Castelnuovo-Mumford regularity and Gorensteinness of fiber cone

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Joint work with Ramakrishna Nanduri

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For an ideal I in a local ring $(R, \mathfrak{m})$, Fiber cone $F(I) := \oplus_{n\geq 0} I^n/\mathfrak{m}I^n$

These graded algebras are called blowup algebras.

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In this work, we concentrate on the Castelnuovo-Mumford regularity and Gorensteinness of fiber cone.

For a graded algebra $S = \oplus_{n \geq 0} S_n$ over a commutative Notherian ring S_0 and a finitely generated graded S-module $M = \oplus_{n \geq 0} M_n$, define a-invariant:

$$a(M) := \begin{cases} \max\{n \mid M_n \neq 0\} & \text{if } M \neq 0 \\ -\infty & \text{if } M = 0. \end{cases}$$

For $i \geq 0$, set

$$a_i(M) := a(H^i_{S_+}(M)),$$

where $H^i_{S_+}(M)$ is the i-th local cohomology module of M with respect to the ideal S_+.

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Castelnuovo-Mumford regularity is defined to be
$\text{reg}(M) := \max\{a_i(M) + i \mid i \geq 0\}$

- Let $(A, \mathfrak{m})$ be a local ring and I be any ideal. The Castelnuovo-Mumford regularity of $R(I)$ and $G(I)$ have been well studied in the past. Ooishi proved that $\operatorname{reg} R(I) = \operatorname{reg} G(I)$, [Ooishi-87, Planas-Vilanova-98, Trung-98].

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- In general $\operatorname{reg} F(I)$ need not be equal to $\operatorname{reg} R(I)$.

- If A is non-Buchsbaum and I is generated by a system of parameters and not a d-sequence then $\operatorname{reg} R(I) > 0$ (Trung-98). Since I is generated by a system of parameters $\operatorname{reg} F(I) = 0$.

Heinzer-Kim-Ulrich Example

Let $A = k[\![X, Y, Z]\!]/(X^2, Y^2, XYZ^2)$ and $I = (\bar{Y}, \bar{Z})A$.

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Looking at the defining ideal of $G(I)$ as an A/I-algebra and using the fact that $\operatorname{reltype} G(I) \leq \operatorname{reg} G(I) + 1$, it can be seen that we have $\operatorname{reg} G(I) \geq 2$.

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$\Rightarrow \quad \operatorname{reg} F(I) < \operatorname{reg} G(I)$.

Question: Under what conditions $\operatorname{reg} F(I) = \operatorname{reg} G(I)$?

A will denote a Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and I an ideal of A.

$\ell := \dim F(I)$, called analytic spread of I.

Let $\mathcal{F}$ denotes the I-good filtration, $A \supset \mathfrak{m} \supset \mathfrak{m}I \supset \mathfrak{m}I^2 \supset \cdots$

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The $R(I)$-modules $R(\mathcal{F}) := R \oplus \mathfrak{m}t \oplus \mathfrak{m}It^2 \oplus \cdots$ and $G(\mathcal{F}) := R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}I \oplus \mathfrak{m}I/\mathfrak{m}I^2 \oplus \cdots$ are finitely generated.

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We have the exact sequences:

$$0 \to R(I) \to R(\mathcal{F}) \to \mathfrak{m}G(I)(-1) \to 0$$

$$0 \to \mathfrak{m}G(I) \to G(I) \to F(I) \to 0$$

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Let $\underline{x^k} := x_1^k, \dots, x_\ell^k$. Then for all $n \in \mathbb{Z}$,

$$\left[H^\ell_{R(I)_+}(R(I))\right]_n \cong \lim_{\substack{\longrightarrow \\ k}} \frac{I^{\ell k+n}}{(\underline{x}^k)I^{(\ell-1)k+n}}$$

and

$$\left[H^\ell_{R(I)_+}(R(\mathcal{F}))\right]_n \cong \lim_{\substack{\longrightarrow \\ k}} \frac{\mathfrak{m}I^{\ell k+n-1}}{(\underline{x}^k)\mathfrak{m}I^{(\ell-1)k+n-1}}.$$

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This implies that $a_\ell(R(\mathcal{F})) - 1 \leq a_\ell(R(I))$.

Theorem: Let $(A, \mathfrak{m})$ be a Noetherian local ring and I be an ideal of A with $\ell = 1$. Then $\operatorname{reg} F(I) \leq \operatorname{reg} G(I)$. Furthermore, if grade $I = 1$, then $\operatorname{reg} F(I) = \operatorname{reg} G(I) = r(I)$, where $r(I)$ denote the reduction number of I.

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Theorem: Let $(A, \mathfrak{m})$ be a Noetherian local ring and I be an ideal of A. Suppose grade $I = \ell$ and grade $G(I)_+ \geq \ell - 1$. Then $\operatorname{reg} F(I) \geq \operatorname{reg} G(I)$. Furthermore, if depth $F(I) \geq \ell - 1$, then $\operatorname{reg} F(I) = \operatorname{reg} G(I)$.

Proof of the first theorem follows by considering the exact sequence:

$$\begin{aligned} 0 \rightarrow & H^0(\mathfrak{m}G(I)) \rightarrow H^0(G(I)) \rightarrow H^0(F(I)) \\ \rightarrow & H^1(\mathfrak{m}G(I)) \rightarrow H^1(G(I)) \rightarrow H^1(F(I)) \rightarrow 0 \end{aligned}$$

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and then comparing the vanishing properties of the graded components of these local cohomology modules.

Proof of the second theorem follows by going modulo $\ell - 1$ superficial sequence and applying the first theorem.

Proposition: Let $(A, \mathfrak{m})$ be a Noetherian local ring and I be an ideal of A. If $\operatorname{reg} R(\mathcal{F}) \leq \operatorname{reg} R(I)$, then $\operatorname{reg} F(I) = \operatorname{reg} R(I)$.

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If $\operatorname{grade}_{R(I)_+} R(\mathcal{F}) \geq \ell$, then $H^i(R(\mathcal{F})) = 0$ for $i < \ell$ and from the above Remark, it follows that $\operatorname{reg} R(\mathcal{F}) \leq \operatorname{reg} R(I)$.

Proposition : Let $(A, \mathfrak{m})$ be a Noetherian local ring and I be an $\mathfrak{m}$-primary ideal of A such that grade $I > 0$. Suppose $I^{n_0} = \mathfrak{m} I^{n_0 - 1}$ for some $n_0 \in \mathbb{N}$. Then $\operatorname{reg} F(I) = \operatorname{reg} G(I)$.

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Proposition : Let $(A, \mathfrak{m})$ be a Noetherian local ring and I be an ideal of A such that grade $I > 0$. Assume that $\mathfrak{m} G(I)$ is a Cohen-Macaulay $R(I)$-module of dimension ℓ. Then

(i) $\operatorname{reg} F(I) \leq \operatorname{reg} R(I)$;

(ii) if $a_\ell(R(\mathcal{F})) - 1 < a_\ell(R(I))$, then $\operatorname{reg} F(I) = \operatorname{reg} R(I)$;

(iii) if $a_\ell(R(\mathcal{F})) - 1 = a_\ell(R(I))$, then $\operatorname{reg} \mathfrak{m} G(I) \leq \operatorname{reg} R(I)$ and $\operatorname{reg} F(I) \leq \operatorname{reg} R(I)$. Furthermore, if $\operatorname{reg} \mathfrak{m} G(I) < \operatorname{reg} G(I)$, then $\operatorname{reg} F(I) = \operatorname{reg} R(I)$.

Gorensteinness

Proposition: Let $(A, \mathfrak{m})$ be a Noetherian local ring and I be an $\mathfrak{m}$-primary ideal such that the associated graded ring $G(I)$ is Cohen-Macaulay. Let $\omega_{G(I)} = \oplus_{n\in\mathbb{Z}}\omega_n$ and $\omega_{F(I)}$ be the canonical modules of $G(I)$ and $F(I)$ respectively. Then

(1) $\omega_{F(I)} \cong \oplus_{n\in\mathbb{Z}}(0 :_{\omega_n} \mathfrak{m})$;

(2) $a(F(I)) = a(G(I)) = r - d$, where r is the reduction number I with respect to any minimal reduction J of I;

(3) for any $k \in \mathbb{N}$, $a(F(I^k)) = [\frac{a(F(I))}{k}] = [\frac{r-d}{k}]$;

(4) if $G(I)$ is Gorenstein, then

$$\omega_{F(I)} \cong \bigoplus_{n\in\mathbb{Z}} \frac{(I^{n+r-d+1} : \mathfrak{m}) \cap I^{n+r-d}}{I^{n+r-d+1}}.$$

Sketch of Proof:

(1) Since $G(I)$ is Cohen-Macaulay and $F(I) = G(I)/\mathfrak{m}G(I)$ is such that $\dim G(I) = \dim F(I) = d$,

$$\begin{aligned} \omega_{F(I)} &\cong Hom_{G(I)}(F(I), \omega_{G(I)}) \cong (0 :_{\omega_{G(I)}} \mathfrak{m}G(I)) \\ &= (0 :_{\omega_{G(I)}} \mathfrak{m}) = \oplus_{n \in \mathbb{Z}}(0 :_{\omega_n} \mathfrak{m}). \end{aligned}$$

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$\Rightarrow (0 :_{\omega_n} \mathfrak{m}) \neq 0$ if and only if $\omega_n \neq 0$. Therefore $a(F(I)) = a(G(I))$.

(3) Follows from (2) and a result of Hoa and Zarzuela on the a-invariant of $G(I^k)$.

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(4) Follows by computing $\omega_{G(I)}$ explicitly and then use (1) to compute $\omega_{F(I)}$.

Corollary:

1. If $G(I)$ and $F(I)$ are Cohen-Macaulay, then $\operatorname{reg}\omega_{G(I)} = \operatorname{reg}\omega_{F(I)}$. Moreover, if either $F(I)$ or $G(I)$ is Gorenstein, then the regularity is equal to $d = \dim A$.

Corollary:

1. If $G(I)$ and $F(I)$ are Cohen-Macaulay, then $\operatorname{reg} \omega_{G(I)} = \operatorname{reg} \omega_{F(I)}$. Moreover, if either $F(I)$ or $G(I)$ is Gorenstein, then the regularity is equal to $d = \dim A$.

2. If $F(I)$ is Gorenstein, then

$$\lambda \left(\frac{(J^{n+1} : \mathfrak{m} I^r) \cap (J^n : I^r)}{(J^{n+1} : I^r)} \right) = \lambda \left(\frac{I^n}{\mathfrak{m} I^n} \right)$$

for all $n \in \mathbb{Z}^+$.

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Theorem: Let $(A, \mathfrak{m})$ be a Noetherian local ring, I be an $\mathfrak{m}$-primary ideal and J be a minimal reduction of I with reduction number r. Assume that T is a Gorenstein ring with the canonical module $\omega_T = T(-b)$ for some $b \in \mathbb{Z}$. Suppose $F(I)$ is Cohen-Macaulay.

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$$\lambda\left(\frac{((I^{n+1}+J):\mathfrak{m})\cap I^n}{I^{n+1}+JI^{n-1}}\right) = \lambda\left(\frac{I^n}{\mathfrak{m}I^n + JI^{n-1}}\right)$$

for $0 \leq n \leq r$. In this situation the canonical module $\omega_{F(I)} = F(I)(-b-1)$.

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"If part": Assume the equality of the given lengths.

Let $\eta : P \to \omega_{F(I/J)}$ be the natural surjective map from a graded free $F(I/J)$-module P of rank equal to $\mu(\omega_{F(I/J)})$.

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$\Rightarrow F(I/J)$ is Gorenstein with $\omega_{F(I/J)} = F(I/J)(r)$. Since $JF(I)$ is generated by a regular sequence $F(I)$ is Gorenstein with $\omega_{F(I)} = F(I)(r-d)$.

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4. If $(A, \mathfrak{m})$ is Noetherian local, I is $\mathfrak{m}$-primary with $G(I)$, $F(I)$ Cohen-Macaulay, then $e_0(\omega_{F(I)}) \leq e_0(\omega_{G(I)})$. Moreover, if $G(I)$ is Gorenstein, then the equality happens if and only if $I = \mathfrak{m}$.

THANK YOU