Stabilization of multigraded Betti numbers

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Outlines

1. Motivation - asymptotic linearity of regularity
2. Multigraded (or G-graded) situation
3. Problem and approach
4. Equi-generated case
5. General case

Asymptotic linearity of regularity

- R a standard graded algebra over a field k,
- $\mathfrak{m}$ its maximal homogeneous ideal,
- M a finitely generated graded R-module.
- $\mathrm{end}(M) := \max\{l \mid M_l \neq 0\}$,
- The regularity of M is

$$\mathrm{reg}(M) = \max\{\mathrm{end}(H^i_{\mathfrak{m}}(M)) + i\}.$$

Theorem (Cutkosky-Herzog-Trung (1999), Kodiyalam (2000), Trung-Wang (2005))

Let R be a standard graded k-algebra, let $I \subseteq R$ be a homogeneous ideal and let M be a finitely generated graded R-module. Then $\mathrm{reg}(I^q M)$ is asymptotically a linear function in q, i.e., there exist a and b such that for $q \gg 0$,

$$\mathrm{reg}(I^q M) = aq + b.$$

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G-graded Betti numbers

- G a finitely generated abelian group, k a field.
- $R = k[x_1, \dots, x_n]$ a G-graded polynomial ring.
- M a finitely generated G-graded module over R.
- The minimal G-graded free resolution of M:

$$0 \to \bigoplus_{\eta \in G} R(-\eta)^{\beta_{p,\eta}(M)} \to \cdots \to \bigoplus_{\eta \in G} R(-\eta)^{\beta_{0,\eta}(M)} \to M \to 0.$$

- The numbers $\beta_{i,\eta}(M)$ are called the G-graded Betti numbers of M.
- $\beta_{i,\eta}(M) = \dim_k \operatorname{Tor}_i^R(M,k)_\eta$
  $\rightsquigarrow$ study the support $\operatorname{Supp}_G(\operatorname{Tor}_i^R(M,k))$.

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Problem

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Let $I_1, \ldots, I_s$ *be* G*-graded homogeneous ideal in* R*, and let* M *be a finitely generated* G*-graded* R*-module. Investigate the asymptotic behavior of* $\mathrm{Supp}_G(\mathrm{Tor}_i^R(I_1^{t_1} \ldots I_s^{t_s} M, k))$ *as* $t = (t_1, \ldots, t_s) \in \mathbb{N}^s$ *gets large.*

Approach to the problem

- $\mathcal{R} = \bigoplus_{t \in \mathbb{N}^s} I_1^{t_1} \dots I_s^{t_s}$, $M\mathcal{R} = \bigoplus_{t \in \mathbb{N}^s} I_1^{t_1} \dots I_s^{t_s} M$.
- $I_i = (F_{i,1}, \dots, F_{i,r_i})$.
- $S = R[T_{i,j} \mid 1 \leq i \leq s, 1 \leq j \leq r_i]$ is $G \times \mathbb{Z}^s$-graded polynomial extension of R, where $\deg_{G \times \mathbb{Z}^s}(a) = (\deg_G(a), 0) \ \forall \ a \in R, \deg_{G \times \mathbb{Z}^s}(T_{i,j}) = (\deg_G(F_{i,j}, \mathbf{e}_i)$.
- $M\mathcal{R}$ is a finitely generated $G \times \mathbb{Z}^s$-graded module over S, and

$$I_1^{t_1} \dots I_s^{t_s} M = (M\mathcal{R})_{(*,t)} = (M\mathcal{R})_{G \times t}.$$

- For a finitely generated $G \times \mathbb{Z}^s$-graded module $\mathcal{M}$ over S, study

$$\mathrm{Tor}_i^R(\mathcal{M}_{G \times t}, k).$$

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Approach to the problem

- If $\mathbb{F}_\bullet$ is a $G \times \mathbb{Z}^s$-graded complex of free S-modules, then for $\delta \in \mathbb{Z}^s$,

$$H_i((\mathbb{F}_\bullet)_{G\times\delta} \otimes_R k) = H_i(\mathbb{F}_\bullet \otimes_R k)_{G\times\delta}.$$

- If $\mathbb{F}_\bullet$ is a $G \times \mathbb{Z}^s$-graded free resolution of $\mathcal{M}$, then $(\mathbb{F}_\bullet)_{G\times\delta}$ is a G-graded free resolution of $\mathcal{M}_{G\times\delta}$. Hence

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Equi-generated case

- $I_i = (F_{i,1}, \ldots, F_{i,r_i})$ is generated in degree $\gamma_i \in G$.
- There is a natural map $S \to \mathcal{R} \simeq \bigoplus_{t \in \mathbb{N}^s} I_1^{t_1}(t_1\gamma_1) \ldots I_s^{t_s}(t_s\gamma_s)$
- Let $\mathbb{F}_i = \bigoplus_{\theta,\ell} S(-\theta, -\ell)^{\beta^i_{\theta,\ell}}$ be the ith module of $\mathbb{F}_\bullet$
- $H_i((\mathbb{F}_\bullet)_{G\times\delta} \otimes_R k)_\eta = H_i(\mathbb{F}_\bullet^{[\eta]} \otimes_R k)_\delta$, where

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Equi-generated case

Theorem

There exists a finite set $\Delta_i \subseteq G$ such that

1. *For all* $t = (t_1, \dots, t_s) \in \mathbb{N}^s$, $\operatorname{Tor}_i^R(I_1^{t_1} \cdots I_s^{t_s} M, k)_\eta = 0$ *if* $\eta \notin \Delta_i + t_1\gamma_1 + \cdots + t_s\gamma_s$.
2. *There exists a subset* $\Delta_i' \subset \Delta_i$ *such that* $\operatorname{Tor}_i^R(I_1^{t_1} \cdots I_s^{t_s} M, k)_{\eta + t_1\gamma_1 + \cdots + t_s\gamma_s} \neq 0$ *for* $t \gg 0$ *and* $\eta \in \Delta_i'$, *and* $\operatorname{Tor}_i^R(I_1^{t_1} \cdots I_s^{t_s} M, k)_{\eta + t_1\gamma_1 + \cdots + t_s\gamma_s} = 0$ *for* $t \gg 0$ *and* $\eta \notin \Delta_i'$.
3. *For any* δ, *the function*

$$\dim_k \operatorname{Tor}_i^R(I_1^{t_1} \cdots I_s^{t_s} M, k)_{\delta + t_1\gamma_1 + \cdots + t_s\gamma_s}$$

is polynomial in the t_i*s for* $t \gg 0$.

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General case

- Recall: study $\text{Supp}_{G\times\mathbb{Z}^s}(H_i(\mathbb{F}_\bullet \otimes_R k))$ where $\mathbb{F}_\bullet \otimes_R k$ is viewed as a $G \times \mathbb{Z}^s$-graded module over $B = k[T_{i,j}]$.
- Study, in general, the support of $G \times \mathbb{Z}^s$-graded modules over $B = k[T_{i,j}]$.

Definition

A subset $E \subseteq G$ is said to be linearly independent if E forms a basis for a free submonoid of G.

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Theorem

Let Δ be a finitely generated abelian group, let $B = k[T_1, \dots, T_r]$ be a Δ-graded polynomial ring, and let $\mathcal{M}$ be a finitely generated Δ-graded B-module. Let $\Gamma = \{\deg_\Delta(T_i)\}$. Then there exist a finite collection of elements $\delta_p \in \Delta$ and linear independent subsets $E_p \subseteq \Gamma$ such that

$$\mathrm{Supp}_\Delta(\mathcal{M}) = \bigcup_p (\delta_p + \langle E_p \rangle),$$

where $\langle E_p \rangle$ denotes the free submonoid of Δ generated by E_p.

General case

Example

Let $B = k[x, y]$ with $\deg(x) = 4$ and $\deg(y) = 7$, and let $M = B/(x) \oplus B/(y) \simeq k[y] \oplus k[x]$. Then

$$\mathrm{Supp}_{\mathbb{Z}}(M) = \{4a + 7b \mid a, b \in \mathbb{Z}\}.$$

Independent subsets of $\{4, 7\}$ are $\{4\}$ and $\{7\}$.

General case

- $I_i = (F_{i,1}, \ldots, f_{i,r_i})$ where $\deg_G(F_{i,j}) = \gamma_{i,j}$.
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Theorem

For $\ell \geq 0$, there exist a finite collection of elements $\delta_p^\ell \in G$, a finite collection of integers $t_{p,i}^\ell$, and a finite collection of linearly independent non-empty tuples $E_{p,i}^\ell \subseteq \Gamma_i$, such that if $t_i \geq \max_p\{t_{p,i}^\ell\}$ for all i then

$$\mathrm{Supp}_G(\mathrm{Tor}_\ell^R(I_1^{t_1} \cdots I_s^{t_s} M, k)) =$$
$$= \bigcup_{p=1}^{m} \left(\delta_p^\ell + \bigcup_{\mathbf{c}_i \in \mathbb{Z}_{\geq 0}^{|E_{p,i}^\ell|}, |\mathbf{c}_i| = t_i - t_{p,i}^\ell} \mathbf{c}_1 . E_{p,1}^\ell + \cdots + \mathbf{c}_s . E_{p,s}^\ell\right).$$

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where u_is are Δ-graded homogeneous elements in $\mathcal{M}$, Z_is are subsets of the variables $\{T_1, \ldots, T_r\}$, and $u_i k[Z_i]$ denotes the k-subspace of $\mathcal{M}$ generated by elements of the form $u_i N$ with N being a monomial in the polynomial ring $k[Z_i]$.

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