Regularity of $(n-2)$-plane arrangements in $\mathbb{P}^n$ with a complete bipartite incidence graph

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Linear subspace arrangements

Let $\mathbb{k}$ be an algebraically closed field of characteristic 0. A **linear subspace** of $\mathbb{P}^n$ is a variety $V(I)$ where $I \subset R = \mathbb{k}[x_0, \ldots, x_n]$ is an ideal generated by linear forms. If a linear subspace has dimension d, then we may call it a d-**plane**.

Let $\mathcal{A}$ be an arrangement of linear subspaces. We define

$$V_{\mathcal{A}} = \bigcup_{X \in \mathcal{A}} X$$

$$I_{\mathcal{A}} = \bigcap_{X \in \mathcal{A}} I(X) = I(V_{\mathcal{A}})$$

Liaison Theory

Why study linear subspace arrangements?

Example

Let $\mathcal{A}$ be a 2-plane arrangement in $\mathbb{P}^4$. In this case, $V_{\mathcal{A}}$ is a surface. Given two hypersurfaces Y and Y', we may be able to construct a surface X which is **linked** to $V_{\mathcal{A}}$ by Y and Y', i.e., $V_{\mathcal{A}} \cup X = Y \cap Y'$.
The hope is to construct X satisfying certain invariants in the effort to classify surfaces in $\mathbb{P}^4$.
To determine these invariants, it will be helpful to know something about the cohomology of the ideal sheaf $\widetilde{I_{\mathcal{A}}}$.

Minimal graded free resolutions and graded Betti numbers

Given a graded ideal I, consider a **minimal graded free resolution**

$$0 \to F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} I \to 0$$

For each i, F_i is a **graded free module**, i.e., $F_i \cong \bigoplus_j R(d_j)$

Definition

The **graded Betti numbers** of I are

$$\beta_{i,j} = \# \text{ of copies of } R(-j) \text{ in } F_i$$

Betti tables

We may list all the graded Betti numbers of a minimal graded free resolution using a **Betti table**:

	0	1	$\cdots$	i	$\cdots$
$\vdots$					
1	$\beta_{0,1}$	$\beta_{1,2}$		$\beta_{i,i+1}$	
2	$\beta_{0,2}$	$\beta_{1,3}$		$\beta_{i,i+2}$	
$\vdots$					
j	$\beta_{0,j}$	$\beta_{1,j+1}$		$\beta_{i,i+j}$	
$\vdots$					

Regularity

Definition 1

The **(Castlenuovo-Mumford) regularity** of a graded ideal I is

$$\operatorname{reg} I = \max\{j : \beta_{i,i+j} \neq 0 \text{ for some } i\}$$

Note that this is the index of the last row of the Betti table.

Definition 2

$$\operatorname{reg} I = \min\{j : h^i(\mathbb{P}^n, \tilde{I}(j-i)) = 0 \ \forall i > 0\}$$

What is known

Theorem (Derksen, Sidman (2002))

If $\mathcal{A}$ is a linear subspace arrangement, then

$$\operatorname{reg} I_{\mathcal{A}} \leq |\mathcal{A}|$$

This bound is sharp. For example, an arrangement of d skew lines intersecting a line L (which is not in the arrangement) in d distinct points will have a regularity of d. The Betti table for one such example with $n = 3$, $d = 5$ is below.

	0	1
3	1	.
4	6	9
5	9	2

Incidence graphs

Suppose $\mathcal{A}$ is an $(n-2)$-plane arrangement in $\mathbb{P}^n$, $n \geq 3$. Note that if $X, Y \in A$ are distinct, then either $\operatorname{codim}(X \cap Y) = 3$ or $\operatorname{codim}(X \cap Y) = 4$.

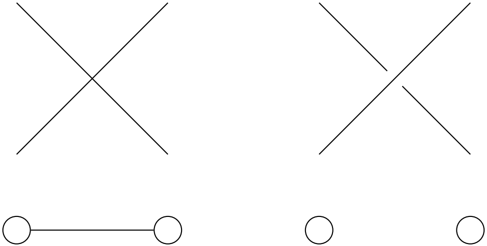
Definition

The **incidence graph** of $\mathcal{A}$ is the graph $\Gamma(\mathcal{A})$ such that

- $V(\Gamma(\mathcal{A})) = \mathcal{A}$
- $E(\Gamma(\mathcal{A})) = \{XY : \operatorname{codim}(X \cap Y) = 3\}$

Lines in $\mathbb{P}^3$

Two lines in $\mathbb{P}^3$ can either intersect in a point or not at all.



Note that the example we saw of d lines with regularity d above has incidence graph dK_1, i.e., no edges.
What happens to the regularity when we impose more structure?

Complete bipartite graphs

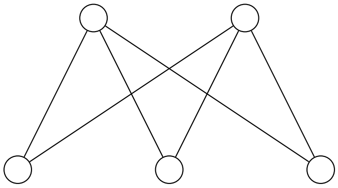
Definition

A graph $G = (V, E)$ is the **complete bipartite graph** $K_{a,b}$ if

- $V = V_1 \cup V_2$ where $|V_1| = a$ and $|V_2| = b$.
- If $u, v \in V_1$ or $u, v \in V_2$, then $uv \notin E$.
- If $u \in V_1$ and $v \in V_2$ or vice versa, then $uv \in E$.

Example

The complete bipartite graph $K_{2,3}$ is as follows:



Question

If $\Gamma(\mathcal{A}) = K_{a,b}$ with $a \leq b$, then what is $\operatorname{reg} I_{\mathcal{A}}$?

Example 1

Using Macaulay 2, we can construct $(n-2)$-plane arrangements with the desired incidence graphs.
If $n = 3$ and $\Gamma(\mathcal{A}) = K_{3,3}$, then $I_{\mathcal{A}}$ has the following Betti table:

	0	1
2	1	.
3	1	.
4	.	1

$$\operatorname{reg} I_{\mathcal{A}} = 4$$

Example 2

If $n = 3$ and $\Gamma(\mathcal{A}) = K_{5,10}$, then $I_{\mathcal{A}}$ has the following Betti table:

	0	1	2
2	1	.	.
3	.	.	.
4	.	.	.
5	.	.	.
6	.	.	.
7	.	.	.
8	.	.	.
9	.	.	.
10	6	10	4

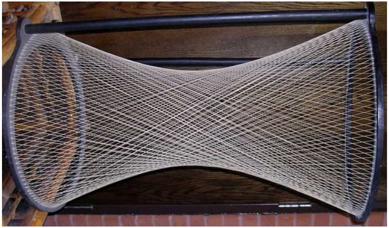
$$\operatorname{reg} I_{\mathcal{A}} = 10$$

The result

Theorem

If $\Gamma(\mathcal{A}) = K_{a,b}$ *with* $a \leq b$*, then* $\operatorname{reg} I_{\mathcal{A}} \leq \max\{a+1, b\}$*, with equality when* $a \geq 3$*.*

Sketch of proof. First, assume $n = 3$ and $a \geq 3$. Then $\mathcal{A}$ consists of rulings of a quadric surface Q.



Sketch of proof, cont.

The result follows from computing cohomologies using the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\cdot Q} \widetilde{I_{\mathcal{A}}} \to \mathcal{I}_{V_{\mathcal{A}} \cap Q, Q} \to 0$$

For $a < 3$, it can be shown that regularity is maximized when $V_{\mathcal{A}}$ lies on a quadric surface.
For $n > 3$, it can be shown that $V_{\mathcal{A}}$ is a cone over $V_{\mathcal{B}} \subset \mathbb{P}^{n-1}$ with $\Gamma(\mathcal{B}) = \Gamma(\mathcal{A})$, and the result follows inductively. □

Thank you!