Gorenstein algebras presented by quadrics

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Set-up

$R = k[x_1, \dots, x_r]$, k a field
$A = R/I = \oplus_{j=1}^{e} [A]_j$, a graded artinian Gorenstein algebra

Hilbert series of $H_A(z) := \sum_{j=0}^{e} \dim_k [A]_j z^j := \sum_{j=0}^{e} h_j z^j$

h-vector or Hilbert function of A is $h = (h_0, h_1, \cdots, h_e)$

Symmetry: $h_i = h_{e-i}$, e the socle degree

WLOG $[I]_1 = 0$, so $h_1 = r$, the codimension

Two scenarios:

(A) I contains a regular sequence of r quadrics.

(B) The minimal generators of I all have degree 2.

Question: What are the possible Hilbert functions of A, assuming condition (A) or (B)?

Basic restrictions

Lemma

Assume A contains a regular sequence of r quadrics. Then

(a) $h_2 \leq \binom{r+1}{2} - r = \binom{r}{2}$.

(b) *$e \leq r$ and $e = r$ if and only if A is a complete intersection of r quadrics. In this case*

$$h = \left(1,\ r,\ \binom{r}{2},\ \binom{r}{3},\ \dots,\ \binom{r}{r-3},\ \binom{r}{r-2},\ r,\ 1\right)$$

(c) *(Kunz) If $e < r$, then I has at least $r + 2$ minimal generators.*

Constructions

Proposition

If A and B are artinian Gorenstein algebras presented by quadrics, then so is $A \otimes_k B$, and its Hilbert series is $H_A(z) \cdot H_B(z)$.

Corollary 1.

If $(h_0, \dots, h_e)$ is the h-vector of a Gorenstein algebra presented by quadrics, then so is the h-vector $(h_0, h_0 + h_1, h_1 + h_2, \dots, h_{e-1} + h_e, h_e)$.

Example

(i) $e = 2$: (Sally) For each $r \geq 1$, $h = (1, r, 1)$ is the h-vector of a Gorenstein algebra presented by quadrics.
(ii) $e = 3$: For each $r \geq 1$, $h = (1, r, r, 1)$ is the h-vector of a Gorenstein algebra presented by quadrics.

Constructions

Corollary 2.

Fix $r \geq 2$. Then, for each e such that $2 \leq e \leq r$, there is an artinian Gorenstein algebra presented by quadrics of codimension $h_1 = r$ with socle degree e.

Recall: $e = r$ iff I is a complete intersection

Corollary 3.

Fix $r \geq 2$. Then, for each e such that $4 \leq e \leq r - 2$, there are at least two artinian Gorenstein algebras presented by quadrics with socle degree e and $h_1 = r$ that have different values of h_2.

When $e = r - 1$?

Submaximal socle degree

Theorem

Assume that $e = r - 1 \geq 4$ and that I contains a regular sequence of r quadrics. Then:

(a) *h_2 must be either $\binom{r}{2} - 2$, $\binom{r}{2} - 1$ or $\binom{r}{2}$, and all of these possibilities do occur.*

(b) *If R/I is presented by quadrics, then $h_2 = \binom{r}{2} - 2$.*

(c) *If $h_2 = \binom{r}{2} - 2$, then R/I is presented by quadrics, and the entire Hilbert function of R/I is uniquely determined. It is*

$$h_j = \binom{r-1}{j} + \binom{r-3}{j-1}.$$

(d) *For $r \geq 7$, if $h_2 = \binom{r}{2} - 1$ or $\binom{r}{2}$, then the Hilbert function of R/I is not uniquely determined, at least if $\operatorname{char} k = 0$.*

For $r = 6$, $\quad h = (1, 6, h_2, h_2, 6, 1)$

Injectivity conjecture

Injectivity Conjecture

Assume R/I is presented by quadrics, has socle degree $e \geq 3$, and $\operatorname{char} k \neq 2$. Let $L \in R$ be a general linear form. Then the multiplication $\times L : [R/I]_1 \to [R/I]_2$ is injective.

Assumption on the characteristic is necessary:
If $I = (x_1^2, \ldots, x_r^2)$ and $\operatorname{char}(k) = 2$, then $L^2 \in I$ for every linear form L.

Socle Lemma (Huneke, Ulrich)

Let $M \neq 0$ be a finitely generated graded R-module. Assume $\operatorname{char}(k) = 0$. Let $L \in [R]_1$ be a general linear form, and consider the exact sequence

$$0 \to K \to M(-1) \xrightarrow{L} M \to C \to 0.$$

If $K \neq 0$, then the initial degrees satisfy $a(K) > a(\operatorname{Soc}(C))$.

Injectivity conjecture

Proposition

Assume that $\operatorname{char} k = 0$. Then, for any complete intersection $I = (Q_1, \ldots, Q_r)$ of quadrics, the Injectivity Conjecture is true.

WLP Conjecture

Assume that $\operatorname{char} k = 0$. Then every artinian Gorenstein algebra presented by quadrics and of socle degree $e \geq 3$ has the Weak Lefschetz Property (WLP), that is, for some linear form L, the multiplication

$$\times L : [A]_i \to [A]_{i+1}$$

has maximal rank for all i (i.e. is injective or surjective).

Note: The Gorenstein assumption can not be dropped. For example, if the ideal I is generated by squares of $r + 1$ general linear forms in an even number of variables, r, then R/I does not have the WLP (Migliore, Miró-Roig, N., 2010).

Injectivity conjecture

Corollary

Any complete intersection of at most 4 quadrics has the WLP, provided $\operatorname{char} k = 0$.

Corollary

Let I be a complete intersection of $r \geq 5$ quadrics, and assume $\operatorname{char} k = 0$. Then, for a general linear form L, the multiplication $\times L : [R/I]_2 \to [R/I]_3$ has at most a 1-dimensional kernel.

For $r = 5$, the h-vector is $(1, 5, 10, 10, 5, 1)$.

Injectivity conjecture

Lemma

Assume that R/I is presented by quadrics and that the Injectivity Conjecture is true for R/I. Let $Q_1, \ldots, Q_{h-1} \in R$ be $h-1$ general quadrics, where $h := \dim_k [R/I]_2$, and consider the ideal $J = (I, Q_1, \ldots, Q_{h-1})$. Then R/J is Gorenstein with Hilbert function $(1, r, 1)$.

Idea of Proof: To show $[\operatorname{Soc} R/J]_1 = 0$, that is, for each $\ell \in [R]_1$, $\ell \cdot [R]_1$ is not in $(I, Q_1, \ldots, Q_{h-1})$. Let

$$\begin{array}{rcl} \Lambda & = & \{\text{hyperplanes in } \mathbb{P}([R]_2) \text{ containing } \mathbb{P}([I]_2)\} \\ \Sigma & = & \{\text{linear subvarieties } \mathbb{P}(\ell \cdot [R]_1) \mid \ell \in [R]_1\} \\ \mathbb{I} & = & \{(A, H) \in \Sigma \times \Lambda \mid A \subset H\} \end{array} .$$

Consider the projections

$$\begin{array}{ccc} & \mathbb{I} \subset \Sigma \times \Lambda & \\ {}^{\phi_1}\swarrow & & \searrow_{\phi_2} \\ \Sigma & & \Lambda \end{array}$$

The Injectivity Conjecture provides $\dim \mathbb{I} = h - 2$, whereas $\dim \Lambda = h - 1$.

Persistence

Proposition

Assume that A is an artinian Gorenstein algebra for which the multiplication by a general linear form on A from degree 1 to degree 2 is an isomorphism (so $h_2 = r$). Then $h_i = r$ for all $i = 1, 2, \ldots, e-1$.
Furthermore, if A is presented by quadrics then $e = 3$.

Corollary

Let A be an artinian Gorenstein algebra presented by quadrics. Assume that $r \geq 3$ and that the Injectivity Conjecture is true for A. Then the following are equivalent:

(a) $h_2 = r$;

(b) $e = 3$;

(c) The h-vector is $(1, r, r, 1)$.

Persistence

Theorem

Let A be presented by quadrics and of socle degree $e \geq 4$. Assume that $h_2 = r < 4e - 6$ and $\operatorname{char} k = 0$. Then multiplication by a general linear form on A from degree 1 to degree 2 must be an isomorphism.

Proof uses the theory of generic initial ideals.

Note: $h = (1, 21, 21, 20, 21, 21, 1)$ is the h-vector of a Gorenstein algebra.

Small codimension

Proposition

For $r \leq 5$, the following are the only h-vectors of artinian Gorenstein algebras presented by quadrics.

r	h-vectors
2	$(1,2,1)$
3	$(1,3,1),(1,3,3,1)$
4	$(1,4,1),(1,4,4,1),(1,4,6,4,1)$
5	$(1,5,1),(1,5,5,1),(1,5,8,5,1),(1,5,10,10,5,1)$

Small codimension

Proposition

For $r = 6$, the following are h-vectors of artinian Gorenstein algebras presented by quadrics and of socle degree e.

e	h-vectors
2	(1,6,1)
3	(1,6,6,1)
4	(1,6,h_2,6,1), where $h_2 \in \{10,11\}$
5	(1,6,13,13,6,1)
6	(1,6,15,20,15,6,1)

Remark

(i) $h_2 \leq 12$, $h_2 \neq 6$.
(ii) If char $k > 0$ is small, then $h_2 = 12$ is possible.

Theorem (Davis, Okun, 2001)

If I is a squarefree, monomial Gorenstein ideal generated by quadrics with h-vector $(1,r,h_2,r,1)$, then $h_2 \geq 2r-2$.