On the Resolutions of (some) Simplicial Forests

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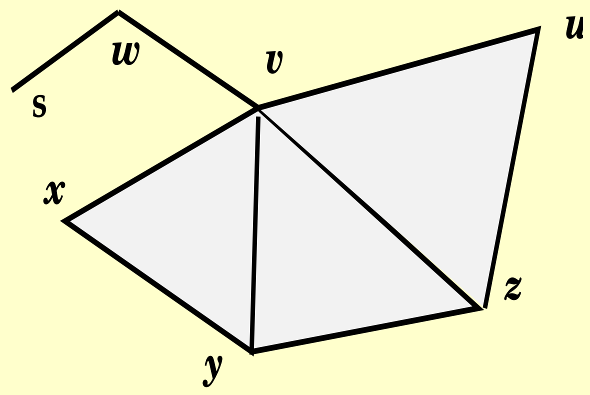
$I = (M_1, \ldots, M_q)$ monomial ideal in polynomial ring.

Question. What are the Betti numbers $\beta_{i,j}(I)$?

Eliahou-Kervaire Splittings: When $I = J + K$ where $\mathcal{G}(J) \cap \mathcal{G}(K) = \emptyset$, and there is a “splitting function” with certain properties, one has a recursive formula:

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)$$

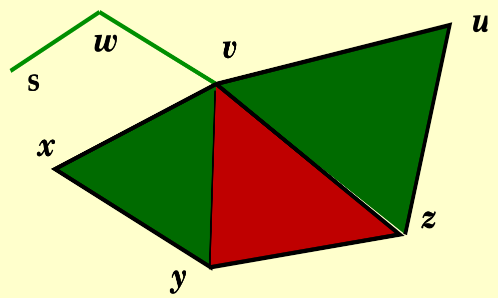


$$I = (xyv, vw, ws, yzv, zuv)$$

Question. Can one give an order to the facets of $\triangle$ so that induces a splitting on the generators of I?

Trees and Good Leafs

Definition. A **leaf** is a facet that intersects the complex in a *face*.



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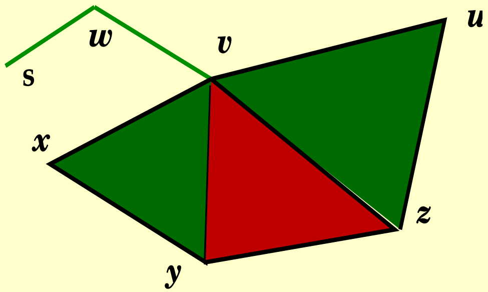
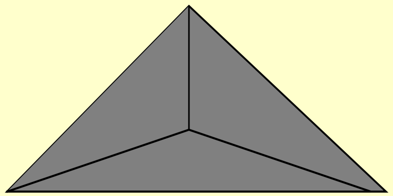


Definition. A **forest** is a complex whose every subset (of facets) has a leaf.

A **tree** is a connected forest.



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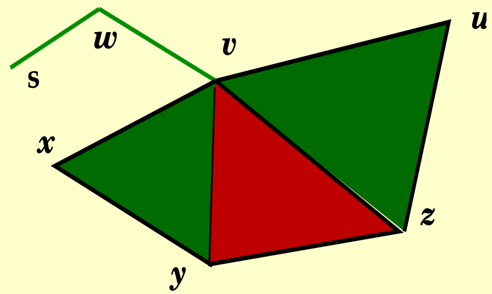

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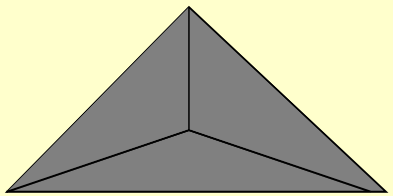

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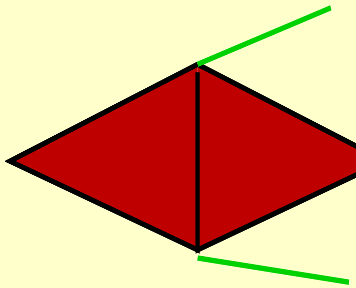

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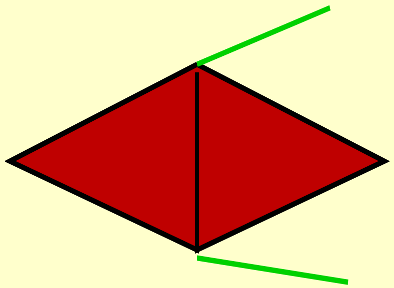

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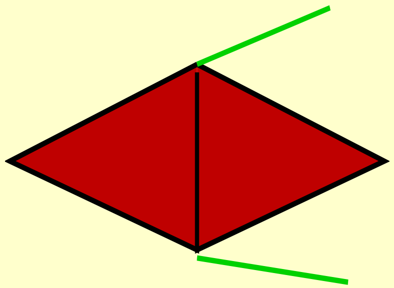

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– $F_0, \ldots, F_q$ where each F_i is the leaf of $\langle F_1, \ldots, F_i \rangle$

– $F_0, F_1, \ldots, F_q$ where F_0 is a good leaf of Δ and

$$F_0 \cap F_1 \supseteq F_0 \cap F_2 \supseteq \cdots \supseteq F_0 \cap F_q$$

Theorem. If Δ is a forest, then its facets can be ordered as $F_0, F_1, \ldots, F_q$ such that

1. F_0 is a good leaf of Δ

2. $F_0 \cap F_1 \supseteq F_0 \cap F_2 \supseteq \cdots \supseteq F_0 \cap F_q$

3. each F_i is a leaf of $\langle F_0, F_1, \ldots, F_i \rangle$ for $0 \leq i \leq q$

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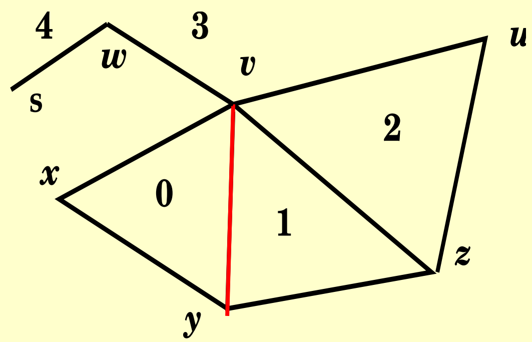

$$vy \supseteq v \supseteq v \supseteq \emptyset$$

Theorem. (Hà - Van Tuyl 2007) If F is a leaf, then there is an Eliahou-Kervaire type splitting for Δ described as follows:

$$\beta_{ij}(\Delta) = \beta_{ij}(\Delta \setminus F) + \sum_{\ell_1=0}^{i} \sum_{\ell_2=0}^{j-|F|} \beta_{\ell_1-1,\ell_2}(\overline{\mathcal{C}}(F))\beta_{i-\ell_1-1,j-|F|-\ell_2}(\Delta/\mathcal{C}(F))$$

where

$$\mathcal{C}(F) = (F' \in \Delta \mid F' \cap F \neq \emptyset)$$

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Note. This formula is *recursive* if Δ is a forest as

$\mathcal{C}(F)$= subset of a forest= also a forest

$\overline{\mathcal{C}}(F)$= localization of a forest = also a forest

Tree $\Delta = (F_0, \quad F_1, \ldots, F_{q-1}, \quad F_q)$

F_0 → good leaf

F_q → leaf

$$\beta_{ij}(\Delta) = \beta_{ij}(F_0, \ldots, F_{q-1})$$

$$+$$

$$\sum_{\ell_1=0}^{i} \sum_{\ell_2=0}^{j-|F_q|} \beta_{\ell_1-1,\ell_2}(\overline{\mathcal{C}}(F_q))\beta_{i-\ell_1-1,j-|F_q|-\ell_2}(\Delta/\mathcal{C}(F_q))$$

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$$\beta_{ij}(F_0, \ldots, F_q)$$

$$= \beta_{ij}(F_0, \ldots, F_{q-1}) + \beta_{i-1,j-|F_q|}(\overline{\mathcal{C}}(F_q))$$

$$= \beta_{ij}(F_0, \ldots, F_{q-2}) + \beta_{i-1,j-|F_{q-1}|}(\overline{\mathcal{C}}(F_{q-1})) + \beta_{i-1,j-|F_q|}(\overline{\mathcal{C}}(F_q))$$

$$\vdots$$

$$= \beta_{ij}(F_0) + \sum_{u=1}^{q} \beta_{i-1,j-|F_u|}(\overline{\mathcal{C}}(F_u))$$

$$\beta_{ij}(F_0,\ldots,F_q)=\beta_{ij}(F_0)+\sum_{u=1}^{q}\beta_{i-1,j-|F_u|}(\overline{\mathcal{C}}(F_u))$$

This formula is **inductive** but not **recursive**!

$$\beta_{ij}(F_0, \ldots, F_q) = \beta_{ij}(F_0) + \sum_{u=1}^{q} \beta_{i-1,j-|F_u|}(\overline{\mathcal{C}}(F_u))$$

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Compute $\beta_{0j}(\Delta)$:

$$\beta_{0,j}(F_0, \ldots, F_q) = \sum_{u=0}^{q} \delta_{j,|F_u|}$$

where $\delta_{a,b}$ is the Kronecker delta function.

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Compute $\beta_{1j}(\Delta)$:

$$\beta_{1j}(F_0,\ldots,F_q) = \sum_{u=1}^{q} \beta_{0,j-|F_u|}(\overline{\mathcal{C}}(F_u))$$

We need to know the generators of $\overline{\mathcal{C}}(F_u)$!

Theorem. Given a good-leaf-ordering $F_0 \cap F_1 \supseteq \cdots \supseteq F_0 \cap F_q$

– $\overline{\mathcal{C}}(F_u) = (F_{i_1} \setminus F_u, \ldots, F_{i_s} \setminus F_u)$ $\quad 0 \leq i_1 < i_2 < \cdots < i_s < u$ is a forest

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– $F_{i_s} \setminus F_u$ has a free vertex and is therefore a “splitting facet” of $\overline{\mathcal{C}}(F_u)$

Moreover if $F_0 \cap F_1 \supsetneq \cdots \supsetneq F_0 \cap F_q$ then

– $i_s = u - 1$

– $\overline{\mathcal{C}}(F_u)$ is connected.

$$\beta_{ij}(F_0,\ldots,F_q) = \beta_{ij}(F_0) + \sum_{u=1}^{q} \beta_{i-1,j-|F_u|}(\overline{\mathcal{C}}(F_u))$$

Compute $\beta_{1j}(\Delta)$:

$$\begin{aligned}\beta_{1j}(\Delta) &= \sum_{u=1}^{q} \beta_{0,j-|F_u|}(\overline{\mathcal{C}}(F_u)) \\ &= \sum_{u=1}^{q} \sum_{v=0}^{u-1} \gamma_{j,|F_u \cup F_v|,\{F_s \cup F_u \mid s<u\}}\end{aligned}$$

where $\gamma_{j,N,A} = \begin{cases} 1 & j = |N|, N' \not| N \text{ for all } N' \in A \\ 0 & \text{otherwise} \end{cases}$

$$\beta_{ij}(F_0,\ldots,F_q) = \beta_{ij}(F_0) + \sum_{u=1}^{q} \beta_{i-1,j-|F_u|}(\overline{\mathcal{C}}(F_u))$$

Compute $\beta_{2j}(\Delta)$:

$$\begin{aligned} \beta_{2j}(\Delta) &= \sum_{u=1}^{q} \beta_{1,j-|F_u|}(\overline{\mathcal{C}}(F_u)) \\ &= \sum \cdots \end{aligned}$$

$$\beta_{ij}(F_0,\ldots,F_q) = \beta_{ij}(F_0) + \sum_{u=1}^{q} \beta_{i-1,j-|F_u|}(\overline{\mathcal{C}}(F_u))$$

More Generally

$$\beta_{ij}(F_0,\ldots,F_q) = \sum_{u_1=1}^{q} \sum_{u_2=0}^{u_1-1} \cdots \sum_{u_{i+1}=0}^{u_i-1} \gamma_{j,|F_{u_1}\cup\cdots\cup F_{u_{i+1}}|,\{F_{u_1}\cup\cdots\cup F_{u_i}\cup F_s \mid s<u_{i+1}\}}$$

where

$$\gamma_{j,N,A} = \begin{cases} 1 & j = |N|, \text{some division properties related to elements of } A \\ 0 & \text{otherwise} \end{cases}$$