Bounding projective dimension and regularity

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Introduction

Given an ideal $I \subset S = K[X_1, \ldots, X_n]$ there are two measures of the computational complexity of finding the resolution of S/I:

	0	1	2	3	4	5	6	7	...
0:	1	$\beta_{1,1}$	$\beta_{2,2}$	$\beta_{3,3}$	$\beta_{4,4}$	$\beta_{5,5}$	$\beta_{6,6}$	$\beta_{7,7}$	...
1:	-	$\beta_{1,2}$	$\beta_{2,3}$	$\beta_{3,4}$	$\beta_{4,5}$	$\beta_{5,6}$	$\beta_{6,7}$	$\beta_{7,8}$	...
2:	-	$\beta_{1,3}$	$\beta_{2,4}$	$\beta_{3,5}$	$\beta_{4,6}$	$\beta_{5,7}$	$\beta_{6,8}$	$\beta_{7,9}$	...
3:	-	$\beta_{1,4}$	$\beta_{2,5}$	$\beta_{3,6}$	$\beta_{4,7}$	$\beta_{5,8}$	$\beta_{6,9}$	$\beta_{7,10}$	...
4:	-	$\beta_{1,5}$	$\beta_{2,6}$	$\beta_{3,7}$	$\beta_{4,8}$	$\beta_{5,9}$	$\beta_{6,10}$	$\beta_{7,11}$	...
5:	-	$\beta_{1,6}$	$\beta_{2,7}$	$\beta_{3,8}$	$\beta_{4,9}$	$\beta_{5,10}$	$\beta_{6,11}$	$\beta_{7,12}$	...
6:	-	$\beta_{1,7}$	$\beta_{2,8}$	$\beta_{3,9}$	$\beta_{4,10}$	$\beta_{5,11}$	$\beta_{6,12}$	$\beta_{7,13}$	...
7:	-	$\beta_{1,8}$	$\beta_{2,9}$	$\beta_{3,10}$	$\beta_{4,11}$	$\beta_{5,12}$	$\beta_{6,13}$	$\beta_{7,14}$	...
8:	-	$\beta_{1,9}$	$\beta_{2,10}$	$\beta_{3,11}$	$\beta_{4,12}$	$\beta_{5,13}$	$\beta_{6,14}$	$\beta_{7,15}$	...
...	-	...	...	...	...	...	...	...	...

- projective dimension="width" of the Betti table (last nonzero column);
- regularity= "height" of the Betti table (last non-zero row).

Stillman's Question

Question (Stillman)

Is there a bound, independent of n, on the projective dimension of ideals in $S = K[X_1, \ldots, X_n]$ which are generated by N homogeneous polynomials of given degrees $d_1, \ldots, d_N$?

Remark

Hilbert's Syzygy Theorem guarantees $pd(S/I) \leq n$, but we seek a bound independent of n.

Stillman's Question

Known cases:

- If $I = (m_1, \ldots, m_N)$ is a monomial ideal, then $pd(S/I) \leq N$ by the Taylor resolution. Note that N does not work in general.

- If $I = (f, g, h)$ with f, g, h quadrics, then $pd(S/I) \leq 4$ by Eisenbud-Huneke (unpublished). This bound is tight.

- If $I = (f, g, h)$ with f, g, h cubics, then $pd(S/I) \leq 36$ by Engheta. The tight bound in this case is likely to be 5.

A bound for ideals of quadrics

Two ideas in pursuing this question:

1. look at ideals generated in small degrees (quadrics, cubics)
2. limit the number of generators (three-generated ideals)

Theorem (Ananyan-Hochster, 2011)

Let $S = K[x_1, \ldots, x_n]$, let $F_1, \ldots, F_N$ be polynomials of degree at most 2 and $I = (F_1, \ldots, F_N)$. Then there is a function $C(N)$ such that I is contained in the K-subalgebra of S generated by a regular sequence of at most $C(N)$ forms of degree at most 2. Consequently the projective dimension of S/I is at most $C(N)$.

Remark

The asymptotic growth of $C(N)$ is of order $2N^{2N}$.

Three-generated ideals

Theorem (Burch-Kohn, 1968)

For any $n \in \mathbb{N}$, *there is a three-generated ideal* $I = (f, g, h)$ *in a polynomial ring* $S = K[x_1, \ldots, x_{2n}]$ *with* $pd(S/I) = n$.

Remark

Engheta computed the degrees of the three generators to be $n-2, n-2, 2n-2$

Three-generated ideals

Theorem (Bruns, 1976)

Any resolution is the resolution of a three-generated ideal.

Remark (Nguyen, Niu, Sanyal, Torrance, Witt, Zhang)

Note that degrees of the generators of the brunsification of an ideal grow, but can be controlled. e.g. brunsification of $(X_1^d, \ldots, X_n^d)$ *yields three generators of degree at most* $d(n-2)^2$.

Y. Zhang's Question

Question (Y. Zhang)

Assume $I = (f_1, \ldots, f_N)$ *is an ideal of* $S = K[X_1, \ldots, X_n]$. *Is it true that* $pd(S/I) \leq \sum_{i=0}^{N} deg\ f_i$?

The following constructions show this bound is (much) too small.

McCullough's ideals with large projective dimension

Fix integers m, n, d such that $m \geq 1, n \geq 0$ and $d \geq 2$.

Let $Z_1, \ldots, Z_p$ be the $\frac{(m+d-2)!}{(m-1)!(d-1)!}$ monomials of degree $d-1$ in $X_1, \ldots, X_m$.

Example

$$S = K[X_1, \ldots, X_n, Y_{1,1}, \ldots, Y_{p,n}]$$

$$I_{m,n,d} = \left(X_1^d, \ldots, X_n^d, \sum_{i=0}^{p} Z_j Y_{j,1}, \ldots, \sum_{j=0}^{p} Z_j Y_{j,n} \right)$$

is generated by $m+n$ degree d generators

Large projective dimension

Theorem (McCullough,2011)

$$pd(R/I_{m,n,d}) = m + np = m + n\frac{(m+d-2)!}{(m-1)!(d-1)!}.$$

Proof sketch:

Show $depth(R/I_{m,n,d}) = 0$ and apply Auslander-Buchsbaum.

Example: $I_{3,4,2}$

Example

$$S = K[X_1, \dots, X_m, Y_{1,1}, \dots, Y_{3,4}]$$

$$I = \big(X_1^2, X_2^2, X_3^2, X_1Y_{1,1} + X_2Y_{2,1} + X_3Y_{3,1}, X_1Y_{1,2} + X_2Y_{2,2} + X_3Y_{3,2},$$
$$X_1Y_{1,3} + X_2Y_{2,3} + X_3Y_{3,3}, X_1Y_{1,4} + X_2Y_{2,4} + X_3Y_{3,4}\big)$$

I has 7 quadratic generators and $pd(S/I) = \#$ variables $= 15 > 7 \cdot 2$.

The answer to Zheng's question is negative.

A new family

Example (The ideal $I = I_{2,(2,2,2)}$)

$$\mathcal{A}_0 = \{\left(\begin{smallmatrix} 0&0&0 \\ 0&0&0 \end{smallmatrix}\right)\}, \mathcal{A}_1 = \{\left(\begin{smallmatrix} 1&0&0 \\ 1&0&0 \end{smallmatrix}\right)\}, \mathcal{A}_2 = \{\left(\begin{smallmatrix} 1&1&0 \\ 1&1&0 \end{smallmatrix}\right)\},$$
$$\mathcal{A}_3 = \{\left(\begin{smallmatrix} 1&1&2 \\ 1&1&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&1&1 \\ 1&1&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&1&0 \\ 1&1&2 \end{smallmatrix}\right)\}.$$

$$\begin{aligned} f &= X^{\left(\begin{smallmatrix} 0&0&0 \\ 0&0&0 \end{smallmatrix}\right)} x_{1,1}^2 x_{1,2}^5 + X^{\left(\begin{smallmatrix} 0&0&0 \\ 0&0&0 \end{smallmatrix}\right)} x_{2,1}^2 x_{2,2}^5 + X^{\left(\begin{smallmatrix} 1&0&0 \\ 1&0&0 \end{smallmatrix}\right)} x_{1,2}^2 x_{1,3}^3 + X^{\left(\begin{smallmatrix} 1&0&0 \\ 1&0&0 \end{smallmatrix}\right)} x_{2,2}^2 x_{2,3}^3 \\ &\quad + X^{\left(\begin{smallmatrix} 1&1&2 \\ 1&1&0 \end{smallmatrix}\right)} y_{\left(\begin{smallmatrix} 1&1&2 \\ 1&1&0 \end{smallmatrix}\right)} + X^{\left(\begin{smallmatrix} 1&1&1 \\ 1&1&1 \end{smallmatrix}\right)} y_{\left(\begin{smallmatrix} 1&1&1 \\ 1&1&1 \end{smallmatrix}\right)} + X^{\left(\begin{smallmatrix} 1&1&0 \\ 1&1&2 \end{smallmatrix}\right)} y_{\left(\begin{smallmatrix} 1&1&0 \\ 1&1&2 \end{smallmatrix}\right)} \\ &= x_{1,1}^2 x_{1,2}^5 + x_{2,1}^2 x_{2,2}^5 + x_{1,1}x_{2,1}x_{1,2}^2 x_{1,3}^3 + x_{1,1}x_{2,1}x_{2,2}^2 x_{2,3}^3 \\ &\quad + x_{1,1}x_{2,1}x_{1,2}x_{2,2}x_{1,3}^2 y_{\left(\begin{smallmatrix} 1&1&2 \\ 1&1&0 \end{smallmatrix}\right)} + x_{1,1}x_{2,1}x_{1,2}x_{2,2}x_{1,3}x_{2,3} y_{\left(\begin{smallmatrix} 1&1&1 \\ 1&1&1 \end{smallmatrix}\right)} \\ &\quad + x_{1,1}x_{2,1}x_{1,2}x_{2,2}x_{2,3}^2 y_{\left(\begin{smallmatrix} 1&1&0 \\ 1&1&2 \end{smallmatrix}\right)}. \end{aligned}$$

Finally, the ideal $I = \left(x_{1,1}^7, x_{2,1}^7, f\right)$.

Larger projective dimension

Fix $g \geq 2, m_n \geq 0, m_{n-1} \geq 1, m_i \geq 2$ for $1 \leq i \leq n-2$.

$$I = I_{g,(m_1,\ldots,m_{n-1})}$$

Theorem (Beder, McCullough, Nuñez, S-, Snapp, Stone)

$$pd(R/I) = \prod_{i=1}^{n-1} \left(\frac{(m_i + g - 1)!}{(g-1)!(m_i)!} - g \right) \left(\frac{(m_n + g - 1)!}{(g-1)!(m_n)!} \right) + gn.$$

Proof: Count the variables: $g \times n$ X variables and $|\mathcal{A}_n|$ Y variables.

Corollary (Beder,McCullough,Nuñez,S-, Snapp,Stone)

Over any field K and for any positive integer p, there exists an ideal I in a polynomial ring R over K with three homogeneous generators in degree p^2 such that $pd(R/I) \geq p^{p-1}$.

Proof:

$$I = I_{2,(\underbrace{p+1,\dots,p+1}_{p-1 \text{ times}},0)}.$$

Corollary (Beder,McCullough,Nuñez,S-,Snapp,Stone)

Over any field K and for any positive integer p, there exists an ideal I in a polynomial ring R over K with $2p+1$ homogeneous generators in degree $2p+1$ such that $pd(R/I) \geq p^{2p}$.

Proof:

$$I = I_{2p,(\underbrace{2,2,2,\dots,2}_{p \text{ times}})}.$$

$I = I_{2,(4,1)}$

Betti Table:

	0	1	2	3	4	5	6	7	8	9	10
total:	1	3	138	621	1303	1642	1352	740	261	54	5
0:	1	-	-	-	-	-	-	-	-	-	-
1:	-	-	-	-	-	-	-	-	-	-	-
2:	-	-	-	-	-	-	-	-	-	-	-
3:	-	-	-	-	-	-	-	-	-	-	-
4:	-	-	-	-	-	-	-	-	-	-	-
5:	-	3	-	-	-	-	-	-	-	-	-
6:	-	-	-	-	-	-	-	-	-	-	-
7:	-	-	-	-	-	-	-	-	-	-	-
8:	-	-	-	-	-	-	-	-	-	-	-
9:	-	-	-	-	-	-	-	-	-	-	-
10:	-	-	3	-	-	-	-	-	-	-	-
11:	-	-	4	5	-	-	-	-	-	-	-
12:	-	-	26	110	213	256	211	120	45	10	1
13:	-	-	96	480	1064	1376	1140	620	216	44	4
14:	-	-	9	26	26	10	1	-	-	-	-

$I = I_{2,(2,1,2)}$

Betti Table:

	0	1	2	3	4	5	6
total:	1	3	75	247	320	188	42
0:	1	-	-	-	-	-	-
1:	-	-	-	-	-	-	-
2:	-	-	-	-	-	-	-
3:	-	-	-	-	-	-	-
4:	-	-	-	-	-	-	-
5:	-	3	-	-	-	-	-
6:	-	-	-	-	-	-	-
7:	-	-	-	-	-	-	-
8:	-	-	-	-	-	-	-
9:	-	-	-	-	-	-	-
10:	-	-	3	-	-	-	-
11:	-	-	-	-	-	-	-
12:	-	-	-	-	-	-	-
13:	-	-	2	3	-	-	-
14:	-	-	-	-	-	-	-
15:	-	-	-	-	-	-	-
16:	-	-	3	6	3	-	-
17:	-	-	-	-	-	-	-
18:	-	-	1	4	5	2	-
19:	-	-	4	8	4	-	-
20:	-	-	1	4	6	4	1
21:	-	-	2	8	10	4	-
22:	-	-	6	14	11	4	1
23:	-	-	2	8	12	8	2
24:	-	-	4	16	21	10	1
25:	-	-	8	20	18	8	2
26:	-	-	3	12	18	12	3
27:	-	-	6	24	32	16	2
28:	-	-	3	12	18	12	3
29:	-	-	4	16	24	16	4
30:	-	-	3	12	18	12	3
31:	-	-	4	16	24	16	4
32:	-	-	1	4	6	4	1
33:	-	-	4	16	24	16	4
34:	-	-	1	4	6	4	1
35:	-	-	2	8	12	8	2
36:	-	-	1	4	6	4	1
37:	-	-	2	8	12	8	2
38:	-	-	1	4	6	4	1
39:	-	-	2	8	12	8	2
40:	-	-	-	-	-	-	-
41:	-	-	2	8	12	8	2

Stilman's Question - Regularity Version

Question (Stillman)

Is there a bound, independent of n, on the regularity of ideals in $S = K[X_1, \ldots, X_n]$ which are generated by N homogeneous polynomials of given degrees $d_1, \ldots, d_N$?

Caviglia proved:

the regularity question $\Leftrightarrow$ the projective dimension question.

Caution: this does not mean the bounds will be the same.

Caviglia's subfamily

Let

$$C_d = (w^d, x^d, wy^{d-1} + xz^{d-1}) \subset S = K[w, x, y, z]$$

Caviglia showed $reg(S/C_d) = d^2 - 1$.

C_d is a subfamily of the new family: $C_d = I_{2,(1,d-2)}$

Question

What is the asymptotic growth of $reg(I_{2,(2,1,d)})$*?*

Conjectures

Let

$$C_d = (w^d, x^d, wy^{d-1} + xz^{d-1}) \subset S = K[w, x, y, z]$$

Caviglia showed $reg(S/C_d) = d^2 - 1$.

C_d is a subfamily of the new family: $C_d = I_{2,(1,d-2)}$

Conjecture

We believe $reg(I_{2,(2,1,d)})$ exhibits cubic growth in d.

We believe $reg(I_{2,(\underbrace{2,2,2,\ldots,2}_{p\ times},1,d)})$ grows asymptotically as d^{p+2}.

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