Recent results on computability of certain asymptotic quantities

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Slides available at:
`http://www.math.unl.edu/~bharbourne1/VietnamSlides.pdf`

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Abstract

The last few years have seen a lot of work on asymptotic quantities with connections to algebraic geometry, commutative algebra and combinatorics, like resurgences, Waldschmidt constants and H-constants. I will review these quantities and these connections and survey some of this work, and highlight results and open problems regarding the computability of some of these constants.

Set up

K is an algebraically closed field, of arbitrary characteristic.

Fat point scheme: $Z = m_1 p_1 + \cdots + m_s p_s$ for $p_1, \ldots, p_s \in \mathbb{P}^n$.

So Z is scheme defined by the ideal $I(Z) = I(p_1)^{m_1} \cap \cdots \cap I(p_s)^{m_s}$.

Note: $I(p_i) \subset K[\mathbb{P}^n] = K[x_0, \ldots, x_n]$ is the ideal generated by all homogeneous F with $F(p_i) = 0$.

Ordinary powers: $I(Z)^r = \left(I(p_1)^{m_1} \cap \cdots \cap I(p_s)^{m_s}\right)^r$.

Symbolic powers: $I(Z)^{(m)} = I(mZ) = I(p_1)^{mm_1} \cap \cdots \cap I(p_s)^{mm_s}$.

Fundamental Question: How do symbolic powers relate to ordinary powers?

Containment Problem i (CPi)

CP1: Given fat points $Z \subset \mathbb{P}^n$, find all (m, r) with $I(mZ) \subseteq I(Z)^r$.

Facts: Assume $0 \neq Z \subset \mathbb{P}^n$.

- (easy) $I(Z)^r \subseteq I(mZ)$ iff $r \geq m$.
- (easy) $I(mZ) \subseteq I(Z)^r$ always fails for $m < r$.
- (hard) **Thm** (Ein-Lazarsfeld-Smith/Hochster-Huneke: ELS-HH, early 2000s)

$$I(mZ) \subseteq I(Z)^r \text{ always holds for } m \geq nr.$$

(In particular, $I(nrZ) \subseteq I(Z)^r$ always holds.)

CP2: What happens for $r \leq m < nr$?

A strategy

Strategy: Study how big can k be and still have $I(mZ) \subseteq I(Z)^r$ for $m \geq nr - k$.

Example: Take $K = \mathbb{C}$, $n = 2$, $k = 1$. Assume $m \geq nr - k = 2r - 1$.

- Failures of $I(mZ) \subseteq I(Z)^r$ are known when $r = 2$ (since 2013)
- No failures of $I(mZ) \subseteq I(Z)^r$ are known for $r > 2$.

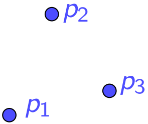
Conjecture (Grifo, 2018): Let $k \geq 0$. Then $I(mZ) \subseteq I(Z)^r$ for all $m \geq nr - k$ for $r \gg 0$.

Question: If true, how big must r be for containment to hold?

Example result

Theorem: Let $Z = p_1 + p_2 + p_3 \subset \mathbb{P}^2$, 3 points in the plane not on a line (so $n = 2$). Then Grifo Conjecture holds for Z, and $r > 3k/2$ suffices.

Z:



How can one prove this?

It's enough to compute an asymptotic quantity, the resurgence.

To do this, we'll need to compute another asymptotic quantity, the Wadschmidt constant.

The resurgence

Bocci-Harbourne, 2010: The resurgence of $I(Z)$ is defined to be

$$\rho(I(Z)) = \sup\left\{\frac{m}{r} : I(mZ) \not\subseteq I(Z)^r\right\}.$$

Comment: Thus $m/r > \rho(I(Z))$ implies $I(mZ) \subseteq I(Z)^r$.

Fact: Let $0 \neq Z \subset \mathbb{P}^n$ be fat points. Then

$$1 \leq \rho(I(Z)) \leq n.$$

Note: $1 \leq \rho(I(Z))$ is easy and $\rho(I(Z)) = 1$ happens (take $Z = p$); $\rho(I(Z)) \leq n$ is a corollary of ELS-HH Theorem.

Open Problem: Does $\rho(I(Z)) = n$ ever happen?

Application

Fact: If $\rho(I(Z)) < n$, then Grifo's conjecture holds for Z.

Proof: Assuming $\rho(I(Z)) < n$ and $m \geq nr - k$, we need to show for $r \gg 0$ that $I(mZ) \subseteq I(Z)^r$, so it's enough to show for $r \gg 0$ that

$$\frac{m}{r} > \rho(I(Z)).$$

But $\frac{m}{r} \geq \frac{nr-k}{r}$, while $\frac{nr-k}{r} > \rho(I(Z))$ simplifies to $r > \frac{k}{n - \rho(I(Z))}$.

Thus $I(mZ) \subseteq I(Z)^r$ holds for all $r > \frac{k}{n - \rho(I(Z))}$.

Example: Let $Z = p_1 + p_2 + p_3 \subset \mathbb{P}^2$, 3 points in the plane not on a line (so $n = 2$). **Claim**: $\rho(I(Z)) = 4/3 < n = 2$.

Thus Grifo's Conjecture holds for Z for $r > \frac{k}{n - \rho(I(Z))} = 3k/2$.

But why is $\rho(I(Z)) = 4/3$?

Computing resurgences

Question: How do you compute $\rho(I(Z))$?

In general it's not known! There are some special case results.

Theorem (Bocci-Harbourne, 2010):

$$\frac{\alpha(I(Z))}{\widehat{\alpha}(I(Z))} \leq \rho(I(Z)) \leq \frac{\mathrm{reg}(I(Z))}{\widehat{\alpha}(I(Z))}$$

$\alpha(I(Z))$: the degree of a nonzero element of $I(Z)$ of least degree.

$\mathrm{reg}(I(Z))$: Castelnuovo-Mumford regularity.

$\widehat{\alpha}(I(Z)) = \lim_{m\to\infty} \frac{\alpha(I(mZ))}{m}$ (the Waldschmidt constant of $I(Z)$)

Back to the example

Example: Let $Z = p_1 + p_2 + p_3 \subset \mathbb{P}^2$, 3 points in the plane not on a line (so $n = 2$). So why is $\rho(I(Z)) = 4/3$?

First it's easy to check that $\alpha(I(Z)) = 2 = \mathrm{reg}(I(Z))$. Hence

$$\frac{2}{\widehat{\alpha}(I(Z))} = \frac{\alpha(I(Z))}{\widehat{\alpha}(I(Z))} \leq \rho(I(Z)) \leq \frac{\mathrm{reg}(I(Z))}{\widehat{\alpha}(I(Z))} = \frac{2}{\widehat{\alpha}(I(Z))}.$$

Thus $\rho(I(Z)) = \dfrac{2}{\widehat{\alpha}(I(Z))}$ so now we just need to compute $\widehat{\alpha}(I(Z))$.

Computing Waldschmidt constants

Question: How can you compute $\widehat{\alpha}(I(Z))$?

In general it's not known! There are some special case results.

The following fact holds for all $m \geq 1$ (due to Waldchmidt and Skoda for $m = 1$ in 1979):

$$\frac{\alpha(I(mZ))}{m+n-1} \leq \widehat{\alpha}(I(Z)) \leq \frac{\alpha(I(mZ))}{m}.$$

This shows you can *in principle* compute $\widehat{\alpha}(I(Z))$ arbitrarily accurately just by computing $\alpha(I(mZ))$ for large enough m.

The upper bound is because $\frac{\alpha(I(mZ))}{m} \geq \frac{\alpha(I(tmZ))}{tm}$ for $t \geq 1$.

The lower bound uses a fact from ELS-HH:

Proof of lower bound

Fact (ELS-HH): For fat points $Z \subset \mathbb{P}^n$ we always have

$$I(r(m+n-1)Z) \subseteq I(mZ)^r.$$

$$(m = 1 \text{ gives the case from before, } I(nrZ) \subseteq I(Z)^r)$$

To show:

$$\frac{\alpha(I(mZ))}{m+n-1} \leq \widehat{\alpha}(I(Z))$$

Proof (Harbourne-Roé): Apply α to $I(r(m+n-1)Z) \subseteq I(mZ)^r$ to get

$$r\alpha(I(mZ)) = \alpha(I(mZ)^r) \leq \alpha(I(r(m+n-1)Z)).$$

Divide by $r(m+n-1)$ and take $\lim_{r\to\infty}$:

$$\frac{\alpha(I(mZ))}{m+n-1} \leq \frac{\alpha(I(r(m+n-1)Z))}{r(m+n-1)} \to \widehat{\alpha}(I(Z)).$$

Chudnovsky-Demailly Conjecture

Conjecture (1981): $\dfrac{\alpha(I(mZ)) + n - 1}{m + n - 1} \leq \widehat{\alpha}(I(Z))$

Chudnovsky proves it for Z reduced, $n = 2$, $m = 1$; thus in this case we get for all $t \geq 1$ that

$$\frac{\alpha(I(Z)) + 1}{2} \leq \widehat{\alpha}(I(Z)) \leq \frac{\alpha(I(tZ))}{t}.$$

Now for Z being 3 noncollinear points in the plane and $t = 2$ we have

$$\frac{2+1}{2} = \frac{\alpha(I(Z)) + 1}{2} \leq \widehat{\alpha}(I(Z)) \leq \frac{\alpha(I(tZ))}{t} = \frac{3}{2}.$$

So $\widehat{\alpha}(I(Z)) = 3/2$, hence $\rho(I(Z)) = 2/\widehat{\alpha}(I(Z)) = 4/3$.

Open Problems

- Compute $\widehat{\alpha}(I(Z))$ for $Z = p_1 + \cdots + p_s \subset \mathbb{P}^n$, $s \gg 0$, p_i generic.
- **Nagata Conj.** (1959): $n = 2$, $s > 9$, p_i generic $\Rightarrow \widehat{\alpha}(I(Z)) = \sqrt{s}$
- **Chudnovsky-Demailly Conj.**: $\dfrac{\alpha(I(mZ)) + n - 1}{m + n - 1} \leq \widehat{\alpha}(I(Z))$
- **Grifo Conj.**: Let $k \geq 0$. Then $I((nr - k)Z) \subseteq I(Z)^r$ for $r \gg 0$.
- Does $\rho(I(Z)) = n$ ever happen?

Other recent work relates to combinatorics

• H.T. Ha and N.V. Trung (AMV 2019, arXiv:1808.05899, in article dedicated to 60th birthday of L.T. Hoa) prove:

Theorem. Let I be a squarefree monomial ideal. Then $\rho(I) \leq \omega(I)$, where $\omega(I)$ is the degree of a generator of maximal degree in a minimal set of generators for I.

• C. Bocci, S. Cooper, E. Guardo, B. Harbourne, M. Janssen, U. Nagel, A. Seceleanu, A. Van Tuyl, T. Vu (JACo 2016 arXiv:1508.00477)

Theorem. For a hypergraph H with a nontrivial edge,

$$\widehat{\alpha}(I(H)) = \frac{\chi^*(H)}{\chi^*(H) - 1}$$

where $\chi^*(H)$ is the fractional chromatic number of H.

Additional recent work

• M. DiPasquale, C.A. Francisco, J. Mermin, J. Schweig (TAMS 2019, arXiv:1808.01547) Shows $\widehat{\rho}(I)$ is the maximum of finitely many ratios involving Waldschmidt-like constants. This reduces computing $\widehat{\rho}(I)$ to computing Waldschmidt-like constants and thus gives an algorithm in some cases.

Note: Guardo-H__-Van Tuyl (2013):

$$\widehat{\rho}(I(Z)) = \sup\left\{ \tfrac{m}{r} : I(mtZ) \not\subseteq I(Z)^{rt},\ t \gg 0 \right\}$$

• S. Tohaneanu, Y. Xie (arXiv:1903.10647).

Theorem: If $0 \neq Z \subset \mathbb{P}^n$ is a reduced point scheme, then

$$\rho(I(tZ)) \leq \frac{t+n-1}{t}.$$

Note: Thus, for $n, t > 1$ and $0 \neq Z \subset \mathbb{P}^n$ a reduced point scheme, we have $\rho(I(tZ)) < n$, so Grifo's conjecture holds for $I(tZ)$.

H-constants and Bounded Negativity

Let $F \in \mathbb{C}[x, y, z]$ be homogeneous, square free of degree d.

Thus $F = 0$ defines a reduced plane curve C.

Let $p_1, \ldots, p_s$ be the singular points of C, $m_i = \mathrm{mult}_C(p_i)$.

Fundamental Question (arXiv:1407.2966): How singular can C be? More precisely, when $s > 0$, how negative can $H(C)$ be, where

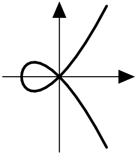
$$H(C) = \frac{d^2 - \sum_i m_i^2}{s}?$$

Let $$H_{\mathbb{P}^2} = \inf_{\substack{\text{reduced, singular} \\ \text{plane curves } C}} H(C).$$

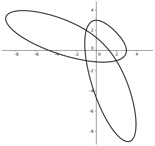
Bounded Negativity Problem: Is $H_{\mathbb{P}^2} = -\infty$?

H-constants for C irreducible

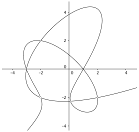
Example: For each d there is an irreducible plane curve C_d with $\deg(C_d) = d$ with $\binom{d-1}{2}$ double points (take a general map of $\mathbb{P}^1$ into $\mathbb{P}^2$).



$d = 3$: 1 node



$d = 4$: 3 nodes



$d = 5$: 6 nodes

$$\text{Thus } H(C_d) = \frac{d^2 - 4\binom{d-1}{2}}{\binom{d-1}{2}} = -2 + \frac{6d-4}{(d-1)(d-2)} \overset{d\to\infty}{\longrightarrow} -2.$$

Open Problem: Does there exist irreducible C with $H(C) \leq -2$?

H-constants for C a union of lines

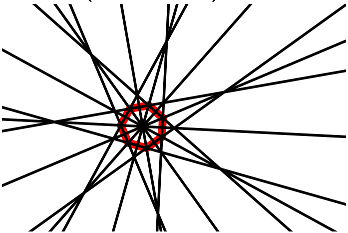
Theorem (arXiv 1407.2966): Let L be a real line arrangement. Then $H(L) > -3$. Moreover, there is a sequence $L_3, L_5, L_7, \ldots$ of real line arrangements such that $H(L_n) \overset{n\to\infty}{\longrightarrow} -3$.

Proof: $H(L) > -3$ follows from a combinatorical result of E. Melchior saying for a nontrivial real line arrangement that

$$t_2 \geq 3 + \sum_{k\geq 3}(k-3)t_k,$$

where t_k is the number of points where exactly k lines cross.

For L_n, take the sides and lines of symmetry of a regular n-gon for n odd (here is L_7):



$d = 2n$

$t_2 = n$ points of multiplicity 2

$t_3 = \binom{n}{2}$ points of multiplicity 3

$t_n = 1$ point of multiplicity n

$$H(L_n) = \frac{d^2 - \sum_{k\geq 2} t_k k^2}{\sum_{k\geq 2} t_k}$$

$$= -3 + \epsilon_n \overset{n\to\infty}{\longrightarrow} -3.$$

Analogous result over $\mathbb{C}$

Theorem (arXiv:1407.2966): Let L be a complex line arrangement. Then $H(L) > -4$.

arXiv:1407.2966: T. Bauer, S. Di Rocco, B. Harbourne, J. Huizenga, A. Lundman, P. Pokora, T. Szemberg: *Bounded Negativity and Arrangements of Lines*, International Math. Res. Notices (2015) [Note: IMRN version has many improvements over arXiv version.]

Open Problem: Suppose L is a line arrangement defined over $\mathbb{C}$. How close to -4 can $H(L)$ be? (Most negative currently known example has $H(L) = -\frac{225}{67} \approx -3.36$.)

Open Problem: Suppose L is a line arrangement defined over $\mathbb{Q}$. How negative can $H(L)$ be? (Most negative currently known example has $H(L) = -\frac{503}{181} \approx -2.779$.)

Another open problem!

These examples of maximally negative known $H(L)$ are very special.

The example with $H(L) = -\frac{225}{67} \approx -3.36$ is called the Wiman arrangement. It has $t_2 = 0$. Only 4 kinds of line arrangements with $t_2 = 0$ are known.

Open Problem: Are there any others? If not, why not?

The 4 known kinds:

(1) $d \geq 3$ concurrent lines

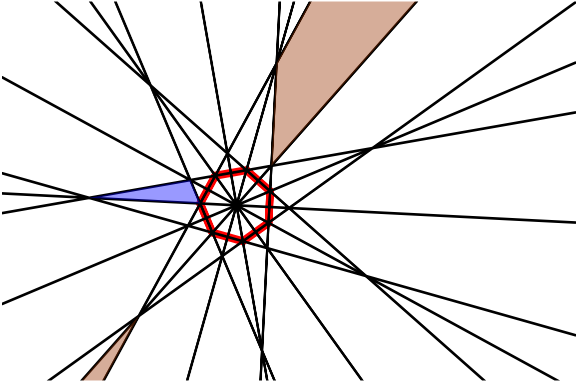
(2) the $d = 3t$ linear factors in $(x^t - y^t)(x^t - z^t)(y^t - z^t)$, $t \geq 3$

(3) Klein arrangement (1879): $d = 21$ lines, $t_3 = 28$, $t_4 = 21$

(4) Wiman arr. (1896): $d = 45$ lines, $t_3 = 120$, $t_4 = 45$, $t_5 = 36$

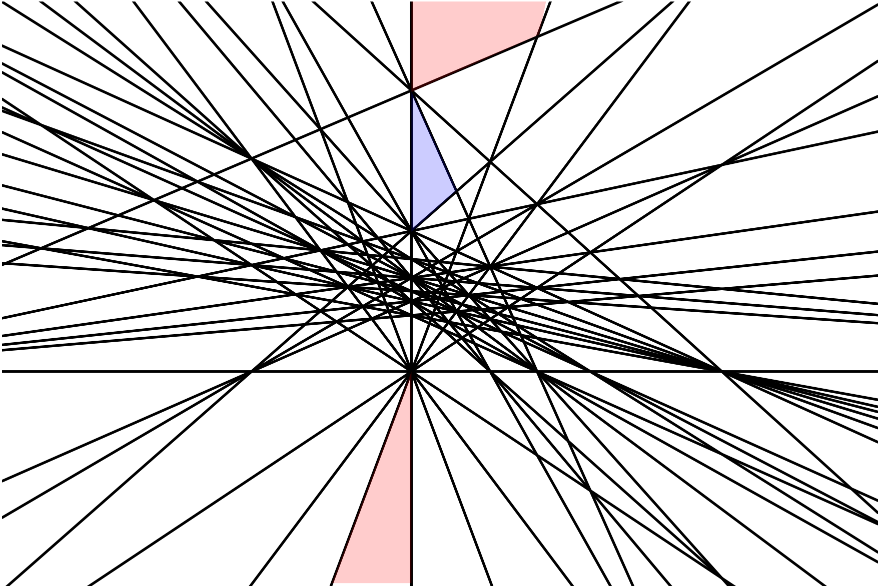
Real simplicial line arrangements

The real L_n with $H(L_n) \to -3$ and the rational L with $H(L) = -\frac{503}{181} \approx -2.779$ are simplicial (which means they triangulate $\mathbb{P}^2_{\mathbb{R}}$), but there are very few known rational simplicial line arrangements (see Grünbaum, 2009, Cuntz arXiv:1108.3000v1).



The rational arrangement L with $H(L) = \frac{-503}{181} \approx -2.779$

This arrangement also is simplicial and has $d = 37$ lines:



Another view

The same arrangement but with one line moved off to infinity: