Generalizing the Borel condition

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Joint work with Jeff Mermin and Jay Schweig

Lincoln, NE
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Also known as strongly stable or 0-Borel ideals.

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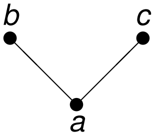

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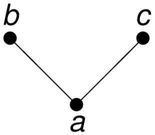

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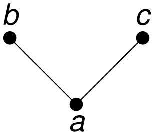

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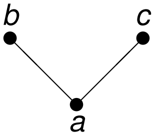

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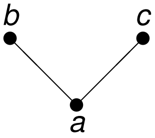

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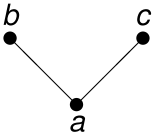

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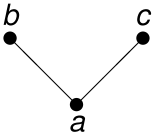

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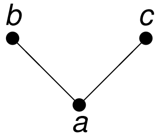

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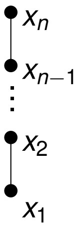
This is not an ordinary Borel ideal because $b^2 \notin I$ ($c \not\to b$).

Extremal cases

Chain C of length n



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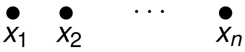

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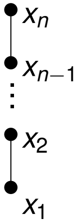

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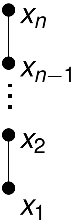

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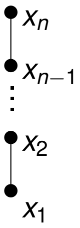

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Guiding idea: The closer Q is to C, the more a Q-Borel ideal should behave like a Borel ideal.

Associated primes of Q-Borel ideals

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Special case: Principal Q-Borel ideals, $I = Q(m)$.

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Theorem A: Suppose

$$I = \prod_{\mathfrak{p} \subset S} \mathfrak{p}^{e_{\mathfrak{p}}},$$

where the $\mathfrak{p}$ are all monomial primes of S, and $e_{\mathfrak{p}} \geq 0$. Then

$$I = \bigcap_{\mathfrak{p} \subset S} \mathfrak{p}^{a_{\mathfrak{p}}},$$

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Get a primary decomposition consisting of powers of monomial primes.

Irredundant primary decomposition

For a monomial m', let $A(m') = \{x_i : x_i \leq_Q x_j \text{ for some } x_j \mid m'\}$.

In English: Variables below any element of $\mathrm{supp}(m')$.

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Theorem B: Let $I = Q(m)$. Let $\mathfrak{p}$ be a prime ideal. Then $\mathfrak{p} \in \mathrm{Ass}(S/I)$ if and only if

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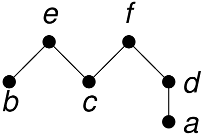
Method: Compute all order ideals corresponding to divisors of m. For the connected ones, use Theorem A to compute exponents.

Principal Q-Borel example

Poset Q:



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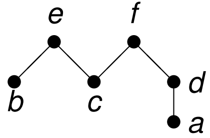

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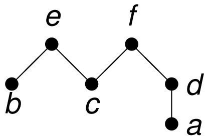

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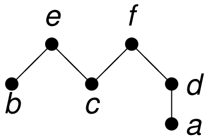

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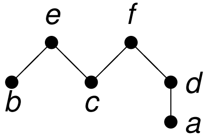

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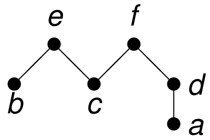

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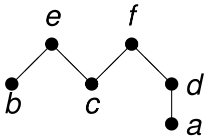

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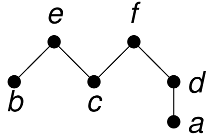

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$$I = (d,a) \cap (e,b,c) \cap (f,c,d,a)^2 \cap (f,c,d,a,e,b)^3$$

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Under above hypotheses, recover part of a Herzog-Hibi result:

Corollary: $I = Q(m)$ Cohen-Macaulay if and only if

- Q is the chain, $m = x_n^{a_n}$ ($I = \mathfrak{m}^{a_n}$), or
- Q is the antichain (I is a principal ideal)

Y-Borel ideals

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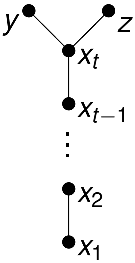

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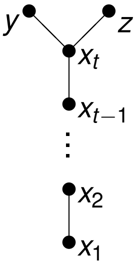

"Close" to the chain C, but Y-Borel ideals may not be componentwise linear.

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  - homological degree $1 + \deg \alpha$
  - multidegree $m\alpha y^{r_m}$
  - r_m minimal such that $m \cdot \frac{y^{r_m}}{z} \in I$.

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  - homological degree $1 + \deg \alpha$
  - multidegree $m\alpha y^{r_m}$
  - r_m minimal such that $m \cdot \frac{y^{r_m}}{z} \in I$.

Induction using Mayer-Vietoris.

If z divides no generator, ideal is Borel in $k[x_1, \ldots, x_t, y]$.

Y-Borel example

$S = k[x_1, x_2, y, z]$, $I = Y(x_1, y^2, z^2)$

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total:	1	6	11	8	2
0:	1	1	.	.	.
1:	.	5	10	6	1
2:	.	.	1	2	1

(Betti diagram of S/I)

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First syzygies

Eliahou-Kervaire symbols: $[x_2^2, x_1]$, $[x_2y, x_1]$, $[x_2y, x_2]$, $[x_2z, x_1]$, $[x_2z, x_2]$, $[y^2, x_1]$, $[y^2, x_2]$, $[z^2, x_1]$, $[z^2, x_2]$

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Other symbols: $\underbrace{[x_2z, 1 \cdot y]}_{\text{usual EK symbol}}$, $[z^2, 1 \cdot y^2]$ (multidegree y^2z^2)