Cohen-Macaulay toric rings arising from finite graphs

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   - If H is an induced subraph of G such that $K[H]$ is not Cohen-Macualay, is the same true for $K[G]$?

The toric ring $K[G]$

$G = (V, E)$ a finite, connected and simple graph.

$$E(G) = \{x_1, \ldots, x_n\} \quad \longleftrightarrow \quad K[\mathbf{x}] = K[x_1, \ldots, x_n]$$
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Define the homomorphism

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- $I_G := \ker \pi$ is the **toric ideal of** G
- $K[G] := K[\mathbf{x}]/I_G$ is the **toric ring of** G

Facts about $K[G]$

- Hibi and Ohsugi (1999) showed $K[G]$ is normal exactly when G satisfies the *odd cycle property*, i.e. any two odd cycles C and C' in G must share a vertex or have an edge between them.

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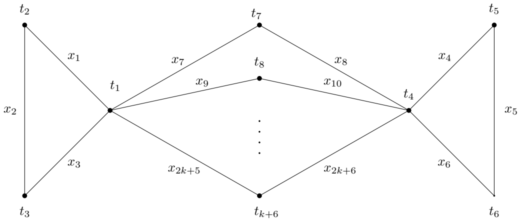
- Generators of I_G arise from even closed paths in G.

- By Auslander-Buchsbaum formula,

$$\operatorname{depth} K[G] = \dim_K K[G] - \operatorname{pd} K[G] = |E(G)| - \operatorname{pd} K[G]$$

A family of graphs

G_{k+6} :



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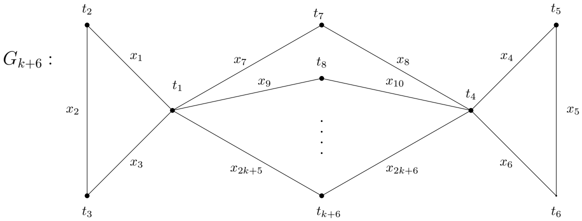

depth $K[G_{k+6}] = 7$ for $k \geq 1$.

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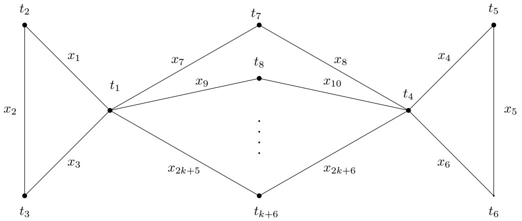

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Theorem (Hibi, Higashitani, Kimura, -)

Let f, d be integers such that $7 \leq f \leq d$. Then there exists a graph G with $|V(G)| = d$ such that $\dim K[G] = d$ and $\operatorname{depth} K[G] = f$.

Sketch of proof

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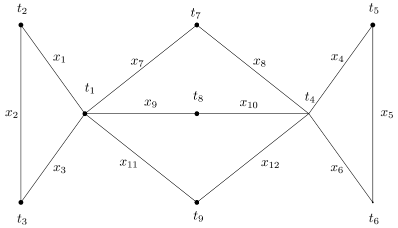
G_9 :



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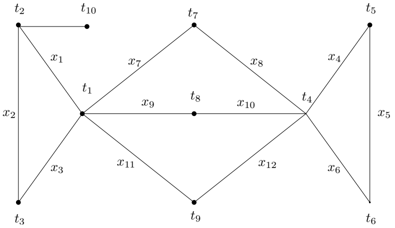

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- $\dim K[G] = 10$

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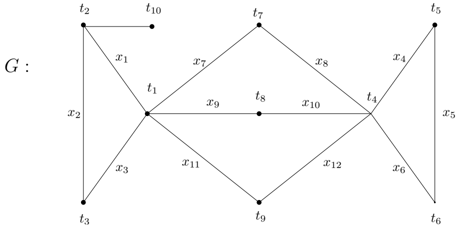

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- $\dim K[G] = 10$
- $\operatorname{pd} K[G] = \operatorname{pd} K[G_9] = |E(G_9)| - 7 = 5$

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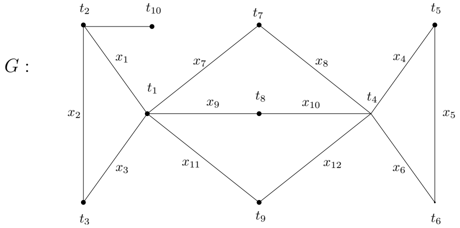

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- $\dim K[G] = 10$
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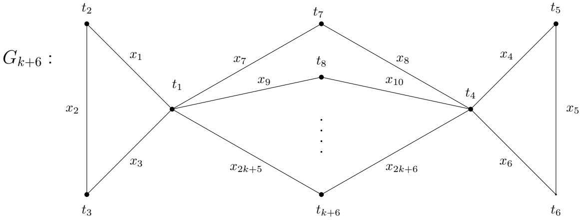
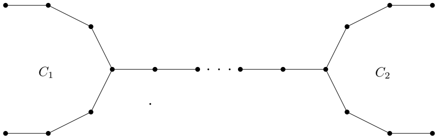
- $\operatorname{depth} K[G] = |E(G)| - 5 = 8$

Let's look again at the family of graphs.



- depth $K[G_{k+6}] = 7 \neq k + 6 = \dim K[G]$ for $k \geq 2$.
- What about these graphs prevent Cohen-Macaulayness?

Suppose G consists of 2 odd cycles, C_1 and C_2, connected by a path of length ≥ 2.



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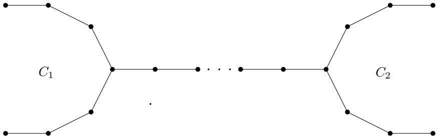

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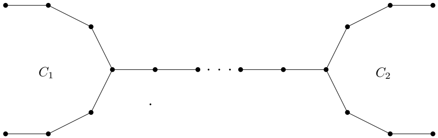

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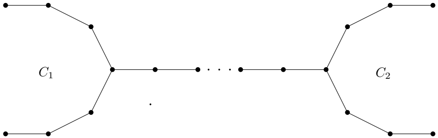

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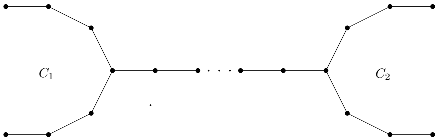

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- $K[G]$ is not normal but is Cohen-Macaulay.

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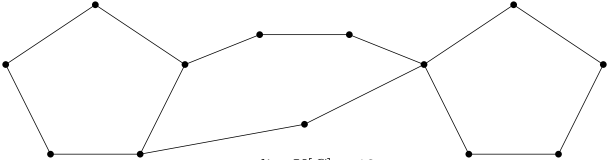

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$\operatorname{depth} K[G] = 12$

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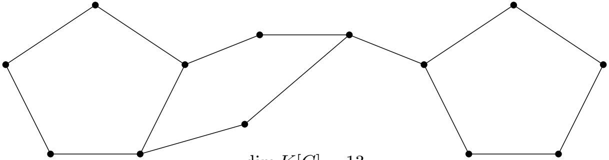

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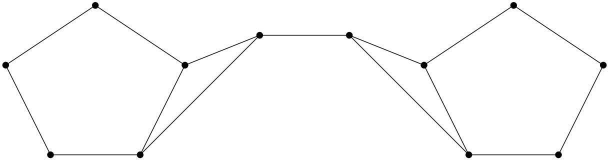

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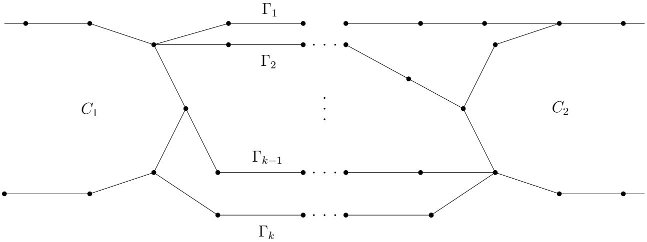
$K[G]$ is Cohen-Macaulay!

Theorem (Hà, -)

Suppose G is comprised of two odd cycles, C_1 and C_2, connected by $k \geq 2$ paths, $\Gamma_1, \ldots, \Gamma_k$, each of length $l_i \geq 2$, for $i = 1, \ldots, k$. Furthermore, suppose that the Γ_i do not share any edges and if a vertex $v \in V(\Gamma_i) \cap V(\Gamma_j)$, then $v \in V(C_1) \cup V(C_2)$. Then $K[G]$ is not Cohen-Macaulay.



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For $\mathbf{s} \in S_G$, define the simplicial complex

$$\Delta_{\mathbf{s}} = \{F \subset [n] : \mathbf{s} - \mathbf{n}_F \in S_G\}, \quad \mathbf{n}_F = \sum_{i \in F} a_i$$

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Briales, Campillo, Marijuán and Pisón (1998) showed that

$$\beta_{j+1,\mathbf{s}}(K[G]) = \dim_K \tilde{H}_j(\Delta_{\mathbf{s}}; K).$$

$\Delta_{\mathbf{s}}$ in terms of G

Choose $\mathbf{s} = (1 + p_i \;:\; p_i = \text{number of paths } t_i \text{ lies on})$.
Then each path determines 2 facets in $\Delta_{\mathbf{s}}$.

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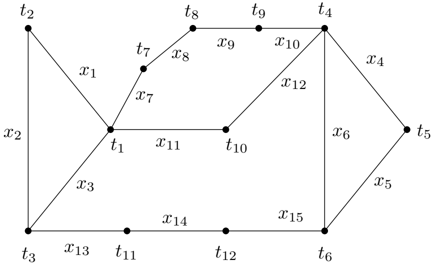

$$\Delta_1 : \begin{cases} F_{1,1} = \{2, 4, 6, 7, 9, 11, 12, 13, 14, 15\} \\ F_{1,2} = \{1, 3, 5, 8, 10, 11, 12, 13, 14, 15\} \end{cases}$$

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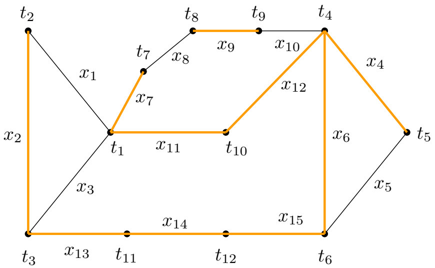

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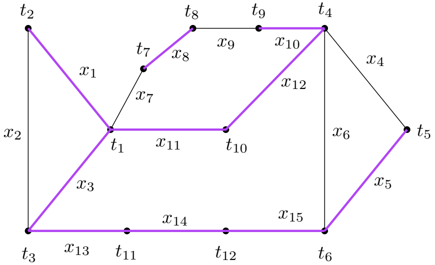

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- Can express $\Delta_{\mathbf{s}}$ as a union of subcomplexes.
- Apply Mayer-Vietoris recursively to get

$$\tilde{H}_k(\Delta_{\mathbf{s}}; K) \cong \tilde{H}_k(F_{1,1} \cap F_{1,2} \cap \Delta_2 \cap \cdots \cap \Delta_k; K) \neq (0).$$

What's next?

- Currently working on showing that if H is an induced subgraph of G such that $K[H]$ is not Cohen-Macaulay, then neither is $K[G]$

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- What about other properties of $K[G]$? When is $K[G]$ Gorenstein?

Thanks!