How do maximal Cohen-Macaulay modules behave over rings of infinite Cohen-Macaulay type?

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Outline

Introduction

Given

- a commutative ring R and
- a class $\mathcal{C}$ of R-modules closed under isomorphism, finite direct sums and direct summands,

what can we say about direct-sum behavior of modules in $\mathcal{C}$?

The **Krull-Remak-Schmidt property** (KRS) holds for $\mathcal{C}$ if, whenever

$$M_1 \oplus M_2 \oplus \cdots \oplus M_r \cong N_1 \oplus N_2 \oplus \cdots \oplus N_s$$

for indecomposable M_i, $N_j \in \mathcal{C}$, then

1. $s = r$ and
2. $M_i \cong N_i$ for all i (after re-indexing).

Introduction

Fact (Swan, 1970). The KRS property holds for the class of finitely generated modules over a **complete** local ring.

- There are examples of non-complete local rings for which direct-sum decompositions of finitely generated modules can be non-unique.

Question

How can we describe the direct-sum behavior of modules over **non-complete** local rings?

Setup

- $(R, \mathfrak{m}, k)$: one-dimensional analytically unramified local ring.
- Class of maximal Cohen-Macaulay R-modules.
  - ◇ In this context, a **maximal Cohen-Macaulay** (MCM) R-module is a non-zero finitely generated torsion-free R-module.

The plan

Goal: study direct-sum behavior of MCM modules.

Approach: describe the **monoid** $\mathfrak{C}(R)$.

$\diamond$ $\mathfrak{C}(R)$: monoid of isomorphism classes of MCM R-modules (together with $[0]$) with operation $[M] + [N] = [M \oplus N]$.

Key: study the rank of modules.

$\diamond$ **Monoid:** commutative cancellative (additive) semigroup with identity 0.
(We always assume $x + y = 0 \Longrightarrow x = y = 0$.)

$\diamond$ **Rank:** The **rank of** M is the tuple $(r_1, \dots, r_s)$, where
$r_i := \dim_{R_{P_i}} M_{P_i}$, $P_i \in \operatorname{MinSpec}(R)$.

The monoid $\mathfrak{C}(R)$

- The natural map

$$R - \mathrm{mod} \to \widehat{R} - \mathrm{mod}, \text{ sending } M \to \widehat{M},$$

  induces an embedding

$$\mathfrak{C}(R) \hookrightarrow \mathfrak{C}(\widehat{R}), \text{ sending } [M] \to [\widehat{M}].$$

- $\mathfrak{C}(\widehat{R}) \cong \mathbb{N}^{(\Lambda)}$.
  - Λ: set of isomorphism classes of indecomposable MCM $\widehat{R}$-modules.
- $q := |\operatorname{MinSpec} \widehat{R}| - |\operatorname{MinSpec} R|$, **splitting number of** R.

The monoid $\mathfrak{C}(R)$

Fact (Levy-Odenthal, 1996).

- $(R, \mathfrak{m})$: one-dimensional analytically unramifed local ring.
- M: finitely generated $\widehat{R}$-module.

Then:
$M \cong \widehat{N}$ for some R-module $N \iff \operatorname{rank}_P M = \operatorname{rank}_Q M$ whenever P, Q are minimal primes of $\widehat{R}$ lying over the same minimal prime of R.

As a consequence:

$$\mathfrak{C}(R) \cong \begin{cases} \mathbb{N}^{(\Lambda)} & \text{if } q = 0, \\ \mathrm{Ker}(\mathcal{A}(R)) \cap \mathbb{N}^{(\Lambda)} & \text{if } q \geq 1. \end{cases}$$

Construction of $\mathcal{A}(R)$

- $P_1, \ldots, P_s$: minimal prime ideals of R
- $Q_{i,1}, \ldots, Q_{i,t_i}$: minimal prime ideals of $\widehat{R}$ lying over P_i.
- M: indecomposable MCM $\widehat{R}$-module of rank

$$(r_{1,1}, \ldots, r_{1,t_1}, \ldots, r_{s,1}, \ldots, r_{s,t_s}), \qquad r_{i,j} = \operatorname{rank}_{Q_{i,j}}(M).$$

- Set $\mathcal{A}(R)$ to be the $q \times |\Lambda|$ integer matrix, where the column indexed by $[M]$ is

$$\begin{bmatrix} r_{1,1} - r_{1,2} & \cdots & r_{1,1} - r_{1,t_1} & \cdots & r_{s,1} - r_{s,2} & \cdots & r_{s,1} - r_{s,t_s} \end{bmatrix}^T.$$

Structure of $\mathfrak{C}(R)$

Theorem (Baeth, Saccon)

- $(R, \mathfrak{m}, k)$: *one-dimensional analytically unramified local ring.*
- $q \geq 1$: *splitting number of* R.
- Λ: *set of isomorphism classes of indecomposable MCM* $\widehat{R}$*-modules.*
- *Assume there is at least one* $Q \in \operatorname{MinSpec}(\widehat{R})$ *such that* $\widehat{R}/Q$ *has infinite CM type.*

Then $\mathfrak{C}(R) \cong \mathrm{Ker}(\mathcal{A}(R)) \cap \mathbb{N}^{(\Lambda)}$, *where*

$$\mathcal{A}(R) = \left[\ \mathcal{T} \mid \mathcal{W} \mid \mathcal{V} \mid \mathcal{U}\ \right].$$

Structure of $\mathfrak{C}(R)$

Theorem (Part 2)

- *Assume either* $q = 1$ *or there is* $P \in \operatorname{MinSpec}(R)$ *such that* $\widehat{R}/Q$ *has infinite CM type for all* $Q \in \operatorname{MinSpec}(\widehat{R})$ *lying over* P.

Then $\mathcal{A}(R)$ *contains* $|k| \cdot |\mathbb{N}|$ *copies of an enumeration of* $\mathbb{Z}^{(q)}$.

Factorizations

Let H be an atomic monoid, and let $h \in H$, $h \neq 0$.

- The **set of lengths of h** is

$$\mathsf{L}(h) := \{n \mid h = a_1 + \cdots + a_n \text{ for atoms } a_i \in H\}.$$

- **Elasticity of $h \in H$**:

$$\rho(h) := \frac{\sup \mathsf{L}(h)}{\inf \mathsf{L}(h)}.$$

- **Elasticity of H**:

$$\rho(H) := \sup\{\rho(h) \mid h \in H \setminus \{0\}\}.$$

- H is **fully elastic** if

$$\mathcal{R}(H) = \mathbb{Q} \cap [1, \rho(H)],$$

where $\mathcal{R}(H) := \{\rho(h) \mid h \in H \setminus \{0\}\}$.

Elasticity of $\mathfrak{C}(R)$

Theorem (Baeth, Saccon)

- $(R, \mathfrak{m}, k)$: *one-dimensional analytically unramified local ring.*
- q: *splitting number of* R.
- *Assume there is at least one* $Q \in \operatorname{MinSpec}(\widehat{R})$ *such that* $\widehat{R}/Q$ *has infinite CM type.*

1. *If* $q = 0$, *then* $\mathfrak{C}(R)$ *is free, and* $\rho(\mathfrak{C}(R)) = 1$.
2. *If* $q \geq 1$, *then* $\rho(\mathfrak{C}(R)) = \infty$.

Elasticity of $\mathfrak{C}(R)$

Sketch of proof.

- Assume $q \geq 1$.
- There exist indecomposable MCM $\widehat{R}$-modules A and B of rank

$$\operatorname{rank} A = (0, 1, 0, \dots, 0) \qquad \text{and} \qquad \operatorname{rank} B = (1, 0, 1, \dots, 1).$$

- Fix positive integers n and $m > n$. There exist indecomposable MCM $\widehat{R}$-modules $C_{m,n}$ and $D_{m,n}$ of rank

$$\begin{aligned} \operatorname{rank} C_{m,n} &= (m+n, m \qquad, m+n, \dots, m+n), \\ \operatorname{rank} D_{m,n} &= (m \qquad, m+n, m \qquad, \dots, m \qquad). \end{aligned}$$

- Consider the following $\widehat{R}$-modules:

$$A \oplus B, \quad C_{m,n} \oplus D_{m,n}, \quad A^{(n)} \oplus C_{m,n}, \quad B^{(n)} \oplus D_{m,n}.$$

Sketch of proof (continued).

- Levy-Odenthal $\Longrightarrow$ there exist indecomposable MCM R-modules X, Y, Z and W such that

$$\widehat{X} \cong A \oplus B, \qquad \widehat{Y} \cong C_{m,n} \oplus D_{m,n},$$
$$\widehat{Z} \cong A^{(n)} \oplus C_{m,n}, \qquad \widehat{W} \cong B^{(n)} \oplus D_{m,n}.$$

- By faithfully flat descent of isomorphism:

$$X^{(n)} \oplus Y \cong Z \oplus W.$$

- Thus $\rho(\mathfrak{C}(R)) = \infty$. $\square$

More results on $\mathfrak{C}(R)$

Theorem (Baeth, Saccon)

- $(R, \mathfrak{m}, k)$: *one-dimensional analytically unramified local ring*
- $q \geq 1$: *splitting number of* R.
- *Assume there is at least one* $Q \in \operatorname{MinSpec}(\widehat{R})$ *such that* $\widehat{R}/Q$ *has infinite CM type.*
- *Assume either* $q = 1$ *or there is* $P \in \operatorname{MinSpec}(R)$ *such that* $\widehat{R}/Q$ *has infinite CM type for all* $Q \in \operatorname{MinSpec}(\widehat{R})$ *lying over* P.

Given an arbitrary non-empty finite set $L \subseteq \{2, 3, \ldots\}$,
there exists a MCM R*-module* M *such that*
M *is the direct sum of* l *indecomposable MCM* R*-modules* $\iff l \in L$.

More results on $\mathfrak{C}(R)$

Corollary

Under the same hypotheses, $\mathfrak{C}(R)$ has infinite elasticity and, in addition, is fully elastic.

Recall: H is **fully elastic** if $\mathcal{R}(H) = \mathbb{Q} \cap [1, \infty)$, where $\mathcal{R}(H) := \{\rho(h) \mid h \in H \setminus \{0\}\}$.