Wild Hypersurfaces

joint work with Andrew Crabbe

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Trichotomy Theorem Template

Let $\mathcal{C}$ be a category of modules. Then (we hope!) exactly one of the following holds:

- $\mathcal{C}$ contains only finitely many indecomposable modules.
- $\mathcal{C}$ has a classification scheme like Jordan canonical form: indecomposables are classified by finitely many discrete parameters (like rank) and one continuous parameter (like an eigenvalue).
- $\mathcal{C}$ has no classification schema: any classification theorem would involve simultaneously classifying the modules over every finite-dimensional algebra. I.e. the category of finite-length $k\langle x_1, \ldots, x_n\rangle$-modules embed into $\mathcal{C}$ for every $n \geqslant 1$.

Call these finite, tame, and wild type, respectively.

Finite-dimensional algebras

Theorem (Drozd 1977, Crawley-Boevey 1988)

Let Λ be a (possibly non-commutative) finite-dimensional algebra over an algebraically closed field. Then Λ-mod has exactly one of finite, tame, or wild representation type.

Standard Examples

- The finite-length modules over the non-commutative polynomial ring $k\langle a, b\rangle$ in two variables have wild type [Gel'fand-Ponomarev 1969]. A classification would solve the simultaneous similarity problem for pairs of matrices; they show the n-matrix problem embed in the 2-matrix one.
- The finite-length modules over $k[a, b]/(a^2, b^2)$ have tame type [Kronecker 1896].

Commutative Examples

Example (Drozd 1972)

The finite-length modules over $k[a,b]/(a^2, ab^2, b^3)$ have wild representation type.

It follows that $k[a_1, \dots, a_n]$ and $k[\![a_1, \dots, a_n]\!]$ have wild finite-length representation type for all $n \geqslant 2$.
($n = 1 \rightsquigarrow$ Jordan canonical form, tame type by definition!)

Maximal Cohen-Macaulay Modules

Reminder

Let S be a regular local ring, f a non-zero non-unit of S, and $R = S/(f)$ a hypersurface ring.

A MCM module over R is a f.g. R-module of depth equal to $\dim R$.

Equivalently, M is of the form $\operatorname{cok}\varphi$, where (φ, ψ) is a matrix factorization of f: square matrices over S such that

$$\varphi\psi = f\, I_n = \psi\varphi\,.$$

We adopt the definitions of finite, tame, wild representation types verbatim for MCM modules/matrix factorizations.

Finite MCM representation type

Theorem (Buchweitz-Greuel-Schreyer-Knörrer 1987)

Let $R = k[\![x_0, \ldots, x_d]\!]/(f)$, where k is an alg. closed field of characteristic $\neq 2, 3, 5$. Then R has finite MCM type if and only if R is isomorphic to the hypersurface defined by

$$g(x_0, x_1) + x_2^2 + \cdots + x_d^2\,,$$

where $g(x_0, x_1)$ is one of the following polynomials.

$$\begin{array}{ll} (A_n) & x_0^2 + x_1^{n+1} \\ (D_n) & x_0^2 x_1 + x_1^{n-1} \\ (E_6) & x_0^3 + x_1^4 \\ (E_7) & x_0^3 + x_0 x_1^3 \\ (E_8) & x_0^3 + x_1^5 \end{array}$$

Finite MCM representation type

The proof of the classification relies on the following Key Step:

Key Step (BGSK)

Let $R = k[\![x_0, \ldots, x_d]\!]/(f)$, where k is an alg. closed field of characteristic $\neq 2, 3, 5$.

If $d \geqslant 2$ and R has finite MCM representation type, then R has multiplicity at most 2, that is, f has order at most 2.

Specifically, if $d \geqslant 2$ and $\operatorname{ord}(f) \geqslant 3$, then R has a $\mathbb{P}_k^{d-1}$ of indecomposable MCMs.

Tame MCM representation type

Question

Can we classify hypersurfaces of tame MCM representation type? In particular, is there an analogue of the Key Step, so we can rule out high multiplicities?

Here are some candidates to replace the ADE polynomials.

Example (Drozd-Greuel 1993)

The one-dimensional hypersurfaces defined by

$$T_{pq}(x, y) = x^p + y^q + x^2y^2 \,,$$

where $p, q \geqslant 2$, have tame MCM representation type.

In fact, a curve singularity of infinite MCM type has tame type if and only if it birationally dominates a T_{pq} hypersurface.

Tame MCM representation type

Example (Drozd-Greuel-Kashuba 2003)

The two-dimensional hypersurfaces defined by

$$T_{pqr}(x, y, z) = x^p + y^q + z^r + xyz\,,$$

where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leqslant 1$, have tame CM representation type.

Potential Key Step

Let $R = k[\![x_0, \ldots, x_d]\!]/(f)$, where k is an alg. closed field of characteristic $\neq 2, 3, 5$.

If $d \geqslant 2$ and $\operatorname{ord}(f) \geqslant 4$, must R have wild MCM representation type?

Result

Theorem (V.V. Bondarenko 2007)

Let $f \in k[\![x_0, x_1, x_2]\!]$ *have order* $\geqslant 4$. *Then* $k[\![x_0, x_1, x_2]\!]/(f)$ *has wild MCM representation type.*

Theorem (Crabbe-Leuschke 2010)

Let $f \in k[\![x_0, \ldots, x_d]\!]$, *with* $d \geqslant 2$, *have order* $\geqslant 4$. *Then* $k[\![x_0, \ldots, x_d]\!]/(f)$ *has wild MCM representation type.*

Sketch of Proof

Let $S = k[\![z, x_1, \ldots, x_d]\!]$, with $d \geqslant 2$. Let $f \in S$ have order at least 4.

Introduce formal parameters $a_1, \ldots, a_d$. Then one can write (formally!)

$$f = z^2 h + (x_1 - a_1 z) g_1 + \cdots + (x_d - a_d z) g_d \,,$$

with $\operatorname{ord}(h) \geqslant 2$ and $\operatorname{ord}(g_i) \geqslant 3$ for each i.

(This is an easy calculation:
$z^2 \mathfrak{m}^2 + (x_1 - a_1 z, \ldots, x_d - a_d z)\mathfrak{m}^3 = \mathfrak{m}^4 \,.$)

Sketch of Proof

So

$$f = z^2 h + (x_1 - a_1 z)g_1 + \cdots + (x_d - a_d z)g_d \,.$$

Note that h and the g_j's involve a_i's.

This is the shape of an (A_1) polynomial in $2d$ variables!

$$\begin{aligned} &= u_1 v_1 + \cdots + u_d v_d \\ &\sim u_1^2 + v_1^2 + \cdots + u_d^2 + v_d^2 \,. \end{aligned}$$

All the non-trivial matrix factorizations of an odd-dimensional (A_1) hypersurface are known: there is exactly one indecomposable one up to equivalence. Call it $(\Phi(\underline{a}), \Psi(\underline{a}))$. It's explicitly given in terms of x_i, z, g_i, and h.

Sketch of Proof

To show that $R = S/(f)$ has wild MCM type, it suffices to embed the category of finite-length $k[a_1, \ldots, a_d]$-modules into $\mathrm{MCM}(R)$.

Let V be a finite-length $k[a_1, \ldots, a_d]$-module, i.e. a k-vector space with operators $A_1, \ldots, A_d \colon V \longrightarrow V$ representing the action of the a_i's.

In the distinguished matrix factorization $(\Phi(\underline{a}), \Psi(\underline{a}))$, replace each a_i by the square matrix A_i, and each x_i and z by $x_i\, I$ and $z\, I$.

Fact

$(\Phi(\underline{A}), \Psi(\underline{A}))$ is a matrix factorization of f.

Theorem

$(\Phi(\underline{A}), \Psi(\underline{A}))$ is indecomposable if V is, and
$(\Phi(\underline{A}), \Psi(\underline{A})) \cong (\Phi(\underline{A'}), \Psi(\underline{A'}))$ iff $V \cong V'$.
Consequently R has wild MCM type.