A. V. Geramita and E. Carlini asked that their slides be combined into a single file, since their talks were meant to be taken as parts 1 and 2 of a single presentation.

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Solution to Warings's Problem for Monomials - I

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Waring's Problem in Number Theory

Begins with

i) Lagrange's observation that every integer is a sum of $\leq$ 4 squares of integers.

ii) Gauss' observation that $n \equiv 7 (mod\ 8)$ is not a sum of three squares.

Waring asserts (and Hilbert proves) that:

> there are integers $g(j)$ such that every integer is a sum of $\leq g(j)$ j^{th} powers.

In particular, Waring asserts that $g(3) = 9$ etc. That is proved but, unlike Gauss' observation, only 23 and 239 need nine cubes.

Waring's Second Problem: Find $G(j)$, the least positive integers so that every sufficiently large integer is a sum of $\leq G(j)$ j^{th} powers.

Are there analogs to Waring's Problems in $S = \mathbb{C}[x_1, \ldots, x_n] = \oplus_{i=o}^{\infty} S_i$*?*

Yes-1, Lagrange analog. Let $F \in S_2$, then

$$F = L_1^2 + \cdots + L_k^2, \quad k \leq n.$$

Moreover, almost every F is a sum of n squares of linear forms. Those which require fewer lie on a hypersurface in $\mathbb{P}(S_2)$.

Yes-2, Hilbert analog. Let $t = \dim S_d$. There are linear forms $L_1, \ldots, L_t$ such that $L_1^d, \cdots, L_t^d$ are a basis for S_d.

Yes-3, Waring Cubes analog. Let $S = \mathbb{C}[x_1, x_2]$, $\mathbb{P}^3 = \mathbb{P}(S_3)$.

i) the points of $\mathbb{P}^3$ of the form $[L^3]$ are the rational normal curve, $\mathcal{C}$, in $\mathbb{P}^3$.

ii) The points $[F]$ not on $\mathcal{C}$, but on its tangent envelope, require 3 cubes.

iii) The general point $[F]$ in $\mathbb{P}^3$, i.e. a point not on the tangent envelope, requires 2 cubes.

Definition: Let $F \in S_d$, a *Waring Decomposition* of F is a way to represent

$$F = L_1^d + \cdots + L_s^d$$

such that no shorter such representation exists. In this case we say that the *(Waring) rank* of F is s.

In 1995, J. Alexander and A. Hirschowitz solved the long outstanding problem of finding the Waring rank of a general form in S_d for any d and any n. (roughly speaking, it is on the order of

$$\dim S_d/(n+1) \qquad).$$

However, it is hard to know when one has a general form! and, as we saw, the Waring Rank of a specific form can be larger than the general rank.

There is a way to find the rank of any specific form, and this involves the use of Macaulay's Inverse System.

Let $F \in S_d$, $S = \mathbb{C}[x_1, \ldots, x_n]$ and let $T = \mathbb{C}[y_1, \cdots, y_n]$. We make S into a graded T-module by

$$y_i \circ F = (\partial/\partial x_i)(F)$$

and extend linearly.

Definition: Given $F \in S_d$, then

$$F^{\perp} = \{\partial \in T \ \mid \ \partial F = 0\}.$$

It is easy to see that $F^{\perp}$ is a homogeneous ideal in T. Less obvious is the fact that it is always a Gorenstein Artinian ideal in T.

Apolarity Lemma: $F \in S_d$ and $I = F^{\perp} \subset T$. If we can find $J \subset I$ where $J = \wp_1 \cap \cdots \cap \wp_s$ is the ideal of a set of s distinct points in $\mathbb{P}^{n-1}$, then

$$F = L_1^d + \cdots + L_s^d.$$

So, it's enough to find the smallest set of distinct points in $\mathbb{P}^{n-1}$ whose defining ideal is in $F^{\perp}$.

There have been several attempts to calculate the Waring Rank of specific forms. In particular, Landsberg-Teitler and Schreyer-Ranestad (among others) have attempted to find the Waring rank of monomials (and succeeded for certain monomials).

Theorem: (Catalisano, Carlini, G..) Let $F = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$. Then the Waring rank of F is exactly

$$s = (b_2 + 1)(b_3 + 1) \cdots (b_n + 1).$$

By the Apolarity Lemma, it will be enough to show that

i) $F^{\perp}$ contains an ideal of s distinct points; and

ii) $F^{\perp}$ does not contain an ideal with fewer than s points.

The first part is simple: It comes from the observation that

$$F^{\perp} = (y_1^{b_1+1}, y_2^{b_2+1}, \cdots, y_n^{b_n+1})$$

in the first instance, and that

$$F_1 = y_2^{b_2+1} - y_1^{b_2+1}, \cdots, F_{n-1} = y_n^{b_n+1} - y_1^{b_n+1}$$

is a regular sequence in T which defines a complete intersection of s distinct points.

The more difficult (and interesting) part of the proof is left to Carlini.

The solution to the Waring problem for monomials - II

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Apolarity Lemma

For a degree d form $F \in S_d$ one can write

$$F = \sum_{i=1}^{s} L_i^d$$

if and only if

there exists a set of s distinct points $\mathbb{X} \subset \mathbb{P}(S_1)$ such that

$$I_{\mathbb{X}} \subset F^{\perp}.$$

The case of monomials

Consider the monomial

$$M = x_1^{b_1} \cdot \ldots \cdot x_n^{b_n}$$

where $1 \leq b_1 \leq \ldots \leq b_n$ and notice that

$$M^{\perp} = (y_1^{b_1+1}, \ldots, y_n^{b_n+1}).$$

Thus we want to study the multiplicity of one dimensional radical ideals I such that

$$I \subset (y_1^{b_1+1}, \ldots, y_n^{b_n+1}).$$

Ideal of points in $(y_1^{a_1}, \dots, y_n^{a_n})$

So we study monomial ideals generated by powers of the variables. Notice that $(y_1^{a_1}, \dots, y_n^{a_n})$ contains the ideal

$$I_{\mathbb{X}} = (y_2^{a_2} - y_1^{a_2}, \dots, y_n^{a_n} - y_1^{a_n})$$

and this is the ideal of a set of points $\mathbb{X}$ which is a complete intersection consisting of $\Pi_2^n(a_i)$ distinct points.

Of course we can find larger set of points, but can we find smaller sets?

Main Theorem

We proved the following

Theorem

Let $n > 1$ and $K = (y_1^{a_1}, \ldots, y_n^{a_n})$ be an ideal of T with $2 \leq a_1 \leq \ldots \leq a_n$. If $I \subset K$ is a one dimensional radical ideal of multiplicity s, then

$$s \geq \prod_{i=2}^{n} a_i.$$

Thus, if $\mathbb{X}$ is set of s distinct points such that $I_{\mathbb{X}} \subset K$, then $s \geq \prod_{i=2}^{n} a_i$.

Idea of the proof

We work out an example.
Let

$$K = (y_1^2, y_2^3, y_3^4)$$

and we look for ideal of points $I_{\mathbb{X}} \subset K$.
Clearly

$$I_{\mathbb{X}} = (y_2^3 - y_1^3, y_3^4 - y_1^4) \subset (y_1^2, y_2^3, y_3^4) = K$$

and $\mathbb{X}$ is a set of 12 distinct points.
We want to show that there is no set of less than 3×4 distinct points such that $I_{\mathbb{X}} \subset K$.

Idea of the proof

Radical (i.e. distinct points) is essential

Remark

Notice that

$$K = (y_1^2, y_2^3, y_3^4) \supset (y_1^2, y_2^3)$$

where the latter is a one dimensional not radical ideal of multiplicity 6.

Hence the result only holds for sets of distinct points.

Idea of the proof

To bound the multiplicity of $I_{\mathbb{X}} \subset K$ we bound its Hilbert function as

$$HF\left(\frac{R}{I_{\mathbb{X}}}, t\right) \leq |\mathbb{X}|$$

for all t. We also notice that

$$I_{\mathbb{X}} + (y_1^2) \subset K = (y_1^2, y_2^3, y_3^4)$$

and hence

$$HF\left(\frac{R}{I_{\mathbb{X}} + (y_1^2)}, t\right) \geq HF\left(\frac{R}{K}, t\right).$$

Idea of the proof

We now consider to cases depending on whether y_1 is a 0-divisor in $\frac{R}{I_{\mathbb{X}}}$.

If y_1 is not a zero divisor in $\frac{R}{I_{\mathbb{X}}}$. Hence

$$HF\left(\frac{R}{I_{\mathbb{X}} + (y_1^2)}, t\right) = HF\left(\frac{R}{I_{\mathbb{X}}}, t\right) - HF\left(\frac{R}{I_{\mathbb{X}}}, t-2\right)$$

and we get the relation

$$HF\left(\frac{R}{I_{\mathbb{X}}}, t\right) \geq HF\left(\frac{R}{K}, t\right) + HF\left(\frac{R}{I_{\mathbb{X}}}, t-2\right)$$

using this expression we obtain the desired bound on $|\mathbb{X}|$.

Idea of the proof

$K = (y_1^2, y_2^3, y_3^4)$ is a complete intersection thus

	0	1	2	3	4	5	6
$HF(R/K, \cdot) =$	1	3	5	6	5	3	1

Now we iterate the relation

$$HF\left(\frac{R}{I_{\mathbb{X}}}, 6\right) \geq HF\left(\frac{R}{K}, 6\right) + HF\left(\frac{R}{I_{\mathbb{X}}}, 4\right)$$

$$HF\left(\frac{R}{I_{\mathbb{X}}}, 6\right) \geq 1 + HF\left(\frac{R}{K}, 4\right) + HF\left(\frac{R}{I_{\mathbb{X}}}, 2\right)$$

$$HF\left(\frac{R}{I_{\mathbb{X}}}, 6\right) \geq 1 + 5 + HF\left(\frac{R}{I_{\mathbb{X}}}, 2\right)$$

Idea of the proof

As $I_{\mathbb{X}} \subset (y_1^2, y_2^3, y_3^4)$ and y_1 is not a zero divisor in $\frac{R}{I_{\mathbb{X}}}$, we have

$$HF\left(\frac{R}{I_{\mathbb{X}}}, 2\right) = 6$$

and hence

$$HF\left(\frac{R}{I_{\mathbb{X}}}, 6\right) \geq 1 + 5 + 6 = 12$$

which proves $|\mathbb{X}| \geq 12$.

Idea of the proof

If y_1 is a zero divisor in $\frac{R}{I_{\mathbb{X}}}$.
Consider the ideal

$$I_{\mathbb{Y}} = I_{\mathbb{X}} : (y_1)$$

and notice that

$$I_{\mathbb{Y}} \subset K : (y_1) = (y_1, y_2^3, y_3^4).$$

As y_1 is not a 0-divisor in $\frac{R}{I_{\mathbb{Y}}}$ we can use the same argument of the previous case and we get

$$|\mathbb{X}| > |\mathbb{Y}| \geq 3 \times 4.$$

Consequences

The rank of any monomial.

Corollary

For integers $m > 1$ and $1 \leq b_1 \leq \ldots \leq b_m$ let M be the monomial

$$x_1^{b_1} \cdot \ldots \cdot x_m^{b_m}$$

then $\mathrm{rk}(M) = \prod_{i=2}^{m}(b_i + 1)$, i.e. M is the sum of $\prod_{i=2}^{m}(b_i + 1)$ power of linear forms and no fewer.

Remark

After we posted our paper on the arXiv we received a draft from W. Buczynska, J. Buczynski and Z. Teitler. This draft contains a statement giving an expression for the rank of any monomial coinciding with the one that we found.

Consequences

On the generic form

Remark

We know in general the degree of the generic degree d form.

We want to compare the maximum rank of a degree d monomial with the generic rank.

Do the monomials provide examples of forms having rank higher than the generic form?

In the case of three variables we showed that

Corollary

$$\max\{\mathrm{rk}(M) : M \in S_d\} \simeq \frac{3}{2}\mathrm{rk}(\text{generic degree } d \text{ form}).$$

For more than three variables this is not true and the monomials have smaller rank than the generic form.

Monomials as sums of powers.

Corollary

For integers $1 \leq b_1 \leq \ldots \leq b_m$ consider the monomial

$$M = x_1^{b_1} \cdot \ldots \cdot x_n^{b_n}.$$

Then

$$M = \sum_{j=1}^{\mathrm{rk}(M)} \gamma_j \left(x_1 + \epsilon_j(2)x_2 + \ldots + \epsilon_j(n)x_n\right)^d$$

where $\epsilon_1(i) \ldots, \epsilon_{\mathrm{rk}(M)}(i)$ are the $(b_i + 1)$-th roots of 1, each repeated $\Pi_{j \neq i,1}(b_i + 1)$ times, and the γ_j are scalars.

Consequences

Remark

W. Buczynska, J. Buczynski and Z. Teitler found the same sum of powers decomposition for monomials and they also determined the coefficients γ_j.

Consequences

An easy example
We consider the monomial $M = x_1x_2x_3$.

In this case $M^\perp = (y_1^2, y_2^2, y_3^2)$ and we can use the complete intersection defined by the ideal

$$(y_2^2 - y_1^2, y_3^2 - y_1^2)$$

defining the four points

$$[1 : 1 : 1], [1 : 1 : -1], [1 : -1 : 1], [1 : -1 : -1]$$

thus we have $24x_1x_2x_3 =$

$$(x_1 + x_2 + x_3)^3 - (x_1 + x_2 - x_3)^3 - (x_1 - x_2 + x_3)^3 + (x_1 - x_2 - x_3)^3.$$