On the Interval Property in algebra and combinatorics

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References

Many colleagues to acknowledge. Discussion will involve:

- FZ: Interval Conjectures for level Hilbert functions, J. Algebra **321** (2009), no. 10, 2705-2715;
- M. Boij, J. Migliore, R. Mirò-Roig, U. Nagel, FZ: On the shape of a pure O-sequence, Memoirs AMS, to appear (arXiv:1003.3825);
- T. Hà, E. Stokes, FZ: Pure O-sequences and matroid h-vectors, preprint (arXiv:1006.0325);
- R. Stanley, FZ: On the rank function of a differential poset, in preparation.

Statement

We say that a class S of (possibly finite) integer sequences has the Interval Property if, when $h, h' \in S$ coincide in all entries but one, say

$$h = (h_0, \dots, h_{i-1}, h_i, h_{i+1}, \dots)$$

and

$$h' = (h_0, \dots, h_{i-1}, h_i + \alpha, h_{i+1}, \dots)$$

for some $\alpha \geq 1$, then

$$(h_0, \dots, h_{i-1}, h_i + \beta, h_{i+1}, \dots)$$

is also in S, for all $\beta = 1, 2, \dots, \alpha - 1$.

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A few well-known examples

Famous classes of sequences coming from graded algebra or combinatorics, where the Interval Property is known to hold or fail, include:

- Hilbert functions of standard graded algebras: holds (Macaulay's theorem).
- f-vectors of simplicial complexes: holds (the Kruskal-Katona theorem).
- h-vectors of Cohen-Macaulay (or shellable) simplicial complexes: holds (Stanley's theorem).
- Matroid h-vectors: fails. Many examples: e.g. $(1, 4, 4)$ and $(1, 4, 6)$ are matroid h-vectors, $(1, 4, 5)$ is not.

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Level and Gorenstein algebras

- Interval Property first conjectured in combinatorial commutative algebra (FZ, J. Algebra, 2009), for the sets of Hilbert functions of graded artinian level and (in an obviously symmetric fashion) Gorenstein algebras.
- All known techniques (algebraic, combinatorial, homological) to study level/Gorenstein Hilbert functions seem to point in the direction of the Interval Property.
- A characterization of such Hilbert functions seems hopeless, and the Interval Property would provide a very natural, strong structural result.
- Some timid progress, but Property still wide open.

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- $A = R/I$ is a standard graded artinian algebra if $R = k[x_1, \ldots, x_r]$, $I \subset R$ is a homogeneous ideal, $\sqrt{I} = (x_1, \ldots, x_r)$, $\deg(x_i) = 1$.
- The Hilbert function of A is $h(A) = (1, h_1, \ldots, h_e)$, with $h_i = \dim_k A_i$ (suppose $h_e \neq 0$, $h_{e+1} = 0$).
- The Socle of A is $\mathrm{soc}(A) = 0 : (\overline{x_1}, \ldots, \overline{x_r}) \subset A$.
- The Socle-vector of A is $s(A) = (0, s_1, \ldots, s_e)$, where $s_i = \dim_k \mathrm{soc}(A)_i$.
- A is level (of type t) if $s(A) = (0, \ldots, 0, t)$.
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Example

- **Example 1.**

$$A = k[x, y, z]/(x^3, y^3, z^3, xyz)$$

has Hilbert function $h = (1, 3, 6, 6, 3)$ and socle-vector $(0, 0, 0, 0, 3)$. Hence A is level of type 3.

- **Example 2.**

$$A = \mathbb{C}[x, y, z]/(z^3, 2x^2z + 3yz^2, 2x^2y + 3y^2z, y^3, x^4)$$

has Hilbert function $h = (1, 3, 6, 6, 3, 1)$ and socle-vector $(0, 0, 0, 0, 0, 1)$. Hence A is Gorenstein (level of type 1).

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Pure O-sequences

- Monomial order ideal: a nonempty, finite set X of (monic) monomials, such that whenever $M \in X$ and N divides M, then $N \in X$. (It's a special case of an order ideal in a poset.)
- The h-vector of X is its degree vector, $h = (1, h_1, \ldots, h_e)$, counting the monomials of X in each degree.
- A pure O-sequence is the h-vector of a monomial order ideal X whose maximal (by divisibility) monomials all have the same degree.

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- The pure monomial order ideal (inside $k[x,y,z]$) with maximal monomials xy^3z and x^2z^3 is:

$$X = \{\mathbf{xy^3z}, \mathbf{x^2z^3}; \quad y^3z, xy^2z, xy^3, xz^3, x^2z^2;$$

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- Equivalently, via Macaulay's inverse systems, pure O-sequences coincide with Hilbert functions of artinian *monomial* level algebras.
- Also for pure O-sequences, the Interval Property in general is open. However, very recent counterexample of M. Varbaro and A. Constantinescu in 4 variables! Conjecture confirmed in some special cases, including for socle degree $e \leq 3$ (as from the Memoir BMMNZ).
- A helpful application: it lead to a different approach to Stanley's matroid h-vector conjecture, and a solution in rank ≤ 3 (preprint with T. Hà and E. Stokes).

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Pure f-vectors

- A collection Δ of subsets of $V = \{v_1, \ldots, v_n\}$ is a simplicial complex if, for each $F \in \Delta$ and $G \subseteq F$, we have $G \in \Delta$. (Again, another order ideal!)
- The elements of Δ are faces, and the maximal faces (under inclusion) are facets.
- The f-vector of Δ is the vector $f(\Delta) = (1, f_0, \ldots, f_{d-1})$, where f_i is the number of cardinality $i + 1$ faces of Δ.
- A pure f-vector is the f-vector of a simplicial complex having all facets of the same cardinality.
- Equivalently, a pure f-vector is a *squarefree* pure O-sequence.
- Precious little is known about the Interval Property for pure f-vectors.

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- A collection Δ of subsets of $V = \{v_1, \ldots, v_n\}$ is a simplicial complex if, for each $F \in \Delta$ and $G \subseteq F$, we have $G \in \Delta$. (Again, another order ideal!)
- The elements of Δ are faces, and the maximal faces (under inclusion) are facets.
- The f-vector of Δ is the vector $f(\Delta) = (1, f_0, \ldots, f_{d-1})$, where f_i is the number of cardinality $i + 1$ faces of Δ.
- A pure f-vector is the f-vector of a simplicial complex having all facets of the same cardinality.
- Equivalently, a pure f-vector is a *squarefree* pure O-sequence.
- Precious little is known about the Interval Property for pure f-vectors.

Differential posets

- What about the Interval Property properly in enumerative combinatorics?

  We just recently looked at the rank functions of r-differential posets, which also appear hopeless to characterize (work in preparation with R. Stanley).
- Fix $r \geq 1$, and let $P = \cup_{n \geq 0} P_n$ be a locally finite, graded poset with a unique element of rank zero. P is r-differential if:
  - **(i)** two distinct elements $x, y \in P$ cover k common elements if and only they are covered by k common elements; and
  - **(ii)** $x \in P$ covers m elements if and only if it is covered by $m + r$ elements.
- The rank function of P is

$$1, p_1 = r, p_2, \ldots, p_n, \ldots,$$

where $p_n = |P_n|$.

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- Famous examples include the Young lattice of integer partitions and the Fibonacci r-differential posets.
- The Interval Property fails in general:
  $1, 4, 14, p_3 = 60, p_4 = 254, p_5, p_6, \dots$ and
  $1, 4, 17, p_3 = 60, p_4 = 254, p_5, p_6, \dots,$ where
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- Do we have the Interval Property in some interesting subcases? Especially, for 1-differential posets?
- P. Byrnes (U. of Minnesota) has computed all rank functions of 1-differential posets up to rank nine. Such sequences seem to clearly suggest that, at least for the initial ranks, the Interval Property does hold.

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