# The concept of geproci subsets of $\mathbb{P}^3$: a timeline

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Algebra Seminar

December 1, 2021

**Abstract**: Interest in geproci sets grew out of work on unexpected hypersurfaces. We define the notion of a geproci set, we give examples and we discuss some recent results, all in the context of a timeline of relevant events.

Slides will be available at my website:
`https://www.math.unl.edu/~bharbourne1/`

# Timeline (in years before present)

$t = -139$ Emmy Noether, born 1882

$t = -95$ Grete Hermann, Noether's 1st student receives PhD (her 1926 thesis laid foundation for computer algebra)

$t = -89$ John von Neumann, proved impossibility of hidden variables in quantum mechanics in 1932

$t = -55$ John Stewart Bell showed von Neumann's proof did not show what was claimed (Bell's 1966 Theorem)

$t = -47$ In 1974 Max Jammer pointed out Hermann had in 1935 already raised the issue Bell addressed (but was largely ignored)

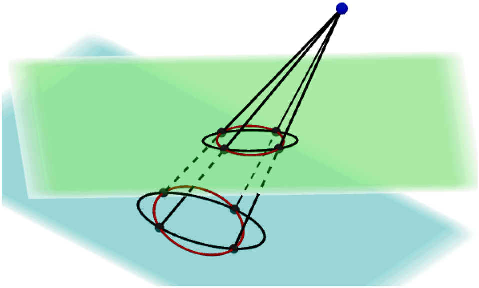
$t = -10$ A question is posted on Math Overflow.

# $t = -10$: A question (6-8-2011).

A "general projection" means projection from a general point.

Let $Z \subset \mathbb{P}^3$ be a finite set of points. We say $Z$ is $(a, b)$-GEPROCI if its GEneral PROjection to a plane is a Complete Intersection of curves of degrees $a$ and $b$ (with $a \leq b$).

Trivial example: A complete intersection (CI) of two curves in the same plane projects isomorphically to its image, so is trivially a CI.

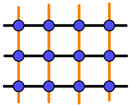


Mathoverflow Quest. 67265 by Francesco Polizzi: Are there nontrivial geproci sets $Z$?

# $t = -9.99988$: Answer by Dmitri Panov (6-8-2011): Grids!

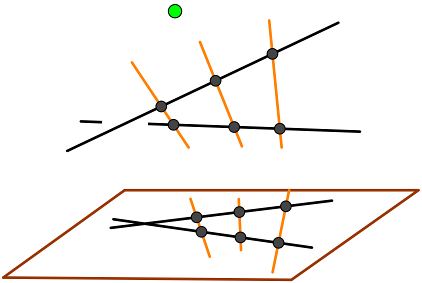
An $(a, b)$-*grid* is $(a, b)$-geproci. What is an $(a, b)$-grid?

It is given by $a$ skew black lines and $b$ skew orange lines, such that each black line meets each orange line in one point. The $ab$ points form the grid. The lines are called *grid lines*.

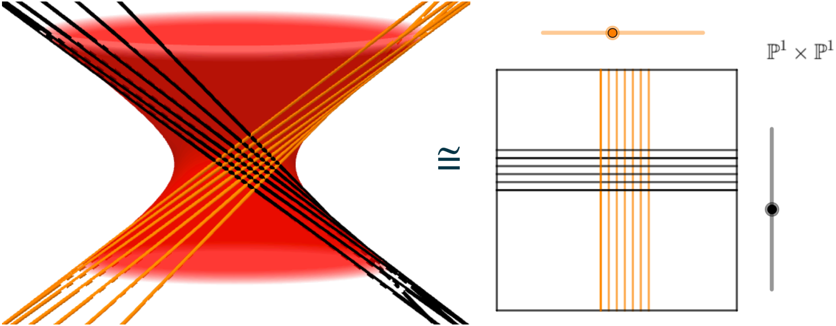


A (3, 4)-grid

Construction is easy when $a = 2$. Here's one with $(a, b) = (2, 3)$.



# What about $a > 2$?




They come from the two rulings on a smooth quadric! And every grid with $2 < a \leq b$ works this way.

# $t = -9.997$: Question edit (6-9-2011)

Based on Panov's construction, Polizzi edited his question:

(1) Are there nontrivial nongrid geproci sets?

(2) Can we classify them (at least for small numbers of points)?

Unexpectedly the answer to both questions is Yes, based on work on unexpected hypersurfaces:

t=-5 CHMN: Cook, H___, Migliore and Nagel introduced the notion of "unexpected curves" (preprint: arXiv:1602.02300; appeared as *Line arrangements and configurations of points with an unexpected geometric property*, Compositio Math. 154:10 (2018) 21502194).

t=-3.5 HMNT: H___, Migliore, Nagel and Teitler extended unexpectedness to hypersurfaces (preprint: arXiv:1805.10626; appeared as *Unexpected hypersurfaces and where to find them*, Mich. Math. J., 2021).

# Recall: unexpected hypersurfaces.

$Z \subset \mathbb{P}^n$: a finite set of points.

$V(Z,t)$: the vector space of forms of degree $t$ vanishing on $Z$.

$d_{Z,t} = \dim V(Z,t)$.

For $F \in V(Z,t)$ to vanish to order $m$ at a point $P$, the $\binom{m+n-1}{n}$ partials of $F$ of order $m-1$ must vanish at $P$.

We say $Z$ has "unexpected hypersurfaces" of degree $t$ with a general point of multiplicity $m$ if for a general point $P$ there is a nonzero $F \in V(Z,t)$ vanishing to order $m$ at $P$ even though $d_{Z,t} \leq \binom{m+n-1}{n}$.

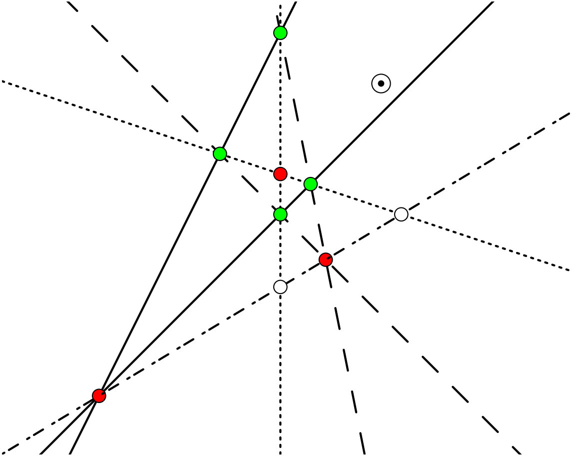
When $m = t$, the unexpected hypersurfaces are cones.

But where should you look for such $Z$?

# The simplest example: a set $Z$ of 9 points in $\mathbb{P}^2$.

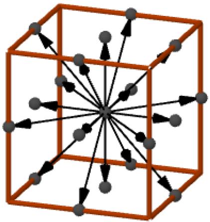
Here for a general point $P$ there is a unique unexpected quartic (so $t = 4$) through the 9 points of $Z$, singular with multiplicity 3 at $P$ (so $m = 3$). We get:

$d_{Z,t} = \dim V(Z,t) = 6$, $\binom{m+n-1}{n} = \binom{3+2-1}{2} = 6$ so $d_{Z,t} \leq \binom{m+n-1}{n}$.



# Another perspective leading to other examples.

These 9 points are the projectivization of the $B_3$ root system:



Here is a 3D view.

HMNT gives additional examples of point sets $Z_R$ with unexpected hypersurface in various $\mathbb{P}^n$ coming from root systems $R$.

Some of them are cones.

# The HMNT examples of unexpected cones

$R = B_{n+1}$: $Z_R \subset \mathbb{P}^n$ has unexpected cones of degree 4 for $n = 3, 4$.

$R = B_{n+1}$: $Z_R \subset \mathbb{P}^n$ has unexpected cones of degrees 3, 4 for $n = 5, 6$.

$R = D_4$: $Z_R \subset \mathbb{P}^3$ has unexpected cones of degrees 3 and 4.

$R = E_7$: $Z_R \subset \mathbb{P}^6$ has unexpected cones of degree 4.

$R = E_8$: $Z_R \subset \mathbb{P}^7$ has unexpected cones of degrees 4 and 5.

$R = F_4$: $Z_R \subset \mathbb{P}^3$ has unexpected cones of degrees 4, 5, 6 and 7.

$R = H_3$: $Z_R \subset \mathbb{P}^2$ has unexpected curves of degrees 6, 7 and 8.

$R = H_4$: $Z_R \subset \mathbb{P}^3$ has unexpected cone of degree 6 (later P. Fraś, M. Zięba, arXiv:2107.08107, showed it had another one of degree 10).

# $t = -3$: Workshop at Levico Terme in 2018

A working group at Levico Terme noticed something interesting for some of these $Z_R$ in $\mathbb{P}^3$:

$R = D_4$: $|Z_R| = 12$ has unexpected cones of degrees 3 and 4.

$R = F_4$: $|Z_R| = 24$ has unexpected cones of degrees 4 and 6.

**Fact** (Workshop working group at Levico Terme, 2018): Let $Z \subset \mathbb{P}^3$ be a finite set of points. If $|Z| = ab$ has unexpected cones of degrees $a$ and $b$ with no components in common, then $Z$ is $(a, b)$-geproci.

The Levico workshop led to:

CM: Chiantini, Migliore, "Sets of points which project to complete intersections," TAMS 374 (2021) 2581–2607 (arXiv:1904.02047).

(Results of the working group are written up in the appendix of CM.)

# The 2018 Levico Terme working group

Alessandra Bernardi

Luca Chiantini

Graham Denham

Giuseppe Favacchio

Brian Harbourne

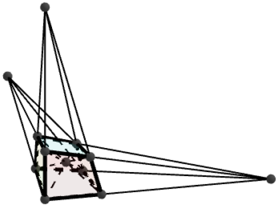
Juan Migliore

Tomasz Szemberg

Justyna Szpond

# Let's look at $R = D_4$ and $R = F_4$.

$Z_{D_4}$ has 12 points and is $(3,4)$-geproci . The 12 points come from a cube in 3 point perspective.



The quartic cone is easy to see. It is the cone with vertex $P$ on 4 skew lines containing $Z_{D_4}$.

The cubic cone comes from a pencil defined by two cubic cones. Here are the two cubic cones. And here is the pencil of cubic cones).

$Z_{F_4}$ is the 24 intersection points of the $\binom{8}{2} = 28$ lines through pairs of vertices of a cube. It is $(4,6)$-geproci and contains $Z_{D_4}$.

# $t = -1$: More examples and a start on classification

We say a geproci $Z$ is a half-grid if it is not a grid but one of the CI curves in its general projection can be taken to be a union of lines.

(1) Examples announced at an MFO workshop in October, 2020:

**Example** (P. Fraś, M. Zięba, arXiv:2107.08107): $Z_{H_4}$ is a nontrivial, nongrid, non-half-grid $(6, 10)$-geproci.

**Example** (P. Pokora, T. Szemberg, J. Szpond, arXiv:2010.08863): A 60 point set due to Klein is a nontrivial $(6, 10)$-geproci half-grid.

(2) The Chiantini-Migliore paper also gave results on classification:

**Theorem** (CM) All nontrivial nongrid geproci sets have at least 12 points (because nontrivial $(a, b)$-geproci sets with $2 = a \leq b$ or $a = b = 3$ are grids).

And new insights on grids:

**Theorem** (CM): Any $(a, b)$-grid with $ab > 4$ has unexpected cones of degrees $a$ and $b$.

# $t < -1$: The Geproci Squad, results and questions

Levico and the 10-2020 MFO workshop led to forming the Geproci Squad to work on geproci questions. Here is some work in progress.

(1) **Theorem** (Geproci Squad): Given $4 \leq a \leq b$, there is a nontrivial half-grid $(a, b)$-geproci set $Z \subset \mathbb{P}^3$.

(2) **Theorem** (Geproci Squad) $Z_{D_4}$ is the unique nontrivial nongrid $(3, b)$-geproci set.

**Some Questions**:

(Q1) Which $a, b$ have a unique nontrivial nongrid $(a, b)$-geproci $Z$? Is $a = 3, b = 4$ (i.e., $Z_{D_4}$) the only one?

(Q2) We know trivial geproci sets and $(2, b)$-grids ($b \geq 2$) do not come from unexpected cones. Do all other $(a, b)$-geproci $Z$ come from unexpected cones of degrees $a$ and $b$?

(Q3) Is $Z_{H_4}$ the only nontrivial nongrid non-half-grid geproci set?

# $t = -.0833$: Late breaking news! (11-3-2021)

Some time ago Squad member Giuseppe Favacchio ran across the fact that $Z_{D_4}$ had been used in giving proofs of Bell's Theorem.

**November 3, 2021**: So Giuseppe searched further and found $Z_{F_4}$ and $Z_{H_4}$ also had been used in giving proofs of Bell's Theorem. And moreover, yet another set of points, based on the Penrose Dodecahedron, was used to prove Bell's Theorem. It's a 40 point set which also turned out to be geproci: it is a nontrivial, nongrid, non-half grid $(5, 8)$-geproci set.

Thus the answer to (Q3) is: $Z_{H_4}$ is not the only nontrivial nongrid non-half-grid geproci set!

Revised question (Q3): For which $a$ and $b$ is there a nontrivial nongrid non-half-grid $(a, b)$-geproci?

New larger question (Q4): What exactly is the connection to Quantum Mechanics?

# The Geproci Squad

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