AMS Special Session Talk, Saturday, October 15, 2011, Lincoln, Nebraska

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The new results discussed here are joint with Wenliang Zhang [HZ].

For simplicity, all rings considered will be assumed to be complete local domains. If R is a domain, R^+ denotes the integral closure of R in an algebraic closure of its fraction field. See [HH2] for an extended treatment. The direct summand conjecture [Ho] asserts that if R is regular, then R is a direct summand of every module-finite extension. In the complete case, this is equivalent to the statement that R is a direct summand of R^+ as an R-module, i.e., that there is an R-linear map $\theta : R^+ \to R$ such that $\theta(1)$ is 1.

We are interested in the case where R is normal but not necessarily regular. In that case we may ask, which elements of R are values of R-linear maps $R^+ \to R$? We call these *target elements*. An elementary argument shows that $r \in R$ is a target element if and only if the inclusion $R \subseteq R + rR^+$ of R-modules splits over R.

Since the direct summand conjecture holds for all normal rings in equal characteristic 0, we henceforth restrict to the case where the residue field has characteristic $p > 0$.

The *plus closure* of an ideal I of R, denoted I^+, is $IR^+ \cap R$. The *test ideal* for plus closure (respectively tight closure) is the ideal of R consisting of all elements of R that kill I^+/I (respectively, I^*/I) for all $I \subseteq R$. See [HH1] for a treatment of test ideals for tight closure.

Theorem 1. *The ideal of target elements in the complete normal domain R is the test ideal for plus closure.*

Question. In characteristic $p > 0$, is the test ideal for plus closure the same as the test ideal for tight closure? (Plus closure is contained in tight closure: by an example of [BrMo], the inclusion can be strict.) These two test ideals agree in Gorenstein rings.

Remark. By a result of [HH3], if S is module-finite over R in characteristic $p > 0$ then the exact sequence $0 \to R \to S \to S/R \to 0$ gives an element of $\mathrm{Ext}^1_R(S/R, S)$, and this element is in the tight closure of 0 in a suitable ambient module. This explains why weakly F-regular rings split from all module-finite extensions, and why one expects a connection between target elements and the test ideal for tight closure.

We want to discuss a strengthened form of the direct summand conjecture. The latter is equivalent to the assertion that for a system of parameters $x_1, \ldots, x_d$ for R, for all t, $\mathfrak{A}_t = (x_1^t, \ldots, x_d^t)R :_R (x_1 \cdots x_d)^{t-1}$ is a proper ideal of R.

Conjecture 1 (strengthened direct summand conjecture). *With notation as just above, $\mathfrak{A}_t$ is contained in the integral closure of $(x_1, \ldots, x_d)R$.*

This follows from Ranganathan's strong direct summand conjecture. Conjecture 1 is true in equal characteristic and in mixed characteristic in dimension at most 3 (it can be deduced from the results of [Heit]).

Theorem 2. *If R is of mixed characteristic p and satisfies Cojecture 1, e.g., if $\dim(R) \leq 3$, and J is the target ideal, then the radical of $J+pR$ contains the defining ideal of the singular locus of R.*

From this result, one expects that target elements are reasonably plentiful. The proof is very delicate. Note that one does *not* know that $\mathrm{Hom}_R(R^+, -)$ commutes with localization, and that after localizing one loses completeness: one may have purity without splitting.

Conjecture 2. *With the same hypothesis as Theorem 2, the radical of J contains the defining ideal of the singular locus.*

We conclude with a tantalizing conjecture that is related to these ideas.

Conjecture 3. *Let R be a complete local domain of dimension d and mixed characteristic $p > 0$. Let $x_1, \ldots, x_d = p$ be a system of parameters. Let $I = (x_1, \ldots, x_{d-1})R :_R p^k$. Let $\overline{R} = R/pR$. Then $I\overline{R}$ is contained in the tight closure of $(x_1, \ldots, x_{d-1})\overline{R}$ in $\overline{R}$.*

This is true if $d = 3$ but the only proof we can find uses Heitmann's proof of the direct summand conjecture in dimension 3. Conjecture 3 implies the direct summand conjecture in dimension d.

References

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