Log-Concavity of Asymptotic Multigraded Hilbert Series

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arXiv:1109.4135

15 October 2011

Motivation

For a graded module M over a standard graded polynomial ring, the Hilbert series of the Veronese submodule $M^{(r)} := \bigoplus_{w \in \mathbb{Z}} M_{rw}$ has the form $\frac{F^{(r)}(t)}{(1-t)^n}$.

Beck-Stapledon (2010):

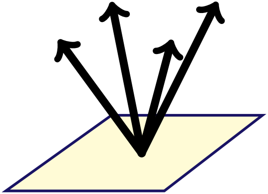
$$\lim_{r \to \infty} \frac{F^{(r)}(t)}{r^{n-1}} = \frac{F(1)}{(n-1)!} \sum \left\langle {n-1 \atop i} \right\rangle t^{i+1}$$

where the Eulerian number $\left\langle {n-1 \atop i} \right\rangle$ counts the permutations of $\{1, \ldots, n-1\}$ with i ascents.

QUESTION: What happens for other gradings?

Multivariate Power Series

Let $A := [\mathbf{a}_1 \cdots \mathbf{a}_n]$ be an integer $(d \times n)$-matrix of rank d such that the only non-negative vector in the kernel is the zero vector.



Equivalently, the rational function $1/\prod_j (1-\mathbf{t}^{\mathbf{a}_j})$ has a unique expansion as a power series.

Let Φ_r operate on $F(\mathbf{t}) \in \mathbb{Z}[\mathbf{t}^{\pm 1}]$ as follows:

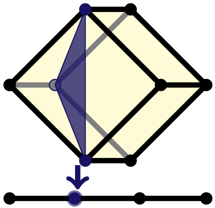
$$\frac{F(\mathbf{t})}{\prod_j (1-\mathbf{t}^{\mathbf{a}_j})} = \sum c_{\mathbf{w}} \mathbf{t}^{\mathbf{w}} \Rightarrow \sum c_{r\mathbf{w}} \mathbf{t}^{\mathbf{w}} = \frac{\Phi_r[F(\mathbf{t})]}{\prod_j (1-\mathbf{t}^{\mathbf{a}_j})}$$

Some Polyhedral Geometry

Let $\alpha:\mathbb{R}^n \to \mathbb{R}^d$ be the linear map determined by A.

The zonotope Z is $\alpha([0,1]^n)$.

For each $\mathbf{u} \in \mathbb{Z}^d$, we set
$P(\mathbf{u}) := \alpha^{-1}(\mathbf{u}) \cap [0,1]^n$.



We say that α is degenerate if there exists $\mathbf{u} \in \mathbb{Z}^d$ in the boundary of Z such that $\dim P(\mathbf{u}) = n-d$.

$\mathrm{vol}_{n-d} P(\mathbf{u})$ equals $(n-d)!$ times the volume of $P(\mathbf{u}) + \mathbf{x} \subseteq \alpha^{-1}(\mathbf{0})$ w/r/t the lattice $\alpha^{-1}(\mathbf{0}) \cap \mathbb{Z}^n$.

Description of the Limit

Let m be the gcd of the d-minors of A.

THEOREM (McCabe-Smith): If $F(\mathbf{t}) \in \mathbb{Z}[\mathbf{t}^{\pm 1}]$ and α is non-degenerate, then we have

$$\limsup_{r\to\infty} \frac{\Phi_r[F(\mathbf{t})]}{r^{n-d}} = \frac{F(\mathbf{1})}{(n-d)!} K_A(\mathbf{t})$$

where $K_A(\mathbf{t}) = \sum_{\mathbf{u}\in \mathrm{int}(Z)\cap\mathbb{Z}^d} \mathrm{vol}_{n-d}(P(\mathbf{u}))\,\mathbf{t}^{\mathbf{u}}$.

The coefficients of $K_A(\mathbf{t})$ are log-concave, quasi-concave, and sum to $m^{n-d}(n-d)!$.

If A is totally unimodular, then $K_A(\mathbf{t}) \in \mathbb{Z}[\mathbf{t}^{\pm 1}]$.

An Explicit Example

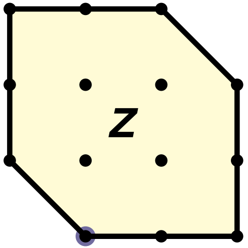
If $A = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ then we have $m = 1$ and

$$\Phi_r[1] = \binom{r-1}{3} t_1 t_2^2 + \left[2\binom{r+2}{3} + \binom{r+1}{2} - 2\binom{r}{1}\right] t_1 t_2 + \left[2\binom{r}{3} + \binom{r-1}{2}\right] t_2^2 + \left[\binom{r+2}{3} + \binom{r-1}{2} - 2\right] t_2 + \binom{r-1}{1} t_1 + 1$$

so $\displaystyle \lim_{r \to \infty} \frac{\Phi_r[1]}{r^3} = \frac{1}{3!}\left(t_1 t_2^2 + 2 t_1 t_2 + 2 t_2^2 + t_2\right).$

$$P(1,2) = \operatorname{conv} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$P(1,1) = \operatorname{conv} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Multigraded Hilbert Series

Let $S := \mathbb{C}[x_1, \ldots, x_n]$ have the grading induced by setting $\deg(x_j) := \mathbf{a}_j \in \mathbb{Z}^d$.

For a finitely generated $\mathbb{Z}^d$-graded S-module M, the Hilbert series has the form $\frac{F(\mathbf{t})}{\prod_j (1-\mathbf{t}^{\mathbf{a}_j})}$.

Applying Φ_r to $F(\mathbf{t})$ corresponds to computing the Hilbert series of the r-th Veronese submodule.

The Theorem implies that there exists a unique asymptotic numerator depending only on the multidegree of M and the matrix A.

Stochastic Matrices

By rescaling the matrix associated to the linear operator Φ_r, one obtains a stochastic matrix $C(r)$ with the following *amazing* properties:

- the stationary vector is $\dfrac{K_A(\mathbf{t})}{(n-d)!}$.
- the eigenvalues are r^{-j} for $0 \leq j \leq n-d$ with explicit eigenvectors independent of r.
- $C(r)\, C(s) = C(rs)$.

QUESTION: Do these matrices correspond to a known Markov chain?