The bi-graded structure of Symmetric Algebras with applications to Rees rings

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This talk is about joint work with Claudia Polini and Bernd Ulrich.

The Data:

• k is a field,

• $R = k[x,y]$,

• I is a height 2 ideal of R minimally generated by 3 forms, g_1, g_2, g_3 of the same positive degree d,

• φ is a homogeneous Hilbert-Burch matrix for $[g_1, g_2, g_3]$. (So, φ is a 3×2 matrix, the columns of φ generate the kernel of $[g_1, g_2, g_3]$ and the signed 2×2 minors of φ are equal to a unit times $[g_1, g_2, g_3]$.)

• the entries of φ in column j have degree d_j with $d_1 \leq d_2$ and $d_1 + d_2 = d$.

The objects of interest:

- the symmetric algebra of I:

$$\mathrm{Sym}(I) = \frac{R[T_1, T_2, T_3]}{I_1([T_1, T_2, T_3]\varphi)},$$

- the Rees algebra of I:

$$\mathcal{R}(I) = R \oplus I \oplus I^2 \oplus I^3 \oplus \cdots = R[It],$$

- and especially,

$$\mathcal{A} = \ker\left(\ \mathrm{Sym}(I) \longrightarrow\!\!\!\!\!\!\twoheadrightarrow \mathcal{R}(I) \ \right).$$

The ideal $\mathcal{A}$ of $\mathrm{Sym}(I)$ is the defining ideal of Rees algebra of I.

The geometric significance of $\mathcal{A}$:

• The data g_1, g_2, g_3 gives rise to a morphism $\Psi : \mathbb{P}^1 \to \mathbb{P}^2$, with

$$\Psi(q) = [g_1(q) : g_2(q) : g_3(q)],$$

for all $q \in \mathbb{P}^1$.

• The image of Ψ is a plane curve $\mathcal{C}$ of degree $d^\dagger$.

• One way to learn about the singularities of $\mathcal{C}$ is to study

$$\text{the graph of } \Psi = \Gamma = \{(q, \Psi(q)) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid q \in \mathbb{P}^1\}.$$

• Notice that $\Gamma = \text{Bi-Proj}\,\mathcal{R}(I)\ ^\ddagger$; so, $\mathcal{A} = I(\Gamma)\text{Sym}(I)$.

• On this slide k is algebraically closed‡ and Ψ is birational† onto its image.

The approach of looking at Γ (or $\mathcal{A}$) to learn about the singularities on $\mathcal{C}$ bears fruit:

• See [Cox,_,Polini,Ulrich] "A study of singularities on rational curves via syzygies", available: now on the arXiv, eventually in the Memoirs.

• Much of [CKPU] is about parameterized plane curves $\mathcal{C}$ of even degree d with a singularity of multiplicity $= d/2$.

• Today I talk about the **algebra** that corresponds to parameterized plane curves $\mathcal{C}$ with a singularity of multiplicity $> d/2$. (The parity of d is not longer relevant.)

• That is, I insist that $d_1 < d_2$ and that φ has a generalized zero in col. 1.

• See the "General Lemma" in [CKPU] or Song-Chen-Goldman (Computer Aided Geometric Design) for the translation from Geometry to Algebra.

"The bi-graded structure of Symmetric Algebras:"

- The symmetric algebra $\mathrm{Sym}(I) = \dfrac{R[T_1, T_2, T_3]}{(f_1, f_2)}$, with $[f_1, f_2] = [T_1, T_2, T_3]\varphi$, is a bi-graded complete intersection.

- Recall $R = k[x, y]$. Let $\mathfrak{m} = (x, y)R$ the homogeneous maximal ideal of R.

- Let $S = k[T_1, T_2, T_3]$ and $B = R \otimes_k S = k[x, y, T_1, T_2, T_3]$.

- Give x and y degree $(1, 0)$ and each T_i degree $(0, 1)$.

- The $\mathrm{Sym}(I)$-ideal $\mathcal{A}$ is readily seen to equal the $\mathrm{Sym}(I)$-ideal

$$H^0_{\mathfrak{m}}(\mathrm{Sym}(I)) = 0 :_{\mathrm{Sym}(I)} \mathfrak{m}^\infty.$$

The ultimate goal is:

- to describe B-structure of $\mathcal{A}$.
- But I don't know that (unless $d_1 = 2$).
- So instead I will tell you the structure of $\mathcal{A}_{(\geq d_1 - 1, *)}$
- first as an S-module, (This is called **Theorem**.)
- then as a B-module. (This is called **Corollary**.)

Recall: $S = k[T_1, T_2, T_3]$, $B = k[x, y, T_1, T_2, T_3]$, $\deg x = \deg y = (1, 0)$, $\deg T_i = (0, 1)$.

Theorem [KPU]

If $d_1 < d_2$ and φ has a generalized zero in column 1, then $\mathcal{A}_{(\geq d_1-1,*)}$ is a free S-module. Furthermore,

1. If $d_1 - 1 \leq i \leq d_2 - 1$, then $\mathcal{A}_{(i,*)} \simeq \bigoplus_{\ell=1}^{d_1} S(-i, -a_\ell)$, where

$$\left\lfloor \frac{d+d_1-1-i}{d_1} \right\rfloor = a_1 \leq \cdots \leq a_{d_1} = \left\lceil \frac{d+d_1-1-i}{d_1} \right\rceil \quad \text{and}$$

$$\sum_{\ell=1}^{d_1} a_\ell = d + d_1 - 1 - i.$$

2. If $d_2 - 1 \leq i \leq d - 2$, then $\mathcal{A}_{(i,*)} \simeq S(-i, -2)^{d-1-i}$.

3. If $d - 1 \leq i$, then $\mathcal{A}_{(i,*)} = 0$.

The following table records the S-module structure of:

$$\mathcal{A}_{(\geq d_1-1,*)} \simeq \bigoplus S(-(i,j))^{n_{i,j}}.$$

So, the minimal homogeneous basis for the free S-module $\mathcal{A}_{(i,j)}$ has $n_{i,j}$ generators of bi-degree (i,j).

$\lceil \frac{d}{d_1} \rceil$	r	$\cdots$										
$\lfloor \frac{d}{d_1} \rfloor$	d_1-r	$\cdots$										
$\vdots$												
$\lfloor \frac{d}{d_1} \rfloor-\lambda+1$		$\cdots$	d_1	d_1-1	d_1-2	$\cdots$						
$\lfloor \frac{d}{d_1} \rfloor-\lambda$				1	2	$\cdots$						
$\vdots$												
3						$\cdots$	1					
2						$\cdots$	d_1-1	d_1	d_1-1	d_1-2	$\cdots$	1
	d_1-1	$\cdots$	λd_1+r-1	λd_1+r	λd_1+r+1	$\cdots$	d_2-2	d_2-1	d_2	d_2+1	$\cdots$	$d-2$

The generator degrees for the free S-module $\mathcal{A}_{(\geq d_1-1,*)}$.

- Define r by $d = d_1 \lfloor \frac{d}{d_1} \rfloor + r$, with $0 \leq r \leq d_1 - 1$.
- The position $(i, j) = (\lambda d_1 + r, \lfloor \frac{d}{d_1} \rfloor - \lambda)$ appears in the table if and only if $1 \leq \lambda \leq \lfloor \frac{d}{d_1} \rfloor - 2$. (These are the “exterior corner points”.)

We describe, in words, the transition of the generator degrees **to $\mathcal{A}_{i-1}$ from $\mathcal{A}_i$**, beginning at the right side of the table.

• The module $\mathcal{A}_{(i,*)} = 0$ for $d-1 \leq i$.

• If $d_2 - 1 \leq i \leq d-2$, then the generators of $\mathcal{A}_{(i,*)}$ are concentrated in the unique degree 2 and rank $\mathcal{A}_{(i-1,*)}$ is rank $\mathcal{A}_{(i,*)} + 1$. The actual generators of $\mathcal{A}_{(\geq d_2-1,*)}$ can be explicitly described by the technique of linkage.

• In the range $d_1 - 1 \leq i \leq d_2 - 1$, the rank of $\mathcal{A}_{(i,*)}$ remains constant at d_1 and the generators of $\mathcal{A}_{(i,*)}$ live in two degrees, or, occasionally, only one degree. As one looks from right to left, one free rank one summand of **lowest** shift in $\mathcal{A}_{(i,*)}$ is replaced by a free rank one summand with shift **one** higher in $\mathcal{A}_{(i-1,*)}$.

The B module structure of $\mathcal{A}_{(\geq d_1-1,*)}$:

Corollary [KPU] As a B-module, $\mathcal{A}_{(\geq d_1-1,*)}$ is minimally generated by the union

$$\left\{ \begin{array}{l} \quad \text{a } k\text{-basis for } \mathcal{A}_{(d_1-1,\lceil \frac{d}{d_1} \rceil)} \\ \cup\, \text{a } k\text{-basis for } \mathcal{A}_{(d_1-1,\lfloor \frac{d}{d_1} \rfloor)} \\ \cup \displaystyle\bigcup_{1\leq\lambda\leq\lfloor \frac{d}{d_1} \rfloor-2} \text{a } k\text{-basis for } \mathcal{A}_{(\lambda d_1+r,\lfloor \frac{d}{d_1} \rfloor-\lambda)}. \end{array} \right.$$

The B module structure of $\mathcal{A}_{(\geq d_1-1,*)}$, Part 2:

• One direction of the proof is obvious. The other direction is some amusing linear algebra over k and S. (Remember, we do not know formulas for any of these basis elements!)

• Conjecture: We conjecture that if $j < d_1 - 1$, then $\mathcal{A}_{(i,j)} = 0$ for $i < \lfloor \frac{d}{d_1} \rfloor$. In particular, we conjecture that the basis elements from "the exterior corner points" are part of a minimal generating set for the B-module $\mathcal{A}$.

$\lceil \frac{d}{d_1} \rceil$	$\mathbf{r}$	$\cdots$										
$\lfloor \frac{d}{d_1} \rfloor$	$\mathbf{d_1 - r}$	$\cdots$										
$\vdots$												
$\lfloor \frac{d}{d_1} \rfloor - \lambda + 1$		$\cdots$	d_1	$d_1 - 1$	$d_1 - 2$	$\cdots$						
$\lfloor \frac{d}{d_1} \rfloor - \lambda$				**1**	2	$\cdots$						
$\vdots$												
3						$\cdots$	1					
2						$\cdots$	$d_1 - 1$	d_1	$d_1 - 1$	$d_1 - 2$	$\cdots$	1
	$d_1 - 1$	$\cdots$	$\lambda d_1 + r - 1$	$\lambda d_1 + r$	$\lambda d_1 + r + 1$	$\cdots$	$d_2 - 2$	$d_2 - 1$	d_2	$d_2 + 1$	$\cdots$	$d - 2$

The generator degrees for the free S-module $\mathcal{A}_{(\geq d_1 - 1, *)}$.

- Define r by $d = d_1 \lfloor \frac{d}{d_1} \rfloor + r$, with $0 \leq r \leq d_1 - 1$.
- The position $(i, j) = (\lambda d_1 + r, \lfloor \frac{d}{d_1} \rfloor - \lambda)$ appears in the table if and only if $1 \leq \lambda \leq \lfloor \frac{d}{d_1} \rfloor - 2$. (These are the “exterior corner points”.)

Some ideas from the proof of Theorem:

• The mathematics that sets the project in motion is due to Jouanolou [Formes d'inertie et résultant: un formulaire, Adv. Math. 126 (1997), 119–250] who proved that the multiplication map

$$\mathcal{A}_{(i,*)} \otimes \mathrm{Sym}(I)_{(d-2-i,*)} \longrightarrow \mathcal{A}_{(d-2,*)} \simeq S(-2) \qquad (*)$$

gives a perfect pairing of S-modules.

• The "Sylvester form" of bi-degree $(d-2,2)$ is a basis for the free S-module $A_{(d-2,*)}$. (Wolmer Vasconcelos would call this form the "Jacobian dual of φ".)

• Jouanolou uses "Morley forms" to exhibit dual bases for the modules of (*).

• We met (*) in a paper Busé posted on the arXiv on December 17, 2007.

Some ideas from the proof of Theorem, Part II:

• Almost immediately after reading Busé's paper we understood $\mathcal{A}$ when $d_1 = 2$ or when $d_1 = d_2 = 3$.

• The perfect pairing

$$\mathcal{A}_{(i,*)} \otimes \mathrm{Sym}(I)_{(d-2-i,*)} \longrightarrow \mathcal{A}_{(d-2,*)} \simeq S(-2) \qquad (*)$$

shows that the S-module structure of $\mathcal{A}_{(i,*)}$ is completely determined by the S-module structure of $\mathrm{Sym}(I)_{(d-2-i,*)}$. The symmetric algebra $\mathrm{Sym}(I)$ is a complete intersection defined by the regular sequence f_1, f_2

$$(\text{Recall: } [f_1, f_2] = [T_1, T_2, T_3]\varphi);$$

so, the S-module structure of $\mathrm{Sym}(I)_{d-2-i}$ depends on the relationship between $d-2-i$, d_1, and d_2.

Some ideas from the proof of Theorem, Part III:

• The part of $\mathrm{Sym}(I)$ that corresponds to $\mathcal{A}_{\geq d_1-1}$, under the duality of (*), is $\mathrm{Sym}(I)_{\leq d_2-1}$. There is no contribution from f_2 to the S-module $\mathrm{Sym}(I)_{\leq d_2-1}$ in the bi-homogeneous B-resolution of $\mathrm{Sym}(I)$. So, basically, we may ignore f_2.

• On the other hand, the hypothesis that the first column of φ has a generalized zero allows us to make the critical calculation over a subring U of S, where U is a polynomial ring in two variables.

• In the proof of Theorem we decompose various bi-graded complexes over $R \otimes_k U$ into their R-graded components and their U-graded components.

Some ideas from the proof of Theorem, Part IV:

• Ultimately,

> the critical calculation is to produce a lower bound for the degrees of the syzygies of a U-module homomorphism. (★)

• Once (★) is carried out, then Hilbert series tricks lead to a complete description of all of the syzygy degrees of the relevant modules.

• During the process of proving this result we were inspired by a classification of matrices whose entries are linear forms from U due to Weierstrass (in the non-singular case) and Kronecker (in the general case); see Gantmacher "Matrix Theory", Chapter XII.