The Generalized Auslander-Reiten Conjecture and Derived Equivalences

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Introduction

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Remark: This has also been shown independently by S. Pan and J. Wei.

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Lemma (Thick-Perp)

For any classes $\mathsf{B}, \mathsf{C} \subseteq \mathsf{T}$*:*

1. ${}^{\perp}\mathsf{B}$ *and* $\mathsf{B}^{\perp}$ *are thick, and*
2. $\mathsf{B} \perp \mathsf{C}$ *if and only if* $\mathrm{Thick}(\mathsf{B}) \perp \mathrm{Thick}(\mathsf{C})$.

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This gives the following:

Theorem

The Gen. AR Conj. holds for R if and only if D^b(Gen. AR Conj.) holds for R.

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Another application of Rickard's Lemma gives $M \in \mathrm{Thick}(R)$, as desired.

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From the previous slide: If $F : \mathrm{D}^b(R) \to \mathrm{D}^b(S)$ is an equivalence.

Lemma (Rickard)

The restriction of F gives an equivalence between $\mathrm{Thick}(R)$ and $\mathrm{Thick}(S)$.

Suppose that S satisfies the conjecture below, we will show that R does.

Conjecture (D^b(Gen. AR Conj.))

If $M \perp M$ and $M \perp \mathrm{Thick}(R)$, then M is perfect.

If $M \perp M$ and $M \perp \mathrm{Thick}(R)$ then $F(M) \perp F(M)$ and $F(M) \perp \mathrm{Thick}(S)$. Since the conjecture holds for S we have $F(M) \in \mathrm{Thick}(S)$.

Another application of Rickard's Lemma gives $M \in \mathrm{Thick}(R)$, as desired.

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$$\mathrm{D}^b_{st}(R) = \frac{\mathrm{D}^b(R)}{\mathrm{Thick}(R)}$$

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Corollary

If R and S are stably derived equivalent Gorenstein rings, then the Gen. AR Conj. holds for R is and only if it holds for S.

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Note that $\mathrm{Thick}(R) = {}^{\perp}\mathrm{D}^b(R)$. Thus, the following statement specializes to $\mathrm{D}^b(\textit{Gen. AR Conj.})$) when $\mathrm{T} = \mathrm{D}^b(R)$:

$$\text{If } M \perp M \text{ and } M \perp {}^{\perp}\mathrm{T} \text{ then } M \in {}^{\perp}\mathrm{T}$$

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