Bounds on the first Hilbert Coefficient

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Conventions

Set-up

- $(R, \mathfrak{m}, k)$: *a Noetherian local ring with maximal ideal* $\mathfrak{m}$, *residue field* k *and* $d = \dim R$.
- I *and* J *will always be ideals.*
- *For a finitely generated* R*-module* M, *we denote the length of* M *by* $\lambda(M)$, *and the minimal number of generators of* M *by* $\mu(M)$.
- *The integral closure of an ideal* I *is denoted* $\bar{I}$.

Hilbert Coefficients

Let I be an $\mathfrak{m}$-primary ideal. For large n, we can write

$$\lambda(R/I^{n+1}) =$$

$$e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d-1} + \ldots + (-1)^d e_d(I)$$

where $e_i(I)$ are all integers with $e_0(I) > 0$.

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Srinivas and Trivedi proved that if R is Cohen-Macaulay, then

$$|e_i(S)| \leq (9e^5)i!$$

where e is the multiplicity of S.

Srinivas and Trivedi also prove that $\lambda(S/\mathfrak{m}^n) = P_S(n)$ if $n \geq (12e^3)(d-1)!$, where $P_S(n)$ denotes the Hilbert polynomial. In particular, there are only finitely many possible Hilbert functions which can occur for Cohen-Macaulay rings of fixed dimension and multiplicity.

We study bounds on e_1.

Let I be $\mathfrak{m}$-primary. A key result is due to Elias, who proved that

$$e_1(I) \leqslant \binom{e_0(I) - k}{2}$$

if $I \subset \mathfrak{m}^k$ and the integral closure of I is not the integral closure of $\mathfrak{m}^k$.

Our work is some improvements on this bound.

Lech: If $(R, \mathfrak{m})$ is a regular local ring of dimension d, and I is $\mathfrak{m}$-primary, then

$$e(I) \leq d!\lambda(R/I).$$

Conjecture: If $(R, \mathfrak{m})$ is a regular local ring of dimension d, and I is $\mathfrak{m}$-primary, then

$$e(I) + \frac{d(d-1)}{2} e(I)^{\frac{d-1}{d}} \leq d!\lambda(R/I).$$

One-dimensional Case

Theorem

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one, and let $I \subset R$ be an $\mathfrak{m}-$primary ideal. Suppose that there exist distinct integrally closed ideals $J_1, ..., J_{k-1}$ such that $\mathfrak{m} \supsetneq J_{k-1} \supseteq J_{k-2} \supseteq .... \supseteq J_1 \supsetneq \overline{I}$. Then $e_1(I) \leqslant \binom{e_0(I)-k}{2}$.

Theorem

Let $(R, \mathfrak{m}, k)$ *be a one-dimensional analytically unramified local domain with infinite residue field* k *and integral closure* S. *Set* $t = dim_k(S/Jac(S))$, *where* $Jac(S)$ *is the Jacobson radical of* S. *Let* I *be an integrally closed ideal of* R. *Then there exists a chain of distinct integrally closed ideals,* $\mathfrak{m} \supset J_{n-1} \supset \ldots \supset J_0 = I$ *where* $n = \lfloor \frac{\lambda(R/I)-1}{t} \rfloor$.

Corollary

Let $(R, \mathfrak{m}, k)$ *be a one-dimensional analytically irreducible Cohen-Macaulay local domain with algebraically closed residue field* k. *Let* I *be an* $\mathfrak{m}$*-primary ideal of* R. *Then*

$$e_1(I) \leq \binom{e_0(I) - \lambda(R/\bar{I}) + 1}{2}.$$

Examples

Example

$R = K[[x,y]]/(xy(x-y))$ and $I_k = (x^{k+1}, y)$, $k \geqslant 1$. In this example, one can prove that $e_0(I_k) = k+3$, and $e_1(I_k) = 2$ Moreover, the ideals I_k are integrally closed for every k. Therefore we have a chain of distinct integrally closed ideals, $\mathfrak{m} \supset I_1 \supset I_2 \supset \ldots \supset I_{k+1}$. Applying our Theorem yields $e_1(I) < \binom{e_0(I)-k}{2}$, but this cannot be improved.

Example

Consider the ring $R = k[[t^7, t^8, t^9, t^{10}]]$ and the ideal $I = (t^9, t^{10}, t^{15})$. Then our bound gives $e_1(I) \leq 15$. The actual value is 9.

A Definition, One-Dimensional Case

We need a definition for our main result.

Definition

Let $(R, \mathfrak{m}, k)$ be a one-dimensional analytically unramified local domain with infinite residue field k. Set S equal to the integral closure of R. We define the *essential rank* of R to be $t = \dim_k(S/\mathrm{Jac}(S))$, where $\mathrm{Jac}(S)$ is the Jacobson radical of S.

Let $(R, \mathfrak{m}, k)$ be an analytically unramified Cohen-Macaulay local domain with an infinite residue field k. Let I be an $\mathfrak{m}$-primary ideal of R, with integral closure $\overline{I}$. Let d be the dimension of R. Choose a minimal reduction $y, x_2, ..., x_d$ of I. This sequence is a regular sequence since R is Cohen-Macaulay. Set $T = R[\frac{x_2}{y}, ..., \frac{x_d}{y}]$. Then $T \cong R[T_2, ..., T_d]/(yT_2 - x_2, ..., yT_d - x_d)$. It follows that the extension of $\mathfrak{m}$ to T is a height one prime ideal; set $A = T_{\mathfrak{m}T}$. Observe that A is a one-dimensional analytically unramified domain with an infinite residue field.

DEFINITION

Let $(R, \mathfrak{m}, k)$ be an analytically unramified Cohen-Macaulay local domain with an infinite residue field k. Let $y, x_1, ..., x_d$ be a regular sequence in R. We define *the essential rank* of $(y, x_2, ..., x_d)$ to be the essential rank of the one-dimensional ring A constructed above. Let I be an $\mathfrak{m}$-primary ideal of R. Define the *essential rank* of I to be the minimum of essential ranks of minimal reductions of I.

Main Theorem

Theorem

Let $(R, \mathfrak{m}, k)$ be an analytically unramified Cohen-Macaulay local domain with infinite residue field k. Let I be an $\mathfrak{m}$-primary ideal of R, with integral closure $\overline{I}$. Let t denote the essential rank of I. Then $e_1(I) \leqslant \binom{e_0(I)-n}{2}$ where $n = \lfloor \frac{\lambda(R/\overline{I})-1}{t} \rfloor$.

Corollary

Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension d and let I be an $\mathfrak{m}$-primary ideal contained in $\mathfrak{m}^k$. Then

$$e_1(I) \leqslant \binom{e_0(I) - k}{2}.$$