Graded Cohen-Macaulayness for commutative rings graded by arbitrary abelian groups

Brian Johnson

University Of Nebraska – Lincoln

14 October 2011

s-bjohns67@math.unl.edu
Introduction

Outline

- Notation
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Outline

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- Properties
Introduction

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- Height & Dimension
Graded Cohen-Macaulayness
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Introduction Outline

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- Grade & Depth
Let $G$ be an abelian group. A (commutative) ring $R$ is \textit{G-graded} if there is a family of subgroups of $R$, $\{R_g\}_{g \in G}$, such that $R = \bigoplus_{g \in G} R_g$, and $R_g R_h \subseteq R_{g+h}$ for all $g, h \in G$. 
Let $G$ be an abelian group. A (commutative) ring $R$ is **$G$-graded** if there is a family of subgroups of $R$, $\{R_g\}_{g \in G}$, such that $R = \bigoplus_{g \in G} R_g$, and $R_g R_h \subseteq R_{g+h}$ for all $g, h \in G$.

For a subgroup $H \leq G$, we set $R_H = \bigoplus_{h \in H} R_h$, which is a $G$- and $H$-graded subring of $R$. More generally,

$$R_{g+H} := \bigoplus_{h \in H} R_{g+h}$$

is a $G$-graded $R_H$-submodule of $R$. 

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Note: the previous definition defines a $G/H$-grading on the ring $R$, using the family $\{R_x\}_{x \in G/H}$.
One can define analogues of many usual properties. For example, a **$G$-field** is a ring in which every homogeneous element is a unit, and a **$G$-maximal** ideal is a homogeneous ideal $I$ such that $R/I$ is a $G$-field (but we omit the $G$ whenever possible).
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\begin{proposition}
Let $R$ be a $G$-graded ring and $H$ a torsion-free subgroup of $G$. Then
\begin{enumerate}
  \item $R$ is a domain if and only if $R$ is a $G/H$-domain.
  \item $R$ is reduced if and only if $R$ is $G/H$-reduced.
\end{enumerate}
\end{proposition}
Theorem

Suppose \( R \) is a \( G \)-graded ring. If \( H \leq G \) is a finitely generated subgroup, the following are equivalent:

1. \( R \) is Noetherian.
2. \( R \) is \( G/H \)-Noetherian.
Introduction

A crucial property

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One more basic piece of notation is the following: If $R$ is $G$-graded, $M$ is a $G$-graded $R$-module, and $N$ is a 0-graded $R$-submodule (i.e., not necessarily $G$-homogeneous) of $M$, we let $N^{*G}$ denote the $R$-submodule of $M$ generated by all the $G$-homogeneous elements contained in $N$. 
Primary Decomposition

Let $R$ be a $G$-graded ring and $N \subseteq M$ graded $R$-modules. Say $N$ is $G$-irreducible if whenever $N = N_1 \cap N_2$ ($N_1, N_2$ graded) then $N_1 = N$ or $N_2 = N$. 


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Call $N$ $G$-primary if for all homogeneous $r \in R$ the map $M/N \rightarrow M/N$ induced by multiplication by $r$ is either injective or nilpotent.
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If $N = \bigcap N_i$ is a primary decomposition, then the prime ideals $P_i$ that occur as radicals of the $\text{Ann}(M/N_i)$ depend only on $M$ and $N$. 
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If $R$ is Noetherian, $P \in \text{Ass } R$ if and only if $P = (0 : f)$ for some homogeneous element $f \in R$. Also, the union of the associated primes of $R$ is, in general, strictly contained in the collection of zerodivisors of $R$. 
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- $\dim^G_R(R) := \sup \{ n \mid P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \text{ is a chain of prime ideals of } R \}$

- $\text{ht}^G_R(I) := \min \{ \dim(R_P) \mid P \supseteq I \text{ and } P \text{ is prime} \}$
Results on Height

The following fact is a generalization of a result of Matijevic-Roberts (1973).

Let $R$ be a $G$-graded ring, and suppose $H \leq G$ is a torsion-free subgroup. If $P \in \text{Spec}^{G/H}(R)$ and $P^* = P^*G$, then $\text{ht}^{G/H}(P/P^*) \leq \text{rank} H$. 

This was extended to $\mathbb{Z}^d$-graded rings and sharpened by Uliczka (2009). A further generalization is:

Theorem

Let $R$ be a $G$-graded ring and $H \leq G$ a torsion-free subgroup of finite rank, and set $P^* = P^*G$. If $P \in \text{Spec}^{G/H}(R)$, then $\text{ht}^{G/H}(P) = \text{ht}^{G/H}(P^*) + \text{ht}^{G/H}(P/P^*)$. 

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$$\text{ht}^{G/H}(P) = \text{ht}^{G/H}(P^*) + \text{ht}^{G/H}(P/P^*).$$
If we add the hypothesis that $R$ is Noetherian to the previous setting and consider $P \in \text{Spec}(R)$ with $\text{ht}^{G/H}(P) = n$, we can show that there exists a chain

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = P$$

such that $P_i \in \text{Spec}(R)$ for $i = 1, \ldots, n$. I.e.,

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The next fact is somewhat unrelated, but is useful when discussing depth and grade.

- If $R$ is a Noetherian graded ring, then for any $p \in \text{Spec} R$, $\text{ht}_R(p) = \text{ht}_R[p][t]$. 
In order to define grade in this setting, we’ll use Čech cohomology. Suppose $R$ is a $G$-graded ring, and $I = (f_1, \ldots, f_n) = (f)$ is a homogeneous ideal. Define

$$\text{grade}^G_I(R) := \min\{i \mid H^i_f(R) \neq 0\}.$$
Grade and Depth

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Then depth is defined in the usual way. If $(R, m)$ is a $G$-graded local Noetherian ring, we set

$$\text{depth}^G(R) := \text{grade}_m(R),$$

and say that $R$ is $G$-Cohen-Macaulay (or just Cohen-Macaulay) if

$$\text{depth}(R) = \text{dim}(R).$$
Setting $S = R[t](mR[t])$ and $\tilde{m} = mR[t](mR[t])$, we then have that $(S, \tilde{m})$ is a graded local ring with the same dimension as $R$. 
Grade and Depth
A useful construction

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In fact, since the extension $R \to S$ is faithfully flat, $\text{depth } R = \text{depth } S$, so that $R$ is Cohen-Macaulay if and only if $S$ is, the advantage being:
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If $R$ is a $G$-graded ring and $I$ is a finitely generated homogeneous ideal, there exist $d \geq 1$ and $t_1, \ldots, t_d$ with $\deg t_i = g_i$ for $g_i \in G$, $i = 1, \ldots, d$, such that $IR[t_1, \ldots, t_d]$ contains a homogeneous $R[t_1, \ldots, t_d]$-regular sequence of length $\text{grade}_I(R)$. 
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Let $R$ be a Noetherian $G$-graded ring, and suppose $H \leq G$ is a finitely generated torsion-free subgroup. TFAE:

1. $R$ is Cohen-Macaulay.
2. $R$ is $G/H$-Cohen-Macaulay.

**Sketch of proof.** For $(2) \Rightarrow (1)$, if $P \in \text{Spec}(R)$, then $P \in \text{Spec}(G/H)(R)$, and we can write $P = (x)$ for a $G$-homogeneous sequence $x$. Then use $H^i x(R)_P = 0$ if and only if $H^i x(R)(P) = 0$. 
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Proof sketch cont.

For \((1) \Rightarrow (2)\), the bulk of the work is contained in a lemma which states:
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- If $P \in \text{Spec}^{G/H}(R)$, and $P^* = P^*G$, then $R_{(P)}$ is $G/H$-CM if and only if $R_{(P^*)}$ is $G/H$-CM.
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- If $P \in \text{Spec}^{G/H}(R)$, and $P^* = P^*^G$, then $R(P)$ is $G/H$-CM if and only if $R(P^*)$ is $G/H$-CM.

The lemma allows us to assume that $(R, m)$ is local and Cohen-Macaulay, and it suffices to show that $R(m)$ is $G/H$-CM.
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Proof sketch cont.

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- If $P \in \text{Spec}^{G/H}(R)$, and $P^* = P^G$, then $R_P$ is $G/H$-CM if and only if $R_{(P^*)}$ is $G/H$-CM.

The lemma allows us to assume that $(R, m)$ is local and Cohen-Macaulay, and it suffices to show that $R_m$ is $G/H$-CM.

That $\dim(R) = \dim_{R_m}^{G/H}(R_m)$ follows from

$ht(m) = ht_{R_m}^{G/H}(m)$, and so we only need to show $\text{grade}_m(R) = \text{grade}_{mR_m}^{G/H}(R_m)$.