New examples of defective secant varieties of Segre-Veronese varieties

(joint work with M. C. Brambilla)

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Notation

- $V = (n+1)$-dimensional vector space over $\mathbb{C}$.
- $\mathbb{P}V$ = projective space of V.
- $[\mathbf{v}] \in \mathbb{P}V$ = equivalence class containing $\mathbf{v} \in V \setminus \{\mathbf{0}\}$.
- $S^dV = d^{\text{th}}$ symmetric power of V.
- $\langle X \rangle$ = linear span of $X \subseteq \mathbb{P}V$.

Secant varieties

- X = projective variety in $\mathbb{P}V$.
- Let $p_1, \ldots, p_s$ be generic points of X. Then $\langle p_1, \ldots, p_s \rangle$ is called a **secant $(s-1)$-plane** to X.
- The s^{th} **secant variety** of X is defined to be the Zariski closure of the union of secant $(s-1)$-planes to X:

$$\sigma_s(X) = \overline{\bigcup_{p_1, \cdots, p_s \in X} \langle p_1, \ldots, p_s \rangle}.$$

Secant dimension and secant defectivity

- A simple parameter count implies the following inequality holds:

$$\dim \sigma_s(X) \leq \min\{s \cdot (\dim X + 1) - 1,\ \dim \mathbb{P}V\}.$$

- If equality holds, we say X has the **expected dimension**.
- $\sigma_s(X)$ is said to be **defective** if it does not have the expected dimension.
- X is said to be **defective** if $\sigma_s(X)$ is defective for some s.

The Alexander-Hrschowitz theorem

- Let $v_d : \mathbb{P}V \to \mathbb{P}S^dV$ be the d^{th} Veronese map, i.e., v_d is the map given by $v_d([\mathbf{v}]) = [\mathbf{v}^d]$.
- Theorem (Alexander-Hirschowitz, 1995)

  $\sigma_s[v_d(\mathbb{P}V)]$ *is non-defective except for the following cases:*

$\dim \mathbb{P}V$	d	s
≥ 2	2	$2 \leq s \leq n$
2	4	5
3	4	9
4	3	7
4	4	14

Secant varieties of Segre-Veronese varieties

- $\mathbf{n} = (n_1, \ldots, n_k)$, $\mathbf{d} = (d_1, \ldots, d_k) \in \mathbb{N}^k$.
- $V_i = (n_i + 1)$-dimensional vector space.
- $\mathrm{Seg} : \prod_{i=1}^k \mathbb{P}V_i \to \mathbb{P}\left(\bigotimes_{i=1}^k V_i\right)$ = Segre map, i.e., the map given by $\mathrm{Seg}([\mathbf{v}_1], \ldots, [\mathbf{v}_k]) = [\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_k]$.
- $X_{\mathbf{n},\mathbf{d}} := \mathrm{Seg}\left(\prod_{i=1}^k v_{d_i}(\mathbb{P}V_i)\right) \hookrightarrow \mathbb{P}\left(\bigotimes_{i=1}^k S^{d_i} V_i\right)$ is called a Segre-Veronese variety.

Conjecturally complete list of defective two factor cases

n	**d**	s
(m,n) with $m \geq 2$	$(d,1)$	$\binom{m+d}{d} - m < s < \min\left\{\binom{m+d}{d}n + 1\right\}$
$(2,2k+1)$	$(1,2)$	$3k+2$
$(4,3)$	$(1,2)$	6
$(1,2)$	$(1,3)$	5
$(1,n)$	$(2,2)$	$n+2 \leq s \leq 2n+1$
$(2,2)$	$(2,2)$	$7,\ 8$
$(2,n)$	$(2,2)$	$\left\lfloor \frac{3n^2+9n+5}{n+3} \right\rfloor \leq s \leq 3n+2$
$(3,3)$	$(2,2)$	$14,\ 15$
$(3,4)$	$(2,2)$	19
$(n,1)$	$(2,2k)$	$kn+k+1 \leq s \leq kn+k+n$

This conjecture is based on:

- already existing results (by many people including E. Carlini and T. Geramita) and
- computational experiments that employ the so-called "Terracini lemma".
- Remark.
  - Terracini's lemma can be used to experimentally detect defective cases.
  - The result of a computation provides strong evidence, but it cannot be used as a rigorous proof of its deficiency.
  - Proving that experimentally determined defective secant varieties are actually defective requires more insight.

What about Segre-Veronese varieties with three or more factors?

Let $\mathbf{n} = (n_1, \cdots n_k), \mathbf{d} = (d_1, \ldots, d_{k-1}, 1) \in \mathbb{N}^k$. Then $(\mathbf{n}, \mathbf{d})$ is said to be unbalanced if

$$n_k \geq \prod_{i=1}^{k-1} \binom{n_i + d_i}{d_i} - \sum_{i=1}^{k-1} n_i + 1.$$

Let $(\mathbf{n}, \mathbf{d})$ be unbalanced. Then $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ is defective if and only if s satisfies the following:

$$\prod_{i=1}^{k-1} \binom{n_i + d_i}{d_i} - \sum_{i=1}^{k-1} n_i < s < \min \left\{ n_k + 1, \prod_{i=1}^{k-1} \binom{n_i + d_i}{d_i} \right\}$$

(Catalisano-Geramita-Gimigliano, 2008).

Defective cases for Segre-Veronese with $k \geq 3$ known before 2010 (modulo the unbalanced case)

$\mathbb{P}^{\mathbf{n}}$	$\mathbf{d}$	s
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$(1,1,2n)$	$2n+1$
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$	$(1,1,2)$	$2n+1$
$\mathbb{P}^n \times \mathbb{P}^1 \times \mathbb{P}^1$	$(1,1,n+1)$	$2n+1$
$\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^1$	$(1,1,2d)$	$\left\lceil \frac{(2d+1)(n+1)}{2} \right\rceil \leq s \leq dn+n+d$
$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$	$(1,1,2)$	11
$\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^2$	$(1,1,2)$	$3n+2$
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$(2,2,2)$	7
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$	$(2,2,2)$	11
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$	$(2,2,2)$	15
$\mathbb{P}^{2n+1} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$(1,1,1,n+1)$	$4n+3$
$\mathbb{P}^2 \times \mathbb{P}^5 \times \mathbb{P}^1 \times \mathbb{P}^1$	$(1,1,1,2)$	11

The main theorem (rough version)

- Theorem (A-Brambilla, 2010)

  Let $k \in \{3,4\}$, let $\mathbf{n} = (n_1, \ldots, n_k)$ and let $\mathbf{d} = (1, \ldots, 1, 2)$. Then there exist infinitely many defective secant varieties of $X_{\mathbf{n},\mathbf{d}}$, which were previously not known.

- Remark. The family we discovered includes some of the defective secant varieties listed one slide ago as special cases.

Defective cases known before 2010 revisited

$\mathbb{P}^{\mathbf{n}}$	$\mathbf{d}$	s
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$(1,1,2n)$	$2n+1$
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$	$(1,1,2)$	$2n+1$
$\mathbb{P}^n \times \mathbb{P}^1 \times \mathbb{P}^1$	$(1,1,n+1)$	$2n+1$
$\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^1$	$(1,1,2d)$	$\left\lceil \frac{(2d+1)(n+1)}{2} \right\rceil \leq s \leq dn+n+d$
$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3$	$(1,1,2)$	11
$\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^2$	$(1,1,2)$	$3n+2$
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$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$	$(2,2,2)$	11
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$	$(2,2,2)$	15
$\mathbb{P}^{2n+1} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$(1,1,1,n+1)$	$4n+3$
$\mathbb{P}^2 \times \mathbb{P}^5 \times \mathbb{P}^1 \times \mathbb{P}^1$	$(1,1,1,2)$	11

Outline of the proof

- Step 1. Find a non-singular subvariety C of $X_{\mathbf{n},\mathbf{d}}$ passing through s generic points.
- Step 2. Use C to provide an upper bound of $\dim \sigma_s(X_{\mathbf{n},\mathbf{d}})$:

$$\dim \sigma_s(X_{\mathbf{n},\mathbf{d}}) \leq s \cdot (\dim X_{\mathbf{n},\mathbf{d}} - \dim C) + \dim\langle C\rangle.$$

- Step 3. Find $(\mathbf{n}, \mathbf{d}, s)$ satisfying

$$s\cdot(\dim X_{\mathbf{n},\mathbf{d}} - \dim C) + \dim\langle C\rangle < \min\left\{ s \cdot (\dim X_{\mathbf{n},\mathbf{d}} + 1), \prod_{i=1}^{k} \binom{n_i + d_i}{n_i} \right\} - 1.$$

Example

- Let $n, d \in \mathbb{N}$, $a \in \{0, \cdots, \lceil n/d \rceil - 1\}$;
- $\mathbf{n} = (n, n+a, 1)$, $\mathbf{d} = (1, 1, 2d) \in \mathbb{N}^3$, and
- $s = (n+a+1)d + k$ for $\forall k \in \{1, \ldots, n - ad\}$.
- Then $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ is defective.
- Remark. This includes the following previously known example as a special case:

$\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^1$	$(1, 1, 2d)$	$\left\lceil \frac{(2d+1)(n+1)}{2} \right\rceil \leq s \leq dn + n + d$

The theorem now implies

$\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^1$	$(1, 1, 2d)$	$d(n+1) + 1 \leq s \leq dn + n + d$

Thank you very much for your attention!