Graded Cohen-Macaulayness for commutative rings graded by arbitrary abelian groups

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Let G be an abelian group. A (commutative) ring R is *G-graded* if there is a family of subgroups of R, $\{R_g\}_{g \in G}$, such that $R = \bigoplus_{g \in G} R_g$, and $R_g R_h \subseteq R_{g+h}$ for all $g, h \in G$.

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For a subgroup $H \leq G$, we set $R_H = \bigoplus_{h\in H} R_h$, which is a G- and H-graded subring of R. More generally,

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Note: the previous definition defines a G/H-grading on the ring R, using the family $\{R_x\}_{x\in G/H}$.

One can define analogues of many usual properties. For example, a *G-field* is a ring in which every homogeneous element is a unit, and a *G-maximal* ideal is a homogeneous ideal I such that R/I is a G-field (but we omit the G whenever possible).

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Proposition

Let R be a G-graded ring and H a torsion-free subgroup of G. Then

1. *R is a domain if and only if R is a G/H-domain.*
2. *R is reduced if and only if R is G/H-reduced.*

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One more basic piece of notation is the following: If R is G-graded, M is a G-graded R-module, and N is a 0-graded R-submodule (i.e, not necessarily G-homogeneous) of M, we let N^{*G} denote the R-submodule of M generated by all the G-homogeneous elements contained in N.

Primary Decomposition

Let R be a G-graded ring and $N \subseteq M$ graded R-modules. Say N is *G-irreducible* if whenever $N = N_1 \cap N_2$ (N_1, N_2 graded) then $N_1 = N$ or $N_2 = N$.

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If R is Noetherian, $P \in \mathrm{Ass}\, R$ if and only if $P = (0 : f)$ for some homogeneous element $f \in R$. Also, the union of the associated primes of R is, in general, strictly contained in the collection of zerodivisors of R.

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- $\dim_R^G(R) := \sup\{n \mid P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ is a chain of prime ideals of $R\}$
- $\mathrm{ht}_R^G(I) := \min\{\dim(R_{(P)}) \mid P \supseteq I$ and P is prime$\}$

Results on Height

The following fact is a generalization of a result of Matijevic-Roberts (1973).

- Let R be a G-graded ring, and suppose $H \leq G$ is a torsion-free subgroup. If $P \in \operatorname{Spec}^{G/H}(R)$ and $P^* = P^{*G}$, then $\operatorname{ht}^{G/H}(P/P^*) \leq \operatorname{rank} H$.

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This was extended to $\mathbb{Z}^d$-graded rings and sharpened by Uliczka (2009). A further generalization is:

Theorem

Let R be a G-graded ring and $H \leq G$ a torsion-free subgroup of finite rank, and set $P^* = P^{*G}$. If $P \in \operatorname{Spec}^{G/H}(R)$, then

$$\operatorname{ht}^{G/H}(P) = \operatorname{ht}^{G/H}(P^*) + \operatorname{ht}^{G/H}(P/P^*).$$

Results on Height

If we add the hypothesis that R is Noetherian to the previous setting and consider $P \in \operatorname{Spec}(R)$ with $\operatorname{ht}^{G/H}(P) = n$, we can show that there exists a chain

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such that $P_i \in \operatorname{Spec}(R)$ for $i = 1, \ldots, n$. I.e.,

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The next fact is somewhat unrelated, but is useful when discussing depth and grade.

- If R is a Noetherian graded ring, then for any $p \in \operatorname{Spec} R$, $\operatorname{ht}_R(p) = \operatorname{ht}_{R[t]}(p[t])$.

Grade and Depth

In order to define grade in this setting, we'll use Čech cohomology. Suppose R is a G-graded ring, and $I = (f_1, \dots, f_n) = (\mathbf{f})$ is a homogeneous ideal. Define

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Then depth is defined in the usual way. If (R, m) is a G-graded local Noetherian ring, we set

$$\operatorname{depth}^G(R) := \operatorname{grade}_m(R),$$

and say that R is *G-Cohen-Macaulay* (or just Cohen-Macaulay) if

$$\operatorname{depth}(R) = \dim(R).$$

Grade and Depth

A useful construction

Setting $S = R[\mathbf{t}]_{(mR[\mathbf{t}])}$ and $\widetilde{m} = mR[\mathbf{t}]_{(mR[\mathbf{t}])}$, we then have that $(S, \widetilde{m})$ is a graded local ring with the same dimension as R.

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In fact, since the extension $R \to S$ is faithfully flat, $\operatorname{depth} R = \operatorname{depth} S$, so that R is Cohen-Macaulay if and only if S is, the advantage being:

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If R is a G-graded ring and I is a finitely generated homogeneous ideal, there exist $d \geq 1$ and $t_1, \ldots, t_d$ with $\deg t_i = g_i$ for $g_i \in G$, $i = 1, \ldots, d$, such that $IR[t_1, \ldots, t_d]$ contains a homogeneous $R[t_1, \ldots, t_d]$-regular sequence of length $\operatorname{grade}_I(R)$.

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Sketch of proof. For $(2) \Rightarrow (1)$, if $P \in \operatorname{Spec}(R)$, then $P \in \operatorname{Spec}^{G/H}(R)$, and we can write $P = (\mathbf{x})$ for a G-homogeneous sequence $\mathbf{x}$. Then use

$$H^i_{\mathbf{x}}(R)_{[P]} = 0 \text{ if and only if } H^i_{\mathbf{x}}(R)_{(P)} = 0.$$

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The lemma allows us to assume that (R, m) is local and Cohen-Macaulay, and it suffices to show that $R_{(m)}$ is G/H-CM.

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That $\dim(R) = \dim^{G/H}_{R_{(m)}}(R_{(m)})$ follows from

$\operatorname{ht}(m) = \operatorname{ht}^{G/H}(m)$, and so we only need to show

$\operatorname{grade}_m(R) = \operatorname{grade}^{G/H}_{mR_{(m)}}(R_{(m)})$. □