The Projective Line Over The Integers

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$\operatorname{Spec}(R) = \{\text{prime ideals of } R\}$, poset under $\subseteq$.

Questions.

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Motivation

For a commutative Noetherian ring R,

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Q2. What is $\mathrm{Spec}(R)$ for a two-dimensional Noetherian domain R related to a polynomial ring?

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R. Wiegand, '86: Characterization of $\mathrm{Spec}(\mathbb{Z}[x])$, the affine line over $\mathbb{Z}$.

5 Axioms for $U := \text{Spec}(\mathbb{Z}[x])$

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(1) (Obvious Axioms) • $|U| = |\mathbb{Z}|$, • $\dim(U) = 2$, • $|u^{\uparrow}| = \infty$,
• $|(u, v)^{\uparrow}| < \infty$, $\forall\ u \neq v$, $\mathrm{ht}(u) = \mathrm{ht}(v) = 1$.

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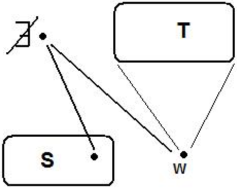
(2) (Subtle Axiom) **(RW)** For $S \subseteq \{\text{ht 1's}\}$, $T \subseteq \{\text{ht 2's}\}$, $0 < |S| < \infty$, $|T| < \infty$, $\exists\ w \in \{\text{ht 1's}\}$,

$$\bigcup_{s \in S} (w, s)^{\uparrow} \subseteq T \subset w^{\uparrow}.$$

Definition. w is a **radical element** for (S, T) if (RW) holds.



Notation. Projective line over $\mathbb{Z}$=Proj $(\mathbb{Z})$

$$\text{Proj}\,(\mathbb{Z}) := \text{Spec}(\mathbb{Z}[x]) \cup \text{Spec}(\mathbb{Z}\left[\tfrac{1}{x}\right]), \text{ with}$$

$$\text{Spec}(\mathbb{Z}[x]) \cap \text{Spec}(\mathbb{Z}\left[\tfrac{1}{x}\right]) \leftrightsquigarrow \text{Spec}(\mathbb{Z}[x, \tfrac{1}{x}]).$$

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e.g. $(4x^3 + 3x^2 - 1) \leftrightsquigarrow (4 + 3 \cdot \dfrac{1}{x} - 1 \cdot \dfrac{1}{x^3})$ in Spec $(\mathbb{Z}[x, \frac{1}{x}])$.

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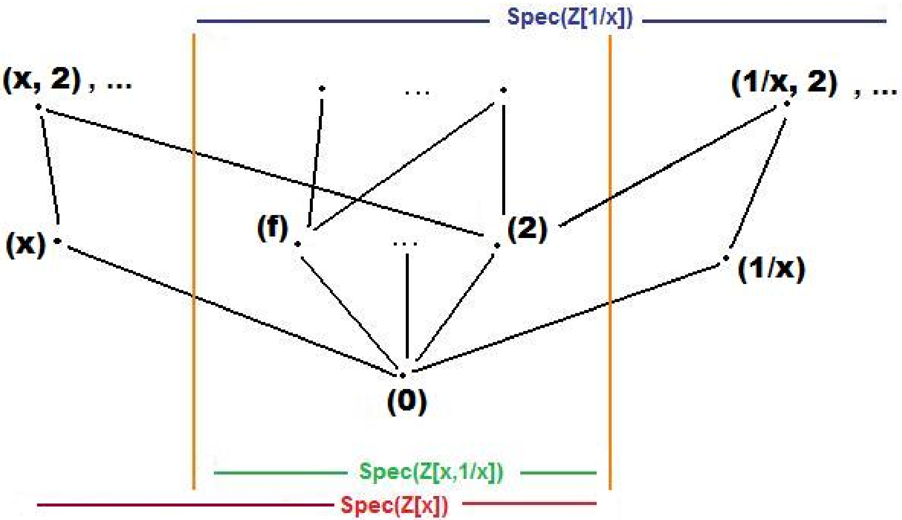
- Characterize Proj $(\mathbb{Z})$?
- Proj $(\mathbb{Z})$ satisfies the obvious axioms.
- What about (RW)? $\exists$ a radical element for finite subsets $\emptyset \neq S$ of height 1's and T of height 2's in Proj $(\mathbb{Z})$?

Proj $(\mathbb{Z})$ – "Roughly"



Preliminaries, Previous Results

Fact. Proj $(\mathbb{Z})$ does not satisfy (RW). [A. Li, S. Wiegand '97]

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Example. For $S := \{(\frac{1}{x}), (2), (5)\}$ and $T := \{(x, 2), (\frac{1}{x}, 2), (\frac{1}{x}, 3)\}$, (S, T) does not have a radical element in Proj$(\mathbb{Z})$.

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Sketch.
If w radical for (S, T) in $\mathrm{Proj}\,(\mathbb{Z})$, then

$w \subseteq (\frac{1}{x}, 2) \cap (\frac{1}{x}, 3) \Longrightarrow w \neq (p)$, $p \in \mathbb{Z}$ prime. Also $w \neq (\frac{1}{x})$.

Thus $w = (g(x))$, for g irreducible, $\deg(g) > 0$.

Since $(5) \in S$, $(g, 5)^{\uparrow} \neq \emptyset$. But $(g, 5)^{\uparrow} \not\subseteq T$, a contradiction.

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Question. When $\exists$ radical elements?

Coefficient Subset

Definition. For U poset, $\dim(U) = 2$, $C \subseteq \{\text{ht 1's}\}$,
C = **coefficient subset of U** if

- $\forall\, p \in C$, $|p^{\uparrow}| = \infty$;
- $\forall\, p \neq q \in C$, $(p, q)^{\uparrow} = \emptyset$;
- $\bigcup_{p \in C} p^{\uparrow} = \{\text{ht 2's}\}$;
- $p \in C$, $u \in \{\text{ht 1's}\} \setminus C \Rightarrow (p, u)^{\uparrow} \neq \emptyset$, $p^{\uparrow} = \bigcup_{(v \in \{\text{ht 1's}\} \setminus C)} (p, v)^{\uparrow}$;

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Proposition 1. [Arnavut] $C_0 = \{p\mathbb{Z}[x] \mid p \in \text{Spec}(Z)\}$ = the unique coefficient subset of Proj $(\mathbb{Z})$.

Conjecture

Proposition 2. [Arnavut] Let $S \subseteq \{\text{ht 1's}\}$, $T \subseteq \{\text{ht 2's}\}$, $0 < |S| < \infty$, $|T| < \infty$. Suppose:

- $S \cap C_0 \subseteq \bigcup_{m \in T}(m^{\downarrow} \cap C_0)$, and
- Either $\bigcup_{s \in S}(s, \frac{1}{x})^{\uparrow} \subseteq T$ or $\bigcup_{s \in S}(s, x)^{\uparrow} \subseteq T$.

Then (S, T) has ∞ radical elements in Proj $(\mathbb{Z})$.

Proposition 2. [Arnavut] Let $S \subseteq \{\text{ht } 1\text{'s}\}$, $T \subseteq \{\text{ht } 2\text{'s}\}$, $0 < |S| < \infty$, $|T| < \infty$. Suppose:

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Conjecture. [Arnavut] Same hypothesis. Suppose:

- $S \cap C_0 \subseteq \bigcup_{m \in T}(m^{\downarrow} \cap C_0)$, and
- $(x) \in S$ and $(\frac{1}{x}) \in S$.

Then $\exists\ \infty$ radical elements for (S, T).

Conjecture when $T = \emptyset$

Proposition 3. [–, E-T] Let $S \subseteq \{\text{ht } 1\text{'s}\}$, $0 < |S| < \infty$ and $T = \emptyset$. Suppose

$$\bullet\ S \cap C_0 = \emptyset; \qquad \bullet\ (x) \in S \quad \& \quad (\tfrac{1}{x}) \in S.$$

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Conjecture when $T = \emptyset$

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Proposition 4. [–, E-T] (S, T) as usual. If

- $T \subseteq (\frac{1}{x})^{\uparrow}$,
- $S \cap C_0 \subseteq \{(p) \mid (\frac{1}{x}, p) \in T\}$, $p \in \mathrm{Spec}(\mathbb{Z})$,
- $(x) \in S$, $(\frac{1}{x}) \in S$,
- $S - C_0 = \{(x), (\frac{1}{x})\} \cup \{(x + \alpha_i),\ \alpha_i \in \mathbb{Z}\}_{i=1}^{m}$,

$\Longrightarrow \exists\ \infty$ radicals for (S, T).

More Results and an Example

Proposition 5. [–, E-T] $S = \{(x), (\frac{1}{x}), (p_1), ..., (p_n)\}$, and $T = \{(x, p_1), ..., (x, p_n), (\frac{1}{x}, p_1), ..., (\frac{1}{x}, p_n)\} \Longrightarrow \exists$ radical elt for (S, T).

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Example of Prop 5. Let $S = \{(x), (\frac{1}{x}), (2), (3), (5)\}$ and

$$T = \{(x, 2), (x, 3), (x, 5), (\tfrac{1}{x}, 2), (\tfrac{1}{x}, 3), (\tfrac{1}{x}, 5)\}.$$

$$S_x := \{(x), (2), (3), (5)\}, \quad T_x := \{(x, 2), (x, 3), (x, 5)\}$$

$$S_{\frac{1}{x}} := \{(\tfrac{1}{x}), (2), (3), (5)\}, \quad T_{\frac{1}{x}} := \{(\tfrac{1}{x}, 2), (\tfrac{1}{x}, 3), (\tfrac{1}{x}, 5)\}$$

$\therefore \{S_x, T_x\} \subseteq \mathrm{Spec}(\mathbb{Z}[x])$, $\{S_{\frac{1}{x}}, T_{\frac{1}{x}}\} \subseteq \mathrm{Spec}(\mathbb{Z}[\frac{1}{x}])$.

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$\Longrightarrow \exists$ rad. elt. for $(S_x, T_x) : v_x = (15x + 2) \in \mathrm{Spec}(\mathbb{Z}[x])$,

$\Longrightarrow \exists$ rad. elt. for $(S_{\frac{1}{x}}, T_{\frac{1}{x}}) : v_{\frac{1}{x}} = (2 + \frac{15}{x}) \in \mathrm{Spec}(\mathbb{Z}[\frac{1}{x}])$.

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Now $(2 + \frac{15}{x}) \leftrightsquigarrow (2x + 15)$.

Example Cont'd

Let y be another variable.

$\gcd((15x+2)(2x+15), 30x) = 1 \implies$
$((15x+2)(2x+15) + y(30x)) \in \mathrm{Spec}(\mathbb{Z}[x,y])$. [Kaplansky ex]

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$\therefore\ w = (r(x)) = (30x^2 + (229+30q)x + 30)$ a rad. elt. for (S, T)

Question

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$$S := \{(p_1), ..., (p_n), (x), (\frac{1}{x}), (x+a), (x+b)\},$$

$$T := \{(x, p_1), ..., (x, p_\ell), (\frac{1}{x}, p_{\ell+1}), ..., (\frac{1}{x}, p_n)\},$$

if $0 \leq \ell \leq n$, $\gcd(ab, p_1 ... p_\ell) = 1$, p_i, a, b distinct primes $\in \mathbb{Z}$?

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Answer. Yes, $|\text{radical elements}| = \infty$.

Theorem 2. [–, E-T] Assume a and b are relatively prime integers and

$$S := \{(p_1), \ldots, (p_n), (x), (\tfrac{1}{x}), (x-a), (x-b)\},$$

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$0 \le \ell \le n$, $\gcd(ab, p_1 ... p_\ell) = 1$, the p_i are distinct prime integers.

Suppose $\mathbf{pq}$ divides

$(1-\mathbf{p}^t)(b^2+ab+a^2)+\mathbf{q}a^3b^3$ and $(1-\mathbf{p}^t+\mathbf{q}b^2a^2)(b+a)$,

$\mathbf{p} = p_1 \ldots p_\ell$, $\mathbf{q} = p_{\ell+1} \ldots p_n$, $t = \text{lcm}(\phi(a^2), \phi(b^2))$, and ϕ is the Euler phi function.

Then $\exists\ \infty$ radical elements for (S, T).

An Example of Theorem 2

For $S = \{(2), (3), (x), (\frac{1}{x}), (x-5), (x-7)\}$ and $T = \{(x, 2), (\frac{1}{x}, 3)\}$,

$$g(x; u, v, w) := 3x^4 + 6ux^3 + 6vx^2 + 6wx + 2^{420} \in \mathbb{Z}[x]$$

generates a radical element for (S, T) for $w = 0$,

$$u = \frac{(1 - 2^{420} + 3 \cdot 5^2 \cdot 7^2)(5+7)}{2 \cdot 3 \cdot 5^2 \cdot 7^2} \in \mathbb{Z} \text{ , and}$$

$$v = \frac{(1 - 2^{420})(5^2 + 35 + 7^2) + 3 \cdot 5^3 \cdot 7^3}{2 \cdot 3 \cdot 5^2 \cdot 7^2} \in \mathbb{Z} \text{ .}$$

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$u \;\&\; v \in \mathbb{Z}$ since $5^2 \cdot 7^2$ divides $1 - 2^{420}$, and also $2 \cdot 3 = 6$ divides $(1 - 2^{420} + 3 \cdot 5^2 \cdot 7^2)(5+7)$ and $(1 - 2^{420})(5^2 + 35 + 7^2) + 3 \cdot 5^3 \cdot 7^3$.

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For $w = 5 \cdot 7 \cdot k = 35k$, $k \in \mathbb{Z}$, we get different integers u and v.

$\therefore \exists \infty$ radical elements for (S, T).

THANKS!