 (M-Xie)

Let $(R, \mathfrak{m})$ be a local CM ring with $|R/\mathfrak{m}| = \infty$, and let I be a j-stretched ideal. Let $J = (x_1, \ldots, x_d)$ be a general minimal reduction of I. Assume either

- *I is $\mathfrak{m}$-primary and $(x_1, \ldots, x_{d-1}) \cap I^2 = (x_1, \ldots, x_{d-1})I$, or*
- *$\ell(I) = \dim R = d$, I satisfies G_d, AN^-_{d-2}, $\mathrm{depth}\,(R/I) \geq 1$.*

TFAE:

(a) *$G = \mathrm{gr}_I(R)$ is Cohen-Macaulay;*

(b) *$I^{K+1} = JI^K$;*

(c) *$I^{K+1} = HI^K$ for some minimal reduction H of I;*

where $K = s_J(I)$, is the index of nilpotency of I with respect to J.

CM-ness of the associated graded ring

Under some residual assumptions, we can characterize the j-stretched ideals for which $\mathrm{gr}_I(R)$ is CM.

Theorem 9 (M-Xie)

Let $(R, \mathfrak{m})$ be a local CM ring with $|R/\mathfrak{m}| = \infty$, and let I be a j-stretched ideal. Let $J = (x_1, \dots, x_d)$ be a general minimal reduction of I. Assume either

- *I is $\mathfrak{m}$-primary and $(x_1, \dots, x_{d-1}) \cap I^2 = (x_1, \dots, x_{d-1})I$, or*
- *$\ell(I) = \dim R = d$, I satisfies G_d, AN^-_{d-2}, $\operatorname{depth}(R/I) \geq 1$.*

TFAE:

(a) *$G = \mathrm{gr}_I(R)$ is Cohen-Macaulay;*

(b) *$I^{K+1} = JI^K$;*

(c) *$I^{K+1} = HI^K$ for some minimal reduction H of I;*

where $K = s_J(I)$, is the index of nilpotency of I with respect to J.

CM-ness of the associated graded ring

Under some residual assumptions, we can characterize the j-stretched ideals for which $\mathrm{gr}_I(R)$ is CM.

Theorem 9 (M-Xie)

Let $(R, \mathfrak{m})$ be a local CM ring with $|R/\mathfrak{m}| = \infty$, and let I be a j-stretched ideal. Let $J = (x_1, \ldots, x_d)$ be a general minimal reduction of I. Assume either

- *I is $\mathfrak{m}$-primary and $(x_1, \ldots, x_{d-1}) \cap I^2 = (x_1, \ldots, x_{d-1})I$, or*
- *$\ell(I) = \dim R = d$, I satisfies G_d, AN^-_{d-2}, $\operatorname{depth}(R/I) \geq 1$.*

TFAE:

(a) *$G = \mathrm{gr}_I(R)$ is Cohen-Macaulay;*

(b) *$I^{K+1} = JI^K$;*

(c) *$I^{K+1} = HI^K$ for some minimal reduction H of I;*

where $K = s_J(I)$, is the index of nilpotency of I with respect to J.

CM-ness of the associated graded ring

Under some residual assumptions, we can characterize the j-stretched ideals for which $\mathrm{gr}_I(R)$ is CM.

Theorem 9 (M-Xie)

Let $(R, \mathfrak{m})$ be a local CM ring with $|R/\mathfrak{m}| = \infty$, and let I be a j-stretched ideal. Let $J = (x_1, \ldots, x_d)$ be a general minimal reduction of I. Assume either

- *I is $\mathfrak{m}$-primary and $(x_1, \ldots, x_{d-1}) \cap I^2 = (x_1, \ldots, x_{d-1})I$, or*
- *$\ell(I) = \dim R = d$, I satisfies G_d, AN^-_{d-2}, $\mathrm{depth}\,(R/I) \geq 1$.*

TFAE:

(a) *$G = \mathrm{gr}_I(R)$ is Cohen-Macaulay;*

(b) *$I^{K+1} = JI^K$;*

(c) *$I^{K+1} = HI^K$ for some minimal reduction H of I;*

where $K = s_J(I)$, is the index of nilpotency of I with respect to J.

CM-ness of the associated graded ring

Under some residual assumptions, we can characterize the j-stretched ideals for which $\mathrm{gr}_I(R)$ is CM.

Theorem 9 (M-Xie)

Let $(R, \mathfrak{m})$ be a local CM ring with $|R/\mathfrak{m}| = \infty$, and let I be a j-stretched ideal. Let $J = (x_1, \ldots, x_d)$ be a general minimal reduction of I. Assume either

- *I is $\mathfrak{m}$-primary and $(x_1, \ldots, x_{d-1}) \cap I^2 = (x_1, \ldots, x_{d-1})I$, or*
- *$\ell(I) = \dim R = d$, I satisfies G_d, AN_{d-2}^{-}, $\mathrm{depth}\,(R/I) \geq 1$.*

TFAE:

(a) *$G = \mathrm{gr}_I(R)$ is Cohen-Macaulay;*

(b) *$I^{K+1} = JI^K$;*

(c) *$I^{K+1} = HI^K$ for some minimal reduction H of I;*

where $K = s_J(I)$, is the index of nilpotency of I with respect to J.

CM-ness of the associated graded ring

Under some residual assumptions, we can characterize the j-stretched ideals for which $\mathrm{gr}_I(R)$ is CM.

Theorem 9 (M-Xie)

Let $(R, \mathfrak{m})$ be a local CM ring with $|R/\mathfrak{m}| = \infty$, and let I be a j-stretched ideal. Let $J = (x_1, \ldots, x_d)$ be a general minimal reduction of I. Assume either

- *I is $\mathfrak{m}$-primary and $(x_1, \ldots, x_{d-1}) \cap I^2 = (x_1, \ldots, x_{d-1})I$, or*
- *$\ell(I) = \dim R = d$, I satisfies G_d, AN_{d-2}^-, $\mathrm{depth}\,(R/I) \geq 1$.*

TFAE:

(a) *$G = \mathrm{gr}_I(R)$ is Cohen-Macaulay;*

(b) *$I^{K+1} = JI^K$;*

(c) *$I^{K+1} = HI^K$ for some minimal reduction H of I;*

where $K = s_J(I)$, is the index of nilpotency of I with respect to J.

CM-ness of the associated graded ring

Under some residual assumptions, we can characterize the j-stretched ideals for which $\mathrm{gr}_I(R)$ is CM.

Theorem 9 (M-Xie)

Let $(R,\mathfrak{m})$ be a local CM ring with $|R/\mathfrak{m}| = \infty$, and let I be a j-stretched ideal. Let $J = (x_1, \dots, x_d)$ be a general minimal reduction of I. Assume either

- *I is $\mathfrak{m}$-primary and $(x_1, \dots, x_{d-1}) \cap I^2 = (x_1, \dots, x_{d-1})I$, or*
- *$\ell(I) = \dim R = d$, I satisfies G_d, AN^-_{d-2}, $\operatorname{depth}(R/I) \geq 1$.*

TFAE:

(a) *$G = \mathrm{gr}_I(R)$ is Cohen-Macaulay;*

(b) *$I^{K+1} = JI^K$;*

(c) *$I^{K+1} = HI^K$ for some minimal reduction H of I;*

where $K = s_J(I)$, is the index of nilpotency of I with respect to J.

Sally's Conjecture for j-stretched ideals

The next result proves Sally's Conjecture for j-stretched ideals, generalizing to any dimension several classical results.

Theorem 10 (M-Xie)

Let $(R, \mathfrak{m})$ be a local CM ring with $|R/\mathfrak{m}| = \infty$, and I be a j-stretched ideal. Let J be a general minimal reduction of I. Assume either

- *I is $\mathfrak{m}$-primary and $(x_1, \ldots, x_{d-1}) \cap I^2 = (x_1, \ldots, x_{d-1})I$, or*
- *$\ell(I) = \dim R = d$, I satisfies G_d, AN^-_{d-2}, $\operatorname{depth}(R/I) \geq 1$.*

If there exists a positive integer p such that

(a) $\lambda(J \cap I^{j+1}/JI^j) = 0$ for every $j \leq p-1$;

(b) $\lambda(I^{p+1}/JI^p) \leq 1$;

$\Rightarrow \operatorname{depth}(\operatorname{gr}_I(R)) \geq \dim R - 1$ (i.e., $\operatorname{gr}_I(R)$ is almost Cohen-Macaulay).

Sally's Conjecture for j-stretched ideals

The next result proves Sally's Conjecture for j-stretched ideals, generalizing to any dimension several classical results.

Theorem 10 (M-Xie)

Let $(R, \mathfrak{m})$ be a local CM ring with $|R/\mathfrak{m}| = \infty$, and I be a j-stretched ideal. Let J be a general minimal reduction of I. Assume either

- *I is $\mathfrak{m}$-primary and $(x_1, \ldots, x_{d-1}) \cap I^2 = (x_1, \ldots, x_{d-1})I$, or*
- *$\ell(I) = \dim R = d$, I satisfies G_d, AN^-_{d-2}, $\operatorname{depth}(R/I) \geq 1$.*

If there exists a positive integer p such that

(a) $\lambda(J \cap I^{j+1}/JI^j) = 0$ for every $j \leq p-1$;

(b) $\lambda(I^{p+1}/JI^p) \leq 1$;

$\Rightarrow \operatorname{depth}(\operatorname{gr}_I(R)) \geq \dim R - 1$ *(i.e., $\operatorname{gr}_I(R)$ is almost Cohen-Macaulay).*

Sally's Conjecture for j-stretched ideals

The next result proves Sally's Conjecture for j-stretched ideals, generalizing to any dimension several classical results.

Theorem 10 (M-Xie)

Let $(R, \mathfrak{m})$ be a local CM ring with $|R/\mathfrak{m}| = \infty$, and I be a j-stretched ideal. Let J be a general minimal reduction of I. Assume either

- *I is $\mathfrak{m}$-primary and $(x_1, \ldots, x_{d-1}) \cap I^2 = (x_1, \ldots, x_{d-1})I$, or*
- *$\ell(I) = \dim R = d$, I satisfies G_d, AN^-_{d-2}, $\operatorname{depth}(R/I) \geq 1$.*

If there exists a positive integer p such that

(a) $\lambda(J \cap I^{j+1}/JI^j) = 0$ *for every $j \leq p-1$;*

(b) $\lambda(I^{p+1}/JI^p) \leq 1$;

$\Rightarrow \operatorname{depth}(\operatorname{gr}_I(R)) \geq \dim R - 1$ *(i.e., $\operatorname{gr}_I(R)$ is almost Cohen-Macaulay).*

Sally's Conjecture for j-stretched ideals

The next result proves Sally's Conjecture for j-stretched ideals, generalizing to any dimension several classical results.

Theorem 10 (M-Xie)

Let $(R, \mathfrak{m})$ be a local CM ring with $|R/\mathfrak{m}| = \infty$, and I be a j-stretched ideal. Let J be a general minimal reduction of I. Assume either

- *I is $\mathfrak{m}$-primary and $(x_1, \ldots, x_{d-1}) \cap I^2 = (x_1, \ldots, x_{d-1})I$, or*
- *$\ell(I) = \dim R = d$, I satisfies G_d, AN^-_{d-2}, $\operatorname{depth}(R/I) \geq 1$.*

If there exists a positive integer p such that

(a) $\lambda(J \cap I^{j+1}/JI^j) = 0$ *for every* $j \leq p-1$;

(b) $\lambda(I^{p+1}/JI^p) \leq 1$;

$\Rightarrow \operatorname{depth}(\operatorname{gr}_I(R)) \geq \dim R - 1$ *(i.e., $\operatorname{gr}_I(R)$ is almost Cohen-Macaulay).*

Sally's Conjecture for j-stretched ideals

The next result proves Sally's Conjecture for j-stretched ideals, generalizing to any dimension several classical results.

Theorem 10 (M-Xie)

Let $(R, \mathfrak{m})$ be a local CM ring with $|R/\mathfrak{m}| = \infty$, and I be a j-stretched ideal. Let J be a general minimal reduction of I. Assume either

- *I is $\mathfrak{m}$-primary and $(x_1, \ldots, x_{d-1}) \cap I^2 = (x_1, \ldots, x_{d-1})I$, or*
- *$\ell(I) = \dim R = d$, I satisfies G_d, AN^-_{d-2}, $\operatorname{depth}(R/I) \geq 1$.*

If there exists a positive integer p such that

(a) $\lambda(J \cap I^{j+1}/JI^j) = 0$ *for every* $j \leq p - 1$;

(b) $\lambda(I^{p+1}/JI^p) \leq 1$;

$\Rightarrow \operatorname{depth}(\operatorname{gr}_I(R)) \geq \dim R - 1$ (*i.e.*, $\operatorname{gr}_I(R)$ *is almost Cohen-Macaulay*).

A concrete example

Example 11

Let $R = k[[t^4, t^6, t^{11}, t^{13}]]$*,* $\mathfrak{m} = (t^4, t^6, t^{11}, t^{13})$*,* $I = (t^4, t^6, t^{11})$*.*
$\Rightarrow$

- *I is an $\mathfrak{m}$-primary ideal,*
- *$J \cap I^2 \neq JI$ for every minimal reduction J of I. In particular, I is not stretched.*
- *I is j-stretched on R.*

Remark. Therefore, j-stretched $\not\Rightarrow$ stretched.
Final Remark. All these definitions and results actually hold, more in general, for (associated graded) modules.

A concrete example

Example 11

Let $R = k[[t^4, t^6, t^{11}, t^{13}]]$, $\mathfrak{m} = (t^4, t^6, t^{11}, t^{13})$, $I = (t^4, t^6, t^{11})$.
$\Rightarrow$

- I *is an* $\mathfrak{m}$*-primary ideal,*
- $J \cap I^2 \neq JI$ *for* every *minimal reduction* J *of* I*. In particular,* I *is not stretched.*
- I *is j-stretched on* R*.*

Remark. Therefore, j-stretched $\not\Rightarrow$ stretched.
Final Remark. All these definitions and results actually hold, more in general, for (associated graded) modules.

A concrete example

Example 11

Let $R = k[[t^4, t^6, t^{11}, t^{13}]]$, $\mathfrak{m} = (t^4, t^6, t^{11}, t^{13})$, $I = (t^4, t^6, t^{11})$.
$\Rightarrow$

- *I is an $\mathfrak{m}$-primary ideal,*
- *$J \cap I^2 \neq JI$ for* every *minimal reduction J of I. In particular, I is not stretched.*
- *I is j-stretched on R.*

Remark. Therefore, j-stretched $\not\Rightarrow$ stretched.
Final Remark. All these definitions and results actually hold, more in general, for (associated graded) modules.

A concrete example

Example 11

Let $R = k[[t^4, t^6, t^{11}, t^{13}]]$, $\mathfrak{m} = (t^4, t^6, t^{11}, t^{13})$, $I = (t^4, t^6, t^{11})$.
$\Rightarrow$

- I *is an* $\mathfrak{m}$*-primary ideal,*
- $J \cap I^2 \neq JI$ *for* every *minimal reduction* J *of* I*. In particular,* I *is not stretched.*
- I *is* j*-stretched on* R*.*

Remark. Therefore, j-stretched $\not\Rightarrow$ stretched.
Final Remark. All these definitions and results actually hold, more in general, for (associated graded) modules.

A concrete example

Example 11

Let $R = k[[t^4, t^6, t^{11}, t^{13}]]$, $\mathfrak{m} = (t^4, t^6, t^{11}, t^{13})$, $I = (t^4, t^6, t^{11})$.
$\Rightarrow$

- I *is an* $\mathfrak{m}$*-primary ideal,*
- $J \cap I^2 \neq JI$ *for* every *minimal reduction* J *of* I*. In particular,* I *is not stretched.*
- I *is* j*-stretched on* R*.*

Remark. Therefore, j-stretched $\not\Rightarrow$ stretched.
Final Remark. All these definitions and results actually hold, more in general, for (associated graded) modules.

A concrete example

Example 11

Let $R = k[[t^4, t^6, t^{11}, t^{13}]]$, $\mathfrak{m} = (t^4, t^6, t^{11}, t^{13})$, $I = (t^4, t^6, t^{11})$.
$\Rightarrow$

- *I is an $\mathfrak{m}$-primary ideal,*
- *$J \cap I^2 \neq JI$ for* every *minimal reduction J of I. In particular, I is not stretched.*
- *I is j-stretched on R.*

Remark. Therefore, j-stretched $\not\Rightarrow$ stretched.

Final Remark. All these definitions and results actually hold, more in general, for (associated graded) modules.

A concrete example

Example 11

Let $R = k[[t^4, t^6, t^{11}, t^{13}]]$, $\mathfrak{m} = (t^4, t^6, t^{11}, t^{13})$, $I = (t^4, t^6, t^{11})$.
$\Rightarrow$

- *I is an $\mathfrak{m}$-primary ideal,*
- *$J \cap I^2 \neq JI$ for* every *minimal reduction J of I. In particular, I is not stretched.*
- *I is j-stretched on R.*

Remark. Therefore, j-stretched $\not\Rightarrow$ stretched.
Final Remark. All these definitions and results actually hold, more in general, for (associated graded) <u>modules</u>.