Projective Monomial Curves in $\mathbb{P}^3$

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with suggestions from Dilip Patil and Les Reid.

Reference: D.P. Patil P. Li and L. Roberts, Bases and ideal generators for projective monomial curves. (On my web page).

Projective Monomial Curves in $\mathbb{P}^3$

$$\mathscr{S} = \{a, b, d\}, 0 < a < b < d, \quad \gcd(a, b, d) = 1.$$

Let S be the affine semigroup generated by

$$\boldsymbol{\alpha}_0 = (d, 0), \boldsymbol{\alpha}_1 = (d-a, a), \boldsymbol{\alpha}_2 = (d-b, b), \boldsymbol{\alpha}_3 = (0, d).$$

- $K[S] \cong K[s^d, s^{d-a}t^a, s^{d-b}t^b, t^d]$
- $C = \mathrm{Proj}(K[S])$ is a projective monomial curve embedded in $\mathbb{P}^3_K$ with homogeneous coordinate ring $K[S]$.
- $\{s^d, t^d\}$ is a system of parameters for $K[S]$, and $K[S]$ is Cohen-Macaulay if and only if $\{s^d, t^d\}$ is a regular sequence on $K[S]$ if and only if t^d is a non-zero-divisor in $K[S]/s^d K[S]$. Informally we will say that C (or $\mathscr{S}$) is Cohen-Macaulay if $K[S]$ is Cohen-Macaulay.

Let $R = K[X_0, X_1, X_2, X_3]$. Define $\varphi : R \to K[s,t]$ by $\varphi(X_0) = s^d,\ \varphi(X_1) = s^{d-a}t^a,\ \varphi(X_2) = s^{d-b}t^b,\ \varphi(X_3) = t^d$, so that $K[S] \cong R/\mathfrak{p}$.

Gradings: The rings R and $K[S]$ are graded by

1. S- (or $\mathbb{N}^2$-) grading, $\deg_S(X_i) = \boldsymbol{\alpha}_i,\ 0 \le i \le 3$.
2. $\mathbb{N}$-grading, $\deg(X_i) = 1$.

The ideal $\mathfrak{p}$ has a minimal set $\mathscr{G}$ of pure binomial generators that are homogeneous in both the above gradings. The set $\mathscr{G}$ is not necessarily unique, but $|\mathscr{G}|$ is unique. In each S-degree there is at most one element of $\mathscr{G}$ (and the degrees in which $\mathscr{G}$ is non-empty are unique).

Finding minimal generators of $\mathfrak{p}$ is thus the same as finding the S-degrees in which generators occur, and informally such degrees will be referred to as the generators.

Motivating Problems:

1. What fraction of all projective monomial curves of degree d in $\mathbb{P}^3$ are Cohen-Macaulay.
2. How many minimal generators can $\mathfrak{p}$ have for a given d (both as an upper bound, and an asymptotic average, as $d \to \infty$)?

We study these questions by describing $\mathscr{G}$ in terms of lattice elements on the boundary of certain convex hulls. This leads to easy computer implementation, and is also a theoretical tool.

Where the lattices come from: Because they are homogeneous in both gradings, the elements of $\mathscr{G}$ are one of the following types (with two exceptions)

(1) $X_0^{a_0}X_3^{a_3} - X_1^{a_1}X_2^{a_2},\ a_i > 0$ "interior type one"

(2) $X_0^{a_0}X_2^{a_2} - X_1^{a_1}X_3^{a_3},\ a_i > 0$ "type two"

$\mathscr{L}_{ij}$ = sublattice of $\mathbb{Z}^2$ generated by $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_j,\ i < j$.

Generators (1) have S-degree

$a_0\boldsymbol{\alpha}_0 + a_3\boldsymbol{\alpha}_3 = a_1\boldsymbol{\alpha}_1 + a_2\boldsymbol{\alpha}_2 \in \mathscr{L}_{03} \cap \mathscr{L}_{12} =: \mathscr{L}$.

Generators (2) have S-degree

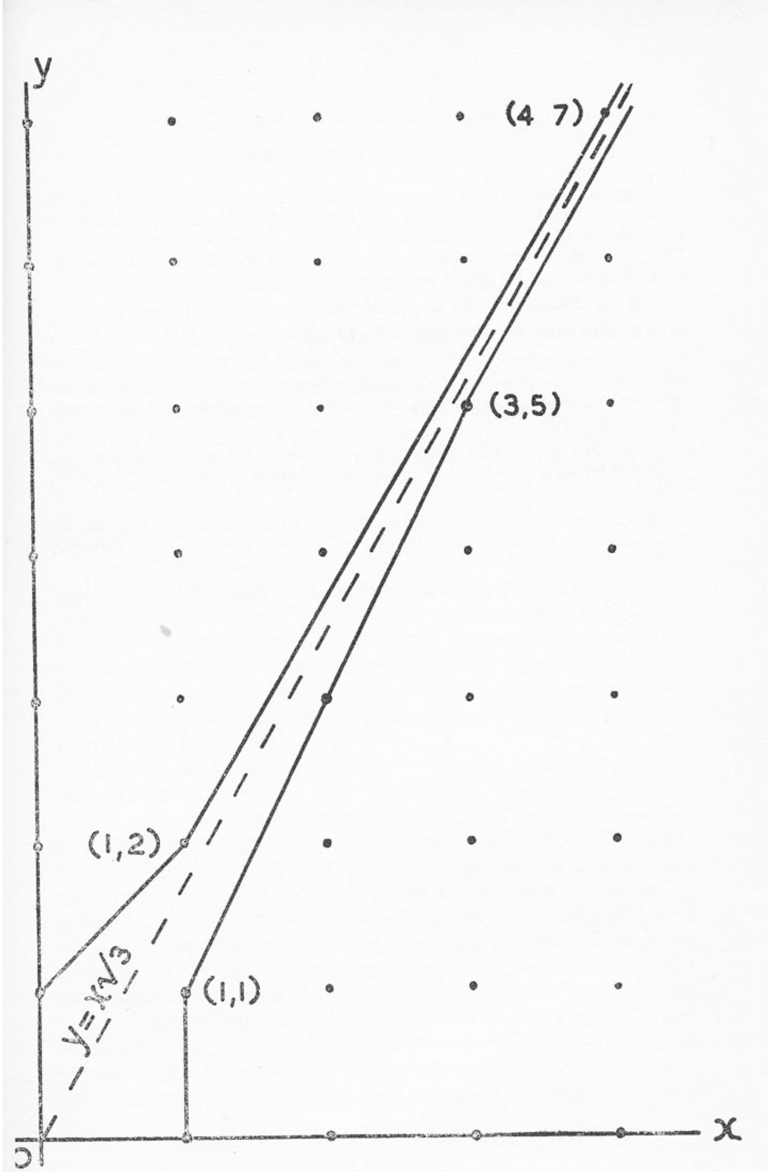
$a_0\boldsymbol{\alpha}_0 + a_2\boldsymbol{\alpha}_2 = a_1\boldsymbol{\alpha}_1 + a_3\boldsymbol{\alpha}_3 \in \mathscr{L}_{02} \cap \mathscr{L}_{13} =: \mathscr{L}'$.

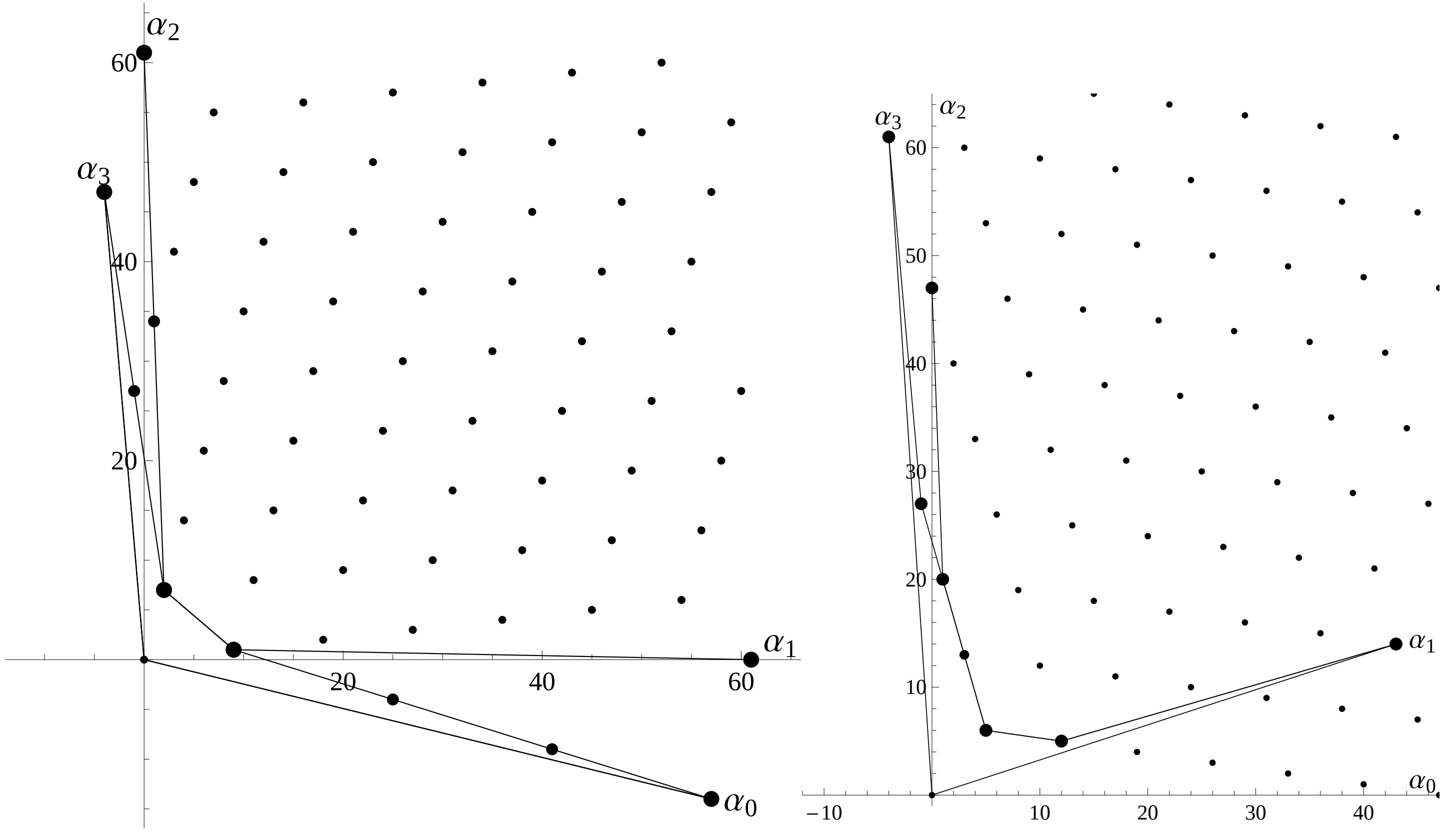
$$C_{ij} = \text{real cone spanned by } \boldsymbol{\alpha}_i \text{ and } \boldsymbol{\alpha}_j.$$

Graphical representation: Plot the S-degrees of the generators (1) in an $\boldsymbol{\alpha}_1$-$\boldsymbol{\alpha}_2$ coordinate system. The generators (1) are (some of the) elements of $\mathscr{L}$ on the boundary of the convex hull of $(\mathscr{L}\backslash\langle 0,0\rangle) \cap C_{12}$.

Plot the S-degrees of the generators (2) in an $\boldsymbol{\alpha}_0$-$\boldsymbol{\alpha}_2$ coordinate system. The generators (2) are (some of the) elements of $\mathscr{L}'$ on the boundary of the convex hull of $(\mathscr{L}'\backslash[\![0,0]\!]) \cap C_{12}$.

We know a basis of $\mathscr{L}$ (in $\boldsymbol{\alpha}_1$-$\boldsymbol{\alpha}_2$ coordinates) and a basis of $\mathscr{L}'$ (in the $\boldsymbol{\alpha}_0$-$\boldsymbol{\alpha}_2$ coordinates). By a suitable integer change of coordinates, finding $\mathscr{G}$ is reduced to finding the vertices of the convex hull of the non-zero integer points in the first quadrant on or below a line through the origin with rational slope r.


y
(4 7)
(3,5)
(1,2)
(1,1)
y=x√3
x



Diagrams for $\mathscr{S} = \{14, 57, 61\}$

$\langle 9, 1\rangle = 9\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = 7\boldsymbol{\alpha}_0 + 3\boldsymbol{\alpha}_3 \Rightarrow X_1^9 X_2 - X_0^7 X_3^3 \in \mathscr{G}$

$\langle -1, 27\rangle = -\boldsymbol{\alpha}_1 + 27\boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_0 + 25\boldsymbol{\alpha}_3 \Rightarrow X_2^{27} - X_0 X_1 X_3^{25} \in \mathscr{G}$

The last generator can be indentified with $\langle 0, 27\rangle$.

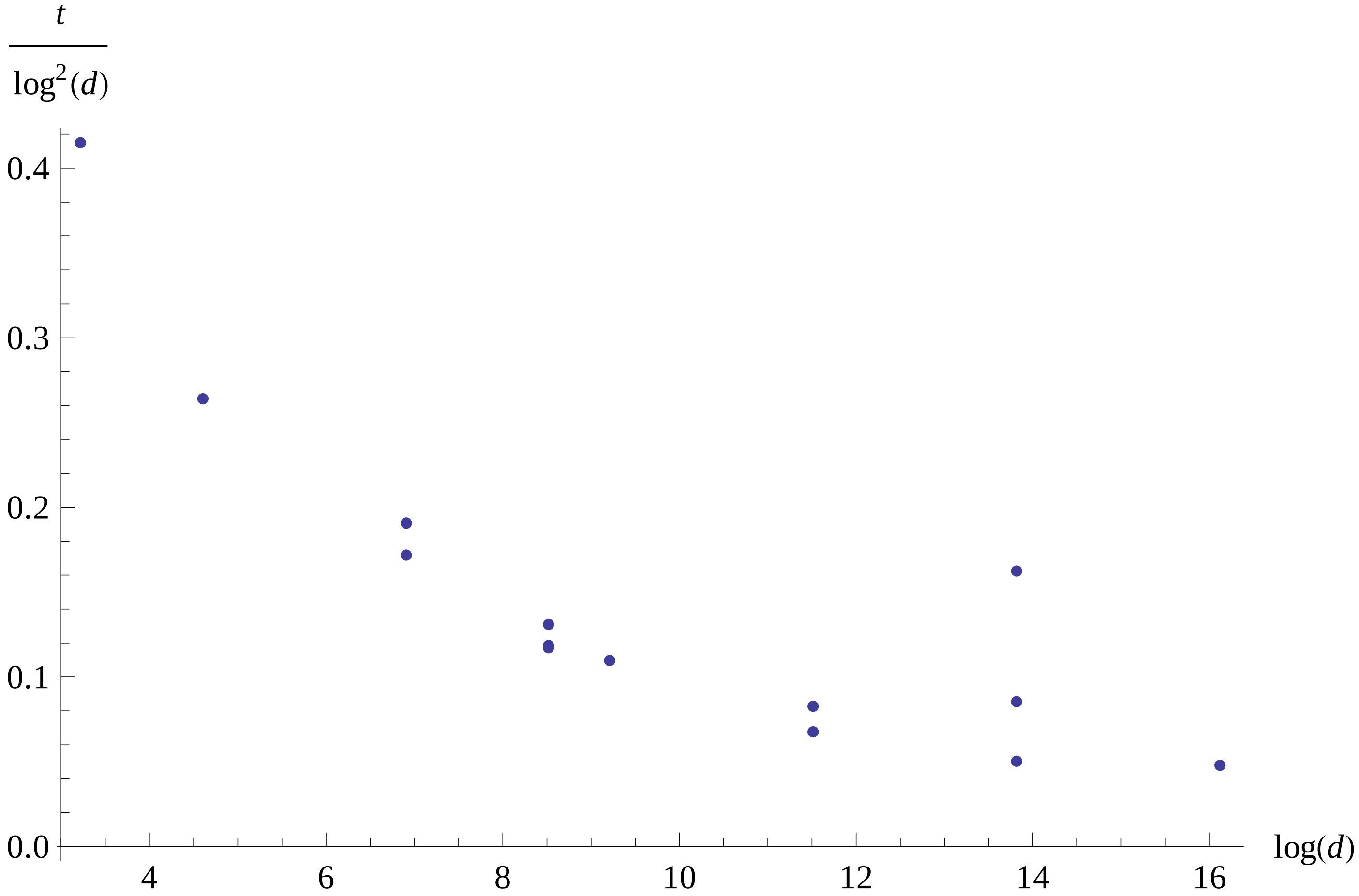
Similarly, form the right diagram $12\boldsymbol{\alpha}_0 + 5\boldsymbol{\alpha}_2 = 16\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_3$ so $X_0^{12} X_2^5 - X_1^{16} X_3 \in \mathscr{G}$. In total there are 8 generators.

Via the change of coordinates we end up with one rational number $r > 0$ with continued fraction expansion $r = \{q_0, q_1, \cdots, q_s\}$ such that the minimal generators of $\mathfrak{p}$ correspond to (some of the) integer points on the convex hulls of non-zero integer points above and below the line through the origin with slope r. The number of such integer points is $N = 2 + q_0 + \cdots q_s$ so $\mathfrak{p}$ has at most N minimal generators. The numerator and denominator of r are at most $d - 2$ from which it follows, using known properties of the quotients in the Euclidean algorithm applied to the numerator and denominator of r, that

- The average value of N is expected to grow linearly in $\log^2(d)$, so that $|\mathscr{G}|$ should have an upper bound linear in $\log^2(d)$.

- N is at most equal to d so $|\mathscr{G}|$ is at most d (realized only for $\mathscr{S} = \{1, d-1, d\}$).

- The maximum number of segments is $s + 2$ which has a bound linear in $\log(d)$. The expected average number of segments also has an upper bound linear in $\log(d)$.
- We do not know what fraction $|\mathscr{G}|$ is of N so we don't have a non-trivial lower bound on the average value of $|\mathscr{G}|$ or of the number of segments. The following plot suggests that the number of average number of generators may not be quite growing linearly in $\log(d)^2$


t / log^2(d)
0.4
0.3
0.2
0.1
0.0
4
6
8
10
12
14
16
log(d)

What fraction of all monomial curves in $\mathbb{P}^3$ are Cohen-Macaulay? First some motivational background.

- Bresinsky (1984) has observed that for fixed $\{a, b\}$, if d is sufficiently large then $\{a, b, d\}$ is Cohen-Macaulay.
- More generally I observed (1995), given any $\{a_1, a_2, \cdots, a_{p-1}\}$, if d is sufficiently large then $\mathscr{S} = \{a_1, a_2, \cdots, a_{p-1}, d\}$ is Cohen-Macaulay.

The above suggest that in some sense most monomial curves are Cohen-Macaulay. However Les Reid and I have proved (2005) that, for fixed d, the fraction of projective monomial curves of degree d (of any embedding dimension) that are *not* Cohen-Macaulay approaches 1.

So what about $\mathbb{P}^3$? Our computational evidence is convincing that, for fixed d, a positive fraction of monomial curves of degree d in $\mathbb{P}^3$ are not Cohen-Macaulay. As d increases, the fraction that are Cohen-Macaulay trends downwards, reaching about 45% for degree near 100, and dropping (on samples of curves) to about 30% when degree is 100,000. But for degree near 1,300,000 the fraction that are Cohen-Macaulay (in samples) is still about 30%. So perhaps the fraction that are Cohen-Macaulay has stabilized at about 30%.

Miscellaneous remarks:

- $\mathscr{S}$ is Cohen-Maculay if and only if $|\mathscr{G}| \leq 3$ (Bresinsky et al)
- If $b - a = \gcd(a, b)$ then $\{a, b, d\}$ is Cohen-Macaulay.
- Let $c = \gcd(a, b)$ and ℓ, h be such that $hb - \ell a = cd$ (h as small as possible non-negative integer). If $\ell - h + c \leq 0$ then $\{a, b, d\}$ is Cohen-Macaulay (however, for large d, $\ell - h + c$ is usually positive).

This *Mathematica* notebook contains code run during my talk, and explanations of the slides that I attempted to give verbally during the talk. The Patil-Li-R paper mentioned on slide 1 will be referred to as [LPR]. References not in [LPR] are available on request and may later be given on my web page.

Slide 7 is a page from Davenport's Higher Arithmetic. Let ℓ be a line through the origin with slope r>0. This picture indicates that the convex hull of integer points in the first quadrant on or below ℓ is given by the lower convergents of the real number r. Davenport was interested in irrational r, but that does not matter. For example, suppose that r=7/4. *Mathematica* conveniently gives the convergents of 7/4:

```
Convergents[7 / 4]
```

$$\left\{1,\ 2,\ \frac{7}{4}\right\}$$

The lower convergents of 7/4 are 1=1/1 and 7/4. The vertices of the convex hull of integer points below ℓ are thus (1,1) and (4,7), together with (1,0). Similarly the vertices of the convex hull of integer points in the first quadrant above ℓ are (1,2) (corresponding to the upper convergent 2=2/1), together with (0,1) and (4,7). The quotients in the continued fraction expansion of r give the number of subdivisions of the opposite segment.

```
ContinuedFraction[7 / 4]
```

```
{1, 1, 3}
```

For example, the quotient 3 corresponds to the three subsegments of (1,1)(4,7) given by the intermediate integer points (2,3) and (3,5).

```
FromContinuedFraction[{1, 1, 2}]
```

$$\frac{5}{3}$$

The last paragraph of slide 4 is based on Theorems 2.7 and 2.9 of [LPR], and illustrated on slide 8 for the curve {14,57,61}. The dots in the left figure are the elements of $\mathcal{L}$ in the first quadrant, plotted in $\alpha_1 - \alpha_2$ coordinates. The boundary of the convex hull of ($\mathcal{L}$\<0,0>)$\bigcap C_{12}$ has vertices <61,0>, <9,1>, <2,7>,<0,61>. These vertices can be found by the change of coordinates indicated in the last paragraph of slide 6, which we have programmed in *Mathematica*. In the right figure the dots are the elements of $\mathcal{L}$' in the first quadrant plotted in $\alpha_0 - \alpha_2$ coordinates. In the right figure the cone C_{12} is the cone generated by the lines labelled $\alpha_1$and α_2. The boundary of the convex hull of ($\mathcal{L}$'\[[0,0]])$\bigcap C_{12}$ has vertices [[43,14]],[[12,5]],[[5,6]],[[1,20]],[[0,47]] (computed by a similar integer change of coordinates).

From the figures on page 8 we see that the ideal of {14,57,61} has 8 minimal generators, of S-degrees <25,0>,<9,1>,<2,7>,<0,27> (left diagram), and [[12,5]],[[5,6]],[[3,13]],[[1,20]] (right diagram). These S-degrees are some of the elements of $\mathcal{L}$, respectively $\mathcal{L}$' on the above convex hull boundaries. The calculation in the bottom half of slide 8 shows how to convert one of these S-degrees, namely <9,1>, into an actual ideal generator. The element <-1,27> $\in\mathcal{L}\bigcap C_{23}$ also yields a minimal generator <0,27>, as also indicated on slide 8. This prevents the lattice points <1,39> and <0,61> from being minimal ideal generators, according to the above mentioned theorem from [LPR]. (We refer to this as "truncation"). Similarly <25,-5>$\in\mathcal{L}\bigcap C_{01}$yields a minimal generator of S-degree <25,0> and prevents <61,0> from being minimal (another type of truncation). <25,0> and <0,27> are the "two exceptions" referred to in slide 5.

```
In[1]:= << "C:\\Users\\Leslie Roberts\\Desktop\\Lincoln-11\\implementation.m"

In[2]:= cur = {14, 57, 61}

Out[2]= {14, 57, 61}
```

Because the minimal ideal generators are on the boundary of a convex hull, they occur on straight line segments. Our two basic programs are typeonevertices and typetwovertices.

```
In[3]:= typeonevertices[{14, 57, 61}]
Out[3]= {{{25, 0}, {9, 1}, {2, 7}, {0, 27}}, {1, 1, 1}}
```

The first coordinate of the result is the vertices of the segments, and the second coordinate is the number of subdivisions of the segments (by elements of $\mathcal{L}$). Here all the subdivisions are 1, which means that the vertices are the only type one generators (including the two exceptions on the axes).

The next few lines work out with *Mathematica* some of the calculations on the bottom of slide 8.

```
In[4]:= {α0, α1, α2, α3} = {{61, 0}, {61 - 14, 14}, {61 - 57, 57}, {0, 61}}
Out[4]= {{61, 0}, {47, 14}, {4, 57}, {0, 61}}
In[5]:= Solve[9 α1 + α2 == x α0 + y α3, {x, y}]
Out[5]= {{x → 7, y → 3}}
```

Therefore $9\,\alpha_1 + \alpha_2 == 7\,\alpha_0 + 3\,\alpha_3$ so X_1^9 X_2 - X_0^7 X_3^3 $\in \mathcal{G}$.

```
In[6]:= arcα2α3
Out[6]= {{0, 61}, {-1, 27}, {-4, 47}}
In[7]:= Solve[-α1 + 27 α2 == x α0 + y α3, {x, y}]
Out[7]= {{x → 1, y → 25}}
```

Therefore $-\alpha_1 + 27\,\alpha_2 == \alpha_0 + 25\,\alpha_3$ so X_2 ^27- X_0^7 X_1 X_3^3 $\in \mathcal{G}$.

The complete convex hull boundaries are not in the explicit output of the basic functions, but are left as global variables so one has access to them if desired. In the second coordinate of the output of the next line, the 2 indicates one intermediate element of $\mathcal{L}$ between <2,7> and <0,61>, namely <1,34>.

```
In[8]:= {arcα1α2, subdivisions12}
Out[8]= {{{61, 0}, {9, 1}, {2, 7}, {0, 61}}, {1, 1, 2}}
In[9]:= ({2, 7} + {0, 61}) / 2
Out[9]= {1, 34}
In[10]:= typetwovertices[{14, 57, 61}]
Out[10]= {{{12, 5}, {5, 6}, {1, 20}}, {1, 2}}
```

The above line gives the vertices of the above mentioned type two generators. The 2 in the second coordinate of the output indicates one intermediate generator between [[5,6]] and [[1,20]], namely [[3,13]]. Again the complete convex hull boundaries are left as global variables. The vertices [[43,14]] on the α_1 – ray and [[0, 47]] on the α_2 – axis never correspond to type 2 generators, again according to [LPR].

```
In[11]:= {arcα1α2x, subdivisionsx12}
Out[11]= {{{43, 14}, {12, 5}, {5, 6}, {1, 20}, {0, 47}}, {1, 1, 2, 1}}
```

The next line is the type two generator calculation at the bottom of slide 8

```
In[12]:= Solve[12 α0 + 5 α2 == x α1 + y α3, {x, y}]
Out[12]= {{x → 16, y → 1}}
```

The following example illustrates the assertion on slide 9 that the curve {1,d-1,d} has d ideal generators, the maximum possible for a curve of degree d.

In[13]:=
```
cur1 = {1, 999, 1000}
```
Out[13]=
```
{1, 999, 1000}
```
In[14]:=
```
typeonevertices[cur1]
```
Out[14]=
```
{{{999, 0}, {1, 1}, {0, 999}}, {1, 1}}
```
The above is the three type one generators.

In[15]:=
```
typetwovertices[cur1]
```
Out[15]=
```
{{{997, 2}, {1, 998}}, {996}}
```
The above indicates one segment of type two generators, with 996 subdivisions, i.e. 997 generators. We note that {997,2}-{1,998} is divisible by 996, with quotient {1,-1}. Below are a few more examples.

In[16]:=
```
cur2 = {10^15 - 1, 10^16 + 17, 10^18}
```
Out[16]=
```
{999 999 999 999 999, 10 000 000 000 000 017, 1 000 000 000 000 000 000}
```
In[17]:=
```
typeonevertices[cur2]
```
Out[17]=
```
{{{370 370 370 370 371, 0}, {17 000, 1000}, {0, 55 583 375 063 037}}, {1, 1}}
```
In[18]:=
```
typetwovertices[cur2]
```
Out[18]=
```
{{{333 333 333 315 361, 37 037 037 038 037}, {10 009, 55 583 375 062 037}}, {18 546 338 024}}
```
cur2 has 18546338025 type two generators on one segment, so it is important not to list them all, but instead to describe them by giving only the vertices. Note also that 18546338025 is a fairly small fraction of d.

In[19]:=
```
cur3 = {10^15 - 1 - 7^10, 10^16 + 17, 10^18};
```
In[20]:=
```
typeonevertices[cur3]
```
Out[20]=
```
{{{6 736 873 403, 0}, {6 361 619 297, 77 935 250},
  {357 553 601, 2 519 463 250}, {0, 2 672 058 750}}, {1, 16, 1}}
```
In[21]:=
```
typetwovertices[cur3]
```
Out[21]=
```
{{{223 809 307, 152 595 500}}, {}}
```
cur3 does not have so many generators, namely 19 type one and one type 2.

In[22]:=
```
cur4 = {10^30 - 1, 10^31 + 17 + 11^3, 10^32};
```
In[23]:=
```
typeonevertices[cur4]
```
Out[23]=
```
{{{14 727 540 500 736 377 025 036 824 412, 0},
  {67 400, 50}, {0, 1 482 679 606 415 959 024 127 240 881}}, {1, 1}}
```
In[24]:=
```
typetwovertices[cur4]
```
Out[24]=
```
{{{13 254 786 450 662 739 322 533 075 704, 1 472 754 050 073 637 702 503 681 931},
  {49 466, 1 482 679 606 415 959 024 127 240 831}}, {198 511 126 846 426 432 471 178}}
```
The last means one segment of type two generators, with 198511126846426432471178 subdivisions, i.e. 1+198511126846426432471178 type two generators.

In[25]:=
```
fib[n_] := Fibonacci[n]
```
For the next example I list only segments, of which there are quite a few, but none with a very large number of subdivisions, so the number of generators is fairly small compared to the degree. Note that the numbers of the two types of segments are about the same.

In[30]:=
```
fib[510] // N
```
Out[30]= 1.71479×10^{106}

In[26]:=
```
cur6 = {1 + fib[54], fib[500], 3 + fib[510]};
```

```
In[27]:= typeonevertices[cur6][[2]]

Out[27]= {1, 1, 1, 1, 1, 15, 3, 21, 1, 6, 2, 3, 2, 1, 1, 2, 2, 1, 3, 131, 2, 2, 1, 1, 3,
          3, 1, 4, 3, 1, 2, 2, 1, 1, 5, 2, 9, 1, 2, 3, 2, 1, 7, 6, 15, 1, 1, 3, 1, 2, 1, 1}

In[28]:= typetwovertices[cur6][[2]]

Out[28]= {5, 3, 4, 4, 3, 1, 12, 1, 6, 2, 2, 1, 5, 4, 2, 7, 1, 1, 9, 5, 2, 1, 1, 1,
          1, 1, 1, 2, 1, 5, 1, 47, 4, 4, 2, 2, 1, 9, 8, 18, 1, 1, 1, 2, 4, 1, 2, 1, 2, 4}

In[29]:= {Length[%%], Length[%]}

Out[29]= {52, 50}
```

Properties of continued fractions suggest that the average number of segments and the number of elements of $\mathcal{L}$ ($\mathcal{L}$') on the two convex hulls is not very large. This is sketched on slides 9 and 10. The suggested growth rates of log (d), respectively $\log^2(d)$, seem to be reasonably consistent with computer experiments, and we think we have actually proved this for d prime. The ideal generators are only some of the elements of $\mathcal{L}$ ($\mathcal{L}$') on these convex hulls so we have upper bounds on the average number of ideal generators, or segments of ideal generators. However the growth rate of the average number of ideal generators seems to be less than $\log^2(d)$, as indicated in the graph on slide 11. The graph in slide 11 is prepared using all projective monomial curves in $\mathbb{P}^3$ of degrees 25 and 100, and a "random" sample of 1000 curves of degrees 1000, 5000, 10000, 100000,1000000, and 10000000 (log is natural log). It is even possible that average number of segments of ideal generators of curves of degree d remains bounded as d→∞, but the average number of generators seems to keep growing as d increases, although fairly slowly.

One of our motivating problems is what fraction of all projective monomial curves of degree d in $\mathbb{P}^3$ is Cohen-Macaulay. Some observations on this are in slides 12,13,14. In addition to the observations on these slides we have proved that Cohen-Macaulay is equivalent to no type two generators.

Fun stuff, related to the observation on slide 14 that b-a=1 implies Cohen-Macaulay. It seems that the larger the difference between a and b the more likely the curve {a,b,d} is to be not Cohen-Macaulay. All curves of degree 101 are plotted as {a,b} in the diagram below, in red if Cohen-Macaulay and blue if not.

```
In[31]:= << "C:\\Users\\Leslie Roberts\\Desktop\\Lincoln-11\\convexhull3.m"

In[32]:= c101 = curves1[101];

In[33]:= cm101 = combCM[c101];
```

The above selects the Cohen-Macaulay curves of degree 101. Running time less than 10 seconds.

```
In[34]:= ncm101 = Complement[c101, cm101];

In[35]:= cm101 = Map[Drop[#, -1] &, cm101];

In[36]:= ncm101 = Map[Drop[#, -1] &, ncm101];
```

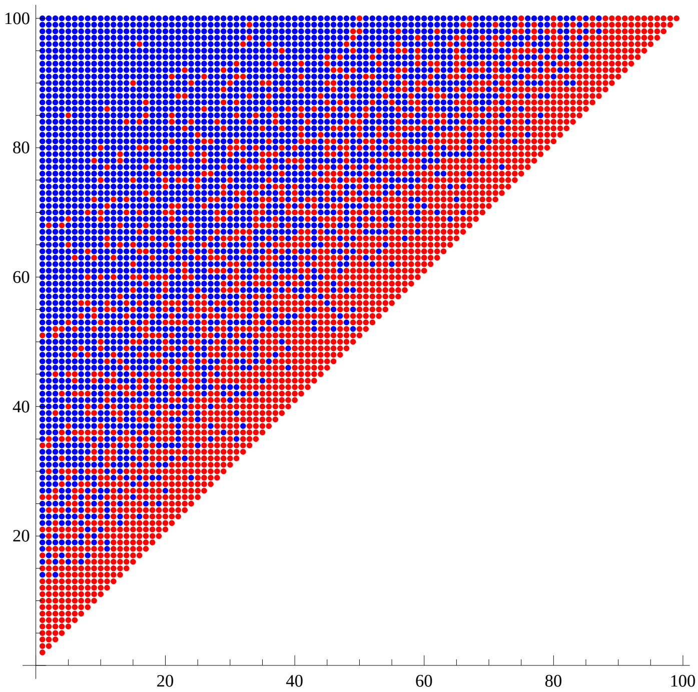
The previous two lines drop the last coordinate, which is always 101.

```
In[37]:= ListPlot[cm101, PlotStyle → RGBColor[1, 0, 0]];

In[38]:= ListPlot[ncm101, PlotStyle → RGBColor[0, 0, 1]];
```

In[39]:= `Show[%37, %38, AspectRatio → Automatic]`

Out[39]=



The above is a plot of {a,b}, for all curves {a,b,101}, red if Cohen-Macaulay, blue if not. In general we have proved Cohen-Macaulay for the bottom edge, and non-Cohen Macaulay for the upper left corner {1,100,101} and a few scattered general types elsewhere. The rest of the diagram is a mystery.