Rees algebras of square-free monomial ideals

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AMS Central Section Meeting
October 15, 2011

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- *Integral Closure: $\overline{\mathcal{R}(I)} = \bigoplus_{i \geq 0} \overline{I^i} t^i = \mathcal{R}(\bar{I})$. So $\bar{I} = [\overline{\mathcal{R}(I)}]_1$.*

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- *Then* $J = \bigoplus_{i=1}^{\infty} J_i$ *is a graded ideal. A minimal generating set of* J *is often referred to as the defining equations of the Rees algebra.*

Example in degree 2

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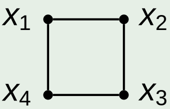

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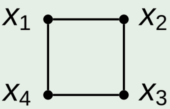

- Notice that $f_1f_3 = f_2f_4 = x_1x_2x_3x_4$ and that the degree 2 relation "comes" from the "even closed walk", in this case the square.

The degree 2 case

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- *Then $J = SJ_1 + S(\bigcup_{i=2}^{\infty} P_s)$, where*

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$$\begin{aligned} P_s &= \{T_\alpha - T_\beta \mid f_\alpha = f_\beta, \text{ for some } \alpha, \beta \in \mathcal{I}_s\} \\ &= \{T_\alpha - T_\beta \mid \alpha, \beta \text{ form an even closed walk } \} \end{aligned}$$

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- *Let* $T_{\alpha,\beta} = \dfrac{f_\beta}{\gcd(f_\alpha, f_\beta)} T_\alpha - \dfrac{f_\alpha}{\gcd(f_\alpha, f_\beta)} T_\beta$, *where* $\alpha, \beta \in \mathcal{I}_s$.

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A degree 3 example

Example (Villarreal)

- Let $R = k[x_1, \ldots, x_7]$ and let I be the ideal of R generated by $f_1 = x_1x_2x_3$, $f_2 = x_2x_4x_5$, $f_3 = x_5x_6x_7$, $f_4 = x_3x_6x_7$.

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- Notice that other than the linear relations there is also a relation in degree 2.

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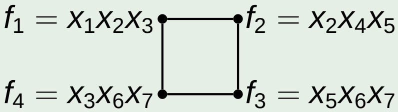

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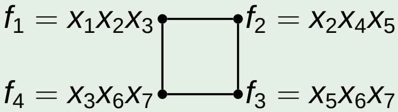

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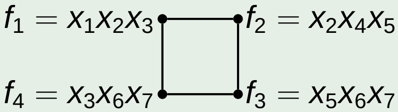

- Recall that the defining equations of $\mathcal{R}(I)$ are generated by J_1 and $x_4T_1T_3 - x_1T_2T_4$.
- Let $\alpha = (1,3)$ and $\beta = (2,4)$. Then $T_{\alpha,\beta} = x_4T_1T_3 - x_1T_2T_4 \in J$.

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Remark

- *By Taylor's Theorem we know that $T_{\alpha,\beta} \in J$.*
- *We show that for all $\alpha, \beta \in \mathcal{I}_s$, where $s \geq 3$ then $T_{\alpha,\beta} \in J_1 \cup J_2$.*

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- *The ideal I is said to be of* **linear type** *if the defining ideal J of the Rees algebra of I is generated in degree one.In other words, $\mathcal{R}(I) \simeq S/J$ and $J = J_1S$.*

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- *The ideal I is said to be of* **linear type** *if the defining ideal J of the Rees algebra of I is generated in degree one.In other words, $\mathcal{R}(I) \simeq S/J$ and $J = J_1S$.*

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Let I be the edge ideal of a connected graph G. Then I is of linear type if and only if G is the graph of a tree or contains a unique odd cycle.

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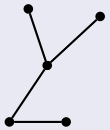

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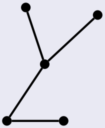

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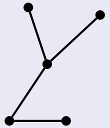

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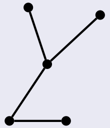

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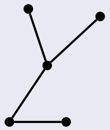

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- *Let I be a square-free monomial ideal generated in the same degree.*
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- *Then I is of linear type.*

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- Let $R = k[x_1, \ldots, x_{15}]$ and let I be generated by $f_1 = x_1x_2x_3x_4$, $f_2 = x_1x_5x_6x_7$, $f_3 = x_2x_8x_9x_{10}$, $f_4 = x_5x_6x_{11}x_{12}$, $f_5 = x_7x_{13}x_{14}x_{15}$.

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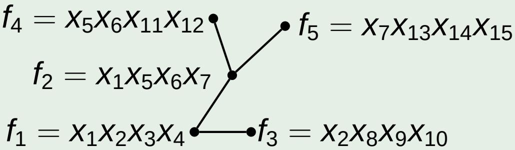

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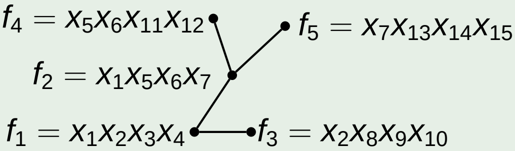

- Then by the previous Theorem, I is of linear type as its generator graph is the graph of a tree.

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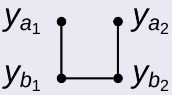

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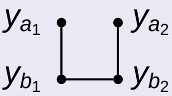

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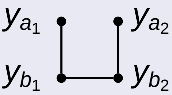

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- *Then it is straightforward to see that*
  $$T_{\alpha,\beta} = \frac{f_{b_2} C T_{a_2}}{\gcd(f_{a_2}, f_{b_2})} [T_{a_1,b_1}] + \frac{f_{a_1} C T_{b_1}}{\gcd(f_{a_1}, f_{b_1})} [T_{a_2,b_2}] \in J_1$$

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- *In the degree 3 Example of Villarreal, the generator graph was a square and the degree 2 relation was $x_4T_1T_3 - x_1T_2T_4 \notin H$. Hence I is not of fiber type.*