The role of line arrangements in some open problems in algebraic geometry

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MFO mini-workshop on Seshadri constants

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Slides available at:
`http://www.math.unl.edu/~bharbourne1/MFO2019Slides.pdf`

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Problem 1: Computability/rationality of multipoint Seshadri Constants

- $Z \subset \mathbb{P}^2$ finite set of points
- $\pi_Z : X_Z \to \mathbb{P}^2$ blow up of Z, E_Z the exceptional locus
- $L' = \pi_Z^* L$ for general line $L \subset \mathbb{P}^2$
- $\varepsilon(Z) = \sup\{t : L' - tE_Z \textit{ is nef}\}$

Question: Let $C \subset \mathbb{P}^2$ be reduced curve, $Z = Z_C = \mathrm{Sing}(C)$, $D_C \subset X$ the proper transform of C:

If $D_C^2 < -1$, must $\varepsilon(Z)$ be rational? Can we compute it?

Test case: take C to be a supersolvable (to be defined below) line arrangement.

Problem 2: Same problem for Waldschmidt constants $\widehat{\alpha}$

Recall: $\widehat{\alpha}(Z) = \inf\{t/m : tL' - mE_Z \textit{ is effective}\}$

Note: $\widehat{\alpha}(Z) \geq |Z|\varepsilon(Z)$.

Test case: take C to be a line arrangement.

Sample open case: take C to be Klein's arrangement of 21 lines.

See: Negative curves on symmetric blowups of the projective plane, resurgences and Waldschmidt constants (IMRN, 2018)

Thomas Bauer, Sandra Di Rocco, Brian Harbourne, Jack Huizenga, Alexandra Seceleanu, Tomasz Szemberg.

Problem 3: Bounded Negativity

Let C be a reduced singular plane curve.

- How negative can $H(C) = D_C^2/|Z_C|$ be?

Facts:

(a) $\inf_C H(C) \leq -2$ (inf taken over reduced, irreducible C, arbitrary char)

(b) $\inf_C H(C) = -3$ (inf taken over real line arrangements C)

(c) $\inf_C H(C) \geq -4$ (inf taken over complex line arrangements C)

See: Bounded Negativity and Arrangements of Lines (IMRN, 2015)

Thomas Bauer, Sandra Di Rocco, Brian Harbourne, Jack Huizenga, Anders Lundman, Piotr Pokora, Tomasz Szemberg.

Problem 3: Bounded Negativity Open Questions

- If C is irreducible (arbitrary char) must we have $H(C) > -2$?

Taking C to be a general image of $\mathbb{P}^1$ in $\mathbb{P}^2$ of degree d gives

$$H(C) = -2 + \frac{6d-4}{(d-1)(d-2)}.$$

- If C is a complex line arrangement must we have $H(C) \geq H(C_W) = -\frac{225}{67} \approx -3.36$, where C_W is Wiman's arrangement of 45 lines?

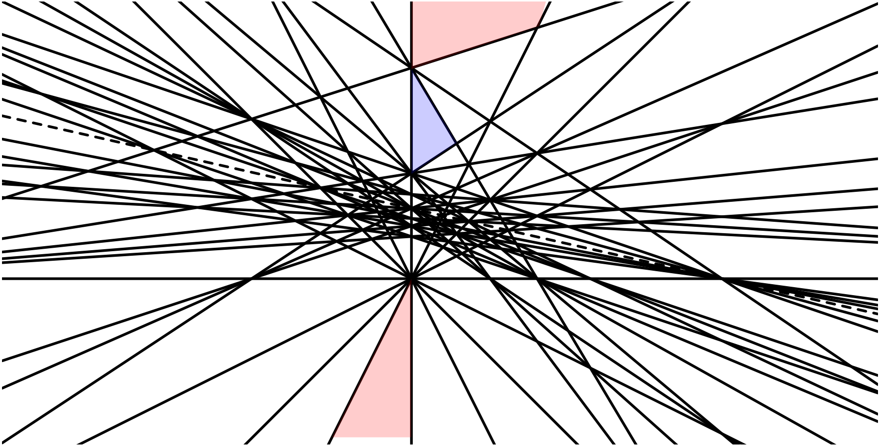
C_W has 120 triple, 45 quadruple and 36 quintuple points.

- What is $\inf_C H(C)$ for rational line arrangements C?

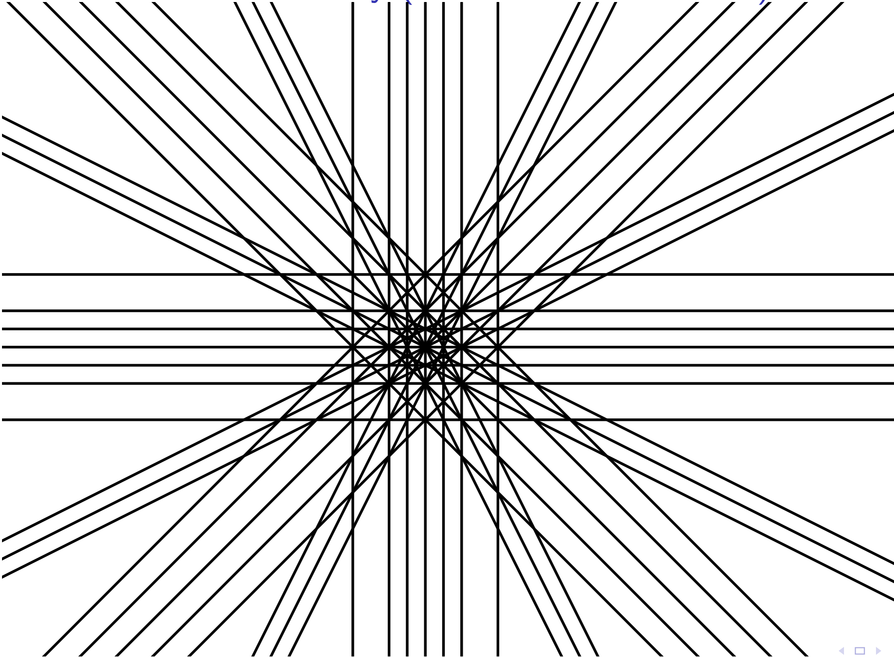
There is a rational C with $H(C) = \frac{-503}{181} \approx -2.779$.

The most negative known rational line arrangement

$d = 37$ lines, 72 double points, 72 triple points, 24 quadruple points, 10 sextuple points, 3 octuple points and $H = \frac{-503}{181} \approx -2.779$:



Same lines drawn differently (dashed line now at ∞)



Problem 4: The Containment Problem (simplest version)

Let $Z \subset \mathbb{P}^2$ be a finite set.

$I_Z^{(m)} = \cap_{p \in Z} I_p^m$.

- Classify Z with $I_Z^{(3)} \not\subseteq I_Z^2$.

Note: All known complex examples have been found by taking $Z \subset \mathrm{Sing}(C)$ for certain line arrangements C.

First known case: C is Dual Hesse (Marcin Dumnicki, Tomasz Szemberg, Halszka Tutaj-Gasińska (2013)).

Problem 5: Unexpected curves

Let $Z \subset \mathbb{P}^2$ be a finite set, $p \in \mathbb{P}^2$ a general point, $Z' = Z \cup \{p\}$.

- Classify all (Z, m) with

$$h^0(X_{Z'}, (m+1)L' - mE_p - E_Z) >$$

$$\max\left(0, h^0(X_{Z'}, (m+1)L' - E_Z) - \binom{m+1}{2}\right).$$

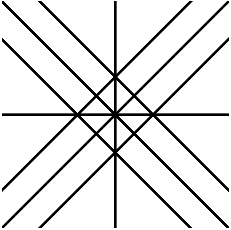
Note: The main technique currently uses properties of the line arrangement dual to Z.

Example: The least m for which there is a Z is $m + 1 = 4$, and this Z is unique (up to projective equivalence), coming from the B_3 line arrangement.

See: On the unique unexpected quartic in $\mathbb{P}^2$ (to appear)
Łucja Farnik, Francesco Galuppi, Luca Sodomaco, William Trok

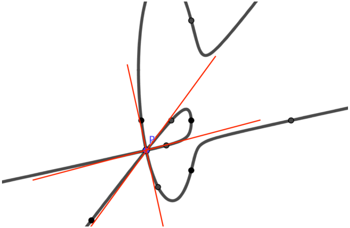
B_3

The B_3 arrangement of 9 lines (the line at infinity is not shown):



(The nine lines of $\mathcal{L}_{B_3}$ are dual to the roots of the B_3 root system.)

The unique unexpected quartic:



A Theorem of Di Marca, Malara and Oneto (DMO)

Note: A *modular* point of a line arrangement is a singular point which can see all other singular points by looking down lines of the arrangement. A *supersolvable* (ss) line arrangement is a line arrangement with at least one modular point.

Example: The B_3 arrangement is ss; it has 3 modular points.

Theorem (DMO: J. Alg. Comb., 2019) Let $\mathcal{L} = \{L_1, \dots, L_r\}$ be supersolvable, $m_{\mathcal{L}}$ the maximum multiplicity among the singular points, and $d_{\mathcal{L}} = r$ the number of lines. Let Let $\mathcal{P}_{\mathcal{L}}$ be the points dual to the lines of $\mathcal{L}$. The following are equivalent:

(a) $\mathcal{P}_{\mathcal{L}}$ has an unexpected curve of degree $d = m + 1$ for some m;
(b) $\mathcal{P}_{\mathcal{L}}$ has an unexpected curve of degree d for $d = m_{\mathcal{L}}$; and
(c) $2m_{\mathcal{L}} < d_{\mathcal{L}}$.

Question: Which supersolvable $\mathcal{L}$ have $2m_{\mathcal{L}} < d_{\mathcal{L}}$?

Question: Can we classify supersolvable $\mathcal{L}$?

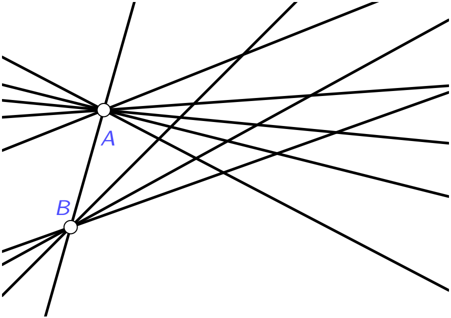
Partial classification of complex supersolvable $\mathcal{L}$

Joint work with Krishna Hanumanthu (HH) arXiv:1907.07712 .

Let $\mathcal{L}$ be supersolvable (ss).

Definition: $\mathcal{L}$ is homogeneous (homog) if every modular point has the same multiplicity.

Theorem (HH): Let $\mathcal{L}$ be ss, non-homog. Then $\mathcal{L}$ looks like the following:



What you are seeing: there is a unique singular point (here it's A) of maximum multiplicity $m_{\mathcal{L}}$, there is at most one other point (here it's B) of multiplicity more than 2, and all other singular points have multiplicity 2.

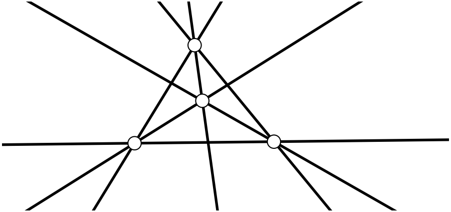
Proof: The key fact is a point of multiplicity $m_{\mathcal{L}}$ must be modular.

Homog ss $\mathcal{L}$

Theorem (HH): Let $\mathcal{L}$ be ss, homog. Then there are at most 4 modular points.

Proof: The key fact is no three modular points are collinear.

Theorem (HH): There is a unique homog $\mathcal{L}$ with 4 modular points; it has $m_{\mathcal{L}} = 3$:



When are there 3 modular points?

Theorem (HH): If $\mathcal{L}$ has exactly 3 modular points, then $m_{\mathcal{L}} > 3$ and (up to choice of coordinates) the lines come from the linear factors of

$$xyz(x^{m-2} - y^{m-2})(x^{m-2} - z^{m-2})(y^{m-2} - z^{m-2}),$$

where $m = m_{\mathcal{L}}$. (Here the three modular points are the coordinate vertices.)

Notes:

(a) If $m = 3$ this gives the case of 4 modular points.

(b) The case $m = 4$ gives the B_3 arrangement.

(c) A lattice theoretic classification has now been given by Dimca for 2 modular points.

(d) The case of 1 modular point is open.

Thanks...

Thank you ...

... from all of us ...

... to all of you!