Primer on unexpected hypersurfaces

Brian Harbourne

Department of Mathematics
University of Nebraska-Lincoln

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Primer on unexpected hypersurfaces: Abstract

Abstract: The topic of unexpected hypersurfaces, defined in the case of curves in $\mathbf{P}^2$ by [CHMN: arXiv:1602.02300]
(Cook, Harbourne, Migliore and Nagel,
Line arrangements and configurations of points with an unexpected geometric property, Compositio (2018) v. 154),
is a rapidly expanding field which brings together aspects of algebraic geometry, commutative algebra, representation theory and computation.

The paper [HMNT: arXiv:1805.10626]
(Harbourne, Migliore, Nagel and Teitler,
Unexpected hypersurfaces and where to find them, preprint 2018),
is a first step to pushing this work to $\mathbf{P}^N$.

Primer on unexpected hypersurfaces

Open Problem: What is the Hilbert function $h_{I^{(m)}}(t)$ of symbolic powers of the ideal $I \subset R = K[\mathbf{P}^N]$ of general points $p_1, \ldots, p_s \in \mathbf{P}^N$?

Recall: $I(p_i) =$ ideal generated by forms vanishing at p_i

$$I = I(p_1) \cap \cdots \cap I(p_s)$$

$$I^{(m)} = I(p_1)^m \cap \cdots \cap I(p_s)^m$$

$$[I^{(m)}]_t = \text{vector space spanned by forms of deg } t$$

$$\text{vanishing to order} \geq m \text{ at each point } p_i$$

$$h_{I^{(m)}}(t) = \dim[I^{(m)}]_t$$

Lower Bound

More generally let $X = m_1 p_1 + \cdots + m_s p_s$ and consider

$$I(X) = I(p_1)^{m_1} \cap \cdots \cap I(p_s)^{m_s}.$$

$$(*) \qquad h_{I(X)}(t) \geq \max\left(0, \binom{N+t}{N} - \sum_i \binom{N+m_i-1}{N}\right)$$

Note:

$$\dim R_t = \binom{N+t}{N}$$

$$\binom{N+m_i-1}{N} = \dim R_t/[I(p_1)^{m_1}]_t \text{ (for } t \geq m_i)$$

$$= \text{conditions imposed on } R_t$$

for $t \geq m_i$ by vanishing on $m_i p_i$

Lower Bound Problems

Easy Exercise: Show

$$h_{I(X)}(t) = \max\left(0, \binom{N+t}{N} - \sum_i \binom{N+m_i-1}{N}\right)$$

for $N = 1$ and any s distinct points $p_i \in \mathbf{P}^{N=1}$.

Open Problem: For $N > 1$ understand when

$$(**) \qquad h_{I(X)}(t) > \max\left(0, \binom{N+t}{N} - \sum_i \binom{N+m_i-1}{N}\right)$$

holds for s general points p_i.

Formal examples with $N = 2$

Formally allow m_i to possibly be negative.

Define $\binom{m+1}{2} = \frac{m^2+m}{2}$ even when $m < 0$.

Define $I(p_i)^{m_i} = R$ when $m_i \leq 0$.

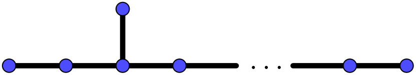
Note that $m_i p_i$ imposes no conditions when $m_i < -1$
(since $I(m_i p_i) = R$ when $m_i \leq 0$) but $\binom{m_i+1}{2} > 0$ when $m_i < -1$.
Hence it is easy to get

$$(**) \qquad h_{I(X)}(t) > \max\left(0, \binom{2+t}{2} - \sum_i \binom{m_i+1}{2}\right)$$

if we let some $m_i < -1$.

$N = 2$ and the Weyl group

There is a group action (induced by Cremona transformations of $\mathbf{P}^2$) of the Weyl group W_s of the s dot Dynkin diagram



on vectors $(t, m_1, \ldots, m_s) \mapsto (t', m_1', \ldots, m_s')$ such that

$$\max\left(0, \binom{2+t}{2} - \sum_i \binom{m_i+1}{2}\right) = \max\left(0, \binom{2+t'}{2} - \sum_i \binom{m_i'+1}{2}\right)$$

and

$$h_{I(X)}(t) = h_{I(X')}(t')$$

where $X = \sum_i m_i p_i$ and $X' = \sum_i m_i' p_i'$.

$N=2$ and the Weyl group

In some cases we have $t>0$, $m_i>0$ for all i but $m_i'<-1$ for some i with

$$h_{I(X')}(t') = \binom{2+t'}{2} - \sum_{m_i'>0} \binom{m_i'+1}{2} > 0$$

hence

$$h_{I(X)}(t) = h_{I(X')}(t') = \binom{2+t'}{2} - \sum_{m_i'>0} \binom{m_i'+1}{2}$$

$$> \max\left(0, \binom{2+t'}{2} - \sum_i \binom{m_i'+1}{2}\right)$$
$$= \max\left(0, \binom{2+t}{2} - \sum_i \binom{m_i+1}{2}\right).$$

$N = 2$ and the SHGH Conjecture

The SHGH Conjecture is essentially the converse:

SHGH Conjecture: If

$$h_{I(X)}(t) > \max\left(0, \binom{2+t}{2} - \sum_i \binom{m_i+1}{2}\right)$$

then there is a $w \in W_s$ such that $w(t, m_1, \ldots, m_s) = (t', m'_1, \ldots, m'_s)$ with $m'_i < -1$ for some i.

One thing making SHGH hard to prove is that sets of general points don't have much structure.

To generalize the problem and add structure suppose we allow some of the points not to be general, as Juan discussed in his talk.

Simplest Generalization

$X = mp$ for a general point $p \in \mathbf{P}^N$

$Z = q_1 + \cdots + q_r$ for fixed points $q_i \in \mathbf{P}^N$

Then $h_{I(X+Z)}(t) \geq \max\left(0, \dim[I(Z)]_t - \binom{N+m-1}{N}\right)$ and (**) becomes:

Understand when

$$(***) \qquad h_{I(X+Z)}(t) > \max\left(0, \dim[I(Z)]_t - \binom{N+m-1}{N}\right).$$

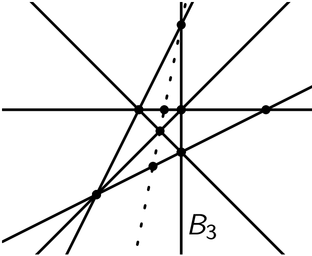
Open Problem: Classify (N, m, Z, t) such that (**) is strict.

Partial Answers: 1

[DMO] Di Marca, Malara, Oneto, *Unexpected curves arising from special line arrangements*, preprint, arXiv:1804.02730: classifies all $(N = 2, m, t = m + 1)$ such that a Z exists for which (**) is strict, in case the lines dual to the points of Z give a supersolvable line arrangement:

Theorem [DMO] Assume the maximum number of collinear points in Z is d and that the lines dual to Z are supersolvable. Then Z admits an unexpected curve of degree d with a general point of multiplicity $m = d - 1$ if and only if $|Z| > 2d$.

The lines dual to Z are *supersolvable* if there is a subset C of collinear points in Z and every line through two of the points of Z contains one of the points of C. The 9 points at right have supersolvable dual; C is the set of 4 points on the dotted line. Also, here $d = 4$ and $|Z| = 9 > 2d$.

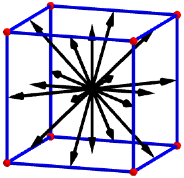


Partial Answers: 2

- Notes: By [DMO], the B_3 configuration of points has an unexpected quartic curve.
- By [FGST] (Farnik, Galuppi, Sodomaco and Trok, *On the unique unexpected quartic in* $\mathbf{P}^2$, preprint arXiv:1804.03590, 2018), B_3 is the unique configuration of points having an unexpected curve of degree 4 or less.
- [BMSS] (Bauer, Malara, Szemberg and Szpond, *Quartic unexpected curves and surfaces.*, Manuscripta Math. (2018) arXiv:1804.03610) pushed this work to higher dimensions, finding the first unexpected hypersurfaces in $\mathbf{P}^3$.
- As Juan mentioned: [HMNT] goes a step further, using mainly cone constructions to classify all (N, m, t) such that a Z exists for which (***) is strict.
- Juan, Halszka Tutaj-Gasińska and I have results replacing the general point P and the points of Z by higher dimensional linear varieties.

More representation theory

[HMNT] also finds additional connections to representation theory: The B_3 configuration of points Z is the projectivization in $\mathbf{P}^2_{\mathbb{R}}$ of the 18 roots of the B_3 root system.



The 18 points in $\mathbb{R}^3$ of the B_3 root system.

GeoGebra: The B3 root system

Projectivizing $B_{n+1} \subset \mathbb{R}^{n+1}$ gives a set Z of $(n+1)^2$ points. Computer checking suggests that Z has an unexpected hypersurface of degree $d = 4$ with a general point of multiplicity $m = 4$ for each $n > 2$.