Quasidualizing Modules and the Auslander and Bass Classes

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Introduction

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Fact

Assume that R is complete. If A is an artinian R-module, then $A^\vee$ is noetherian. If N is a noetherian R-module, then $N^\vee$ is artinian. The modules A and N are Matlis reflexive.

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Example

The R-module R is always semidualizing.

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Example

E is a quasidualizing R-module.

Example

If R is complete, then T is a quasidualizing R-module if and only if $T^\vee$ is a semidualizing R-module.

Hom-tensor adjointness

Fact

Let A, B, and C be R-modules. Then the natural map

$$\psi : \mathrm{Hom}_R(A \otimes_R B, C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C))$$

is an isomorphism. This map is called Hom-tensor adjointness.

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Hom-tensor adjointness explains the first and second steps in the following sequence:

$$\begin{aligned}\mathrm{Hom}_R(T^\vee, T^\vee) &\cong \mathrm{Hom}_R(T^\vee \otimes_R T, E)\\ &\cong \mathrm{Hom}_R(T, \mathrm{Hom}_R(T^\vee, E))\\ &\cong \mathrm{Hom}_R(T, T).\end{aligned}$$

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2. one has $\mathrm{Ext}^i_R(L, M) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(L, M), M)$ for all $i > 0$.

Remark

We write $\mathcal{G}_M^{artin}(R)$ to denote the class of all artinian derived M-reflexive R-modules, $\mathcal{G}_M^{noeth}(R)$ to denote the class of all noetherian derived M-reflexive R-modules, and $\mathcal{G}_M^{mr}(R)$ to denote the class of all Matlis reflexive M-reflexive R-modules.

Remark

When $M = C$ is a semidualizing R-module, the class $\mathcal{G}_M^{noeth}(R)$ is the class of *totally C-reflexive R-modules*, sometimes denoted $\mathcal{G}_C(R)$.

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Let L and M be R-modules. We say that L is in the Auslander class $\mathcal{A}_M(R)$ with respect to M if it satisfies the following:

Auslander Class

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Let L and M be R-modules. We say that L is in the Auslander class $\mathcal{A}_M(R)$ with respect to M if it satisfies the following:

1. the natural homomorphism $\gamma_L^M : L \to \mathrm{Hom}_R(M, M \otimes_R L)$, defined by $l \mapsto \psi_l$ where $\psi_l(m) = m \otimes l$, is an isomorphism; and

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2. one has $\mathrm{Tor}_i^R(M, L) = 0 = \mathrm{Ext}_R^i(M, M \otimes_R L)$ for all $i > 0$.

Theorem

Lemma (–,Leamer, Sather-Wagstaff)

Let A and M be R-modules such that A is artinian and M is Matlis reflexive. Then $A \otimes_R M$ is Matlis reflexive.

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Let A and M be R-modules such that A is artinian and M is Matlis reflexive. Then $A \otimes_R M$ is Matlis reflexive.

Theorem

Assume that R is complete and let T be a quasidualizing R-module. Then there exists an equality of classes

$$\mathcal{G}^{mr}_{T^\vee}(R) = \mathcal{A}^{mr}_T(R).$$

Sketch of Proof

Let M be a Matlis reflexive R-module. We will show that $M \in \mathcal{G}^{mr}_{T^\vee}(R)$ if and only if $M \in \mathcal{A}^{mr}_{T}(R)$.

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We have the following commutative diagram:

$$
\begin{array}{ccc}
M & \xrightarrow{\delta_M^{T^\vee}} & \operatorname{Hom}_R(\operatorname{Hom}_R(M, T^\vee), T^\vee) \\
\Big\downarrow \gamma_M^T & & \Big\downarrow \cong \\
\operatorname{Hom}_R(T, T \otimes_R M) & & \operatorname{Hom}_R(\operatorname{Hom}_R(M, T^\vee) \otimes_R T, E) \\
\Big\downarrow \operatorname{Hom}_R(T, \delta_{T \otimes_R M}) \ \cong & & \Big\downarrow \cong \\
\operatorname{Hom}_R(T, (T \otimes_R M)^{\vee\vee}) & \xrightarrow{\cong} & \operatorname{Hom}_R(T, (\operatorname{Hom}_R(M, T^\vee))^\vee)
\end{array}
$$

Sketch of Proof cont.

Let P be a projective resolution of M.

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Let P be a projective resolution of M. Hom-tensor adjointness explains the second step in the following sequence:

$$\begin{aligned}
\operatorname{Ext}^i_R(M, T^\vee) &\cong \mathrm{H}_{-i}(\operatorname{Hom}_R(P, T^\vee)) \\
&\cong \mathrm{H}_{-i}(\operatorname{Hom}_R(P \otimes_R T, E)) \\
&\cong \operatorname{Hom}_R(\mathrm{H}_i(P \otimes_R T), E) \\
&\cong \operatorname{Hom}_R(\operatorname{Tor}^R_i(M, T), E).
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The third step follows from the fact that E is injective and homology commutes with exact functors.

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The third step follows from the fact that E is injective and homology commutes with exact functors. Since the Matlis dual of a module is zero if and only if the module is zero, we conclude that $\operatorname{Ext}^i_R(M, T^\vee) = 0$ if and only if $\operatorname{Tor}^R_i(M, T) = 0$.

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\operatorname{Ext}^i_R(\operatorname{Hom}_R(M, T^\vee), T^\vee) &\cong \operatorname{Ext}^i_R((M \otimes_R T)^\vee, T^\vee) \\
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The second step follows from the fact that $T^\vee$ is Matlis reflexive and a manifestation of Hom-tensor adjointness. The third step follows from the fact that T and $M\otimes_R T$ are Matlis reflexive.

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The second step follows from the fact that $T^\vee$ is Matlis reflexive and a manifestation of Hom-tensor adjointness. The third step follows from the fact that T and $M \otimes_R T$ are Matlis reflexive. Thus $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M, T^\vee), T^\vee) = 0$ if and only if $\operatorname{Ext}^i_R(T, M \otimes_R T) = 0$ concluding our proof.

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4. $\mathcal{G}_{T^\vee}^{artin}(R) = \mathcal{A}_T^{artin}(R)$.

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are inverse bijections.

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Assume that R is complete and let T be a quasidualizing R-module. Then the following maps are inverse bijections

$$\mathcal{B}^{noeth}_{T^\vee}(R) \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathcal{G}^{artin}_{T}(R)$$

and

$$\mathcal{B}^{artin}_{T}(R) \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathcal{G}^{noeth}_{T^\vee}(R).$$