Various versions of the resurgence and how they are related

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Abstract: Recent research has highlighted the importance of integral closure to understanding asymptotic resurgences and has led to important advances and new questions related to symbolic powers and polynomial interpolation. We discuss these advances and some of these questions. This talk is based on joint work with Jake Kettinger and Frank Zimmitti (HKZ: arXiv:2005.05282), partly inspired by a paper of M. DiPasquale and B. Drabkin (arXiv:2003.06980).

Fat points

We'll focus on ideals $I(Z)$ of *fat points* $Z \subset \mathbb{P}^n$, so $I(Z)$ is a homogeneous ideal in the polynomial ring $k[\mathbb{P}^n] = k[x_0, \dots, x_n]$ where

$$Z = m_1 p_1 + \cdots + m_s p_s$$

and

$$I(Z) = I(p_1)^{m_1} \cap \cdots \cap I(p_s)^{m_s}.$$

Here

$$p_1, \dots, p_s \in \mathbb{P}^n \text{ (distinct points)},$$

$$m_1, \dots, m_s \in \{1, 2, 3, \dots\}.$$

Comparing Powers and Symbolic Powers of Ideals

Commutative Algebra (ordinary powers): $I(Z)^r = \left(\bigcap_i I(p_i)^{m_i}\right)^r$

Algebraic Geom. (*symbolic* powers): $I(Z)^{(m)} = I(mZ) = \bigcap_i \left(I(p_i)^{mm_i}\right)$

How do these compare?

$$I(Z)^r \subseteq I(mZ) \text{ iff } r \geq m$$

$$I(mZ) \subseteq I(Z)^r \text{ implies } m \geq r \text{ so: } m < r \text{ implies } I(mZ) \not\subseteq I(Z)^r$$

(Invent. 2001/02) Ein-Lazarsfeld-Smith, Hochster-Huneke:

Theorem: Let $Z \subset \mathbb{P}^n$. If $C \geq n$, then $\dfrac{m}{r} \geq C$ implies $I(mZ) \subseteq I(Z)^r$.

(JAG 2010) Bocci-Harbourne: For each $C < n$ there is a Z for which the Theorem fails.

So $\frac{m}{r} \geq n$ is optimal for the general containment $I(mZ) \subseteq I(Z)^r$.

The resurgence

But for a specific Z typically you can do better:

Given Z, the least $C_Z \geq 0$ such that

$$\frac{m}{r} > C_Z \text{ implies } I(mZ) \subseteq I(Z)^r$$

exists and is called the *resurgence*, $\rho(I(Z))$.

It is known that $1 \leq \rho(I(Z)) \leq n$:

Since $m < r$ implies $I(mZ) \not\subseteq I(Z)^r$, we have $1 \leq \rho(I(Z))$.

Since $\dfrac{m}{r} \geq n$ implies $I(mZ) \subseteq I(Z)^r$, we have $\rho(I(Z)) \leq n$.

Extremality of the resurgence

No Z are known where you can't do better (i.e., where $C_Z = n$).

Open Problem 1: Does $\rho(I(Z)) = n$ ever happen?

Conjecture 1 (implies Grifo Conjecture (2018)): No.

Theorem (Tohaneanu-Xie (2019)):
$\rho(I(mZ)) \leq \dfrac{m+n-1}{m}$.

Corollary: For $n > 1$, let $m > 1$. Then $\rho(I(mZ)) < n$.

Thus Conjecture 1 holds if $Z = m_1 p + \cdots + m_s p_s$ with $gcd(\{m_i\}) > 1$.

Conjecture 1 often holds when Z is reduced (i.e., $Z = p_1 + \cdots + p_s$), as we will see.

What about the other extreme?

Open Problem 2a: When is $\rho(I(Z)) = 1$?

Theorem (Jafarloo-Zito (2018)): For Z with collinear points $p_i \in \mathbb{P}^n$ and any m_i we have $\rho(I(Z)) = 1$.

Open Problem 2b: Classify all Z such that $\rho(I(Z)) = 1$.

Theorem (Harbourne-Kettinger-Zimmitti (HKZ), 2020): Assume Z is reduced (i.e., $Z = p_1 + \cdots + p_s$). Then $I(Z)$ is a complete intersection if and only if $\rho(I(Z)) = 1$.

Open Problem 2c: What about when Z is not reduced?

Some examples

If $n > 1$, then Z needs to be reduced for $I(Z)$ to be a complete intersection.

But if $I(Z)$ is a complete intersection, then all powers of $I(rZ)$ are symbolic for all $r \geq 1$. I.e., $I(rZ)^m = I(rZ)^{(m)} = I(mrZ)$ for all $m \geq 1$.

If $I(mZ) = I(Z)^m$ for all $m \geq 1$, then $\rho(I(Z)) = 1$.

Open Problem 3a: Is the converse true: If $\rho(I(Z)) = 1$, is $I(mZ) = I(Z)^m$ for all $m \geq 1$?

Aside (Drabkin-DiPasquale (DD), 2020): The answer is no for certain monomial ideals in place of $I(Z)$.

Open Problem 3b: For which Z do we have $I(mZ) = I(Z)^m$ for all $m \geq 1$?

Note: $I(mZ) = I(Z)^m$ for all $m \geq 1$ can occur even when Z is not rY for a complete intersection $I(Y)$.

A new perspective: the *asymptotic resurgence*

Recall: $$\rho(I(Z)) = \sup\left\{\frac{m}{r} : I(mZ) \not\subseteq I(Z)^r\right\}$$

Thus: $$\rho(I(Z)) = \sup\left\{\frac{m}{r} : I(mtZ) \not\subseteq I(Z)^{rt} \text{ for some } t > 0\right\}$$

Now recall the asymptotic resurgence.

Guardo-Harbourne-Van Tuyl (GHVT), 2013:

$$\widehat{\rho}(I(Z)) = \sup\left\{\frac{m}{r} : I(mtZ) \not\subseteq I(Z)^{rt} \text{ for all } t \gg 0\right\}$$

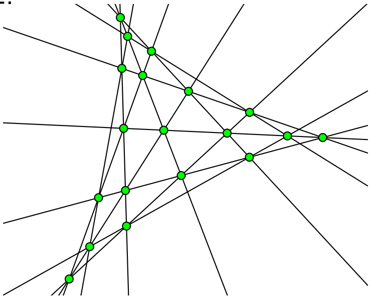
Clearly, $1 \leq \widehat{\rho}(I(Z)) \leq \rho(I(Z))$.

Sometimes $\widehat{\rho}(I(Z)) = \rho(I(Z))$ but $\widehat{\rho}(I(Z)) < \rho(I(Z))$ also happens.

Examples

(1) Bocci-Harbourne (2010), GHVT (2013): If Z is a star configuration (the points of intersection of all subsets of n hyperplanes from a set of $r > n$ general hyperplanes), then $\widehat{\rho}(I(Z)) = \rho(I(Z))$.

(2) (Dumnicki, Harbourne, Nagel, Seceleanu, Szemberg, Tutaj-Gasińska (2015)): There is a set Z of 19 points such that $\widehat{\rho}(I(Z)) = 5/4$ and $\rho(I(Z)) = 3/2$.



Waldschmidt constants

Initial degree of $I(Z)$: $\alpha(I(Z))$ is the least degree of a nonzero element of $I(Z)$.

Facts:

(1) $\alpha(I(Z)^m) = m\alpha(I(Z))$ (since $I(Z)$ is homogeneous)

(2) $\alpha(I(mZ)) \leq \alpha(I(Z)^m) = m\alpha(I(Z))$ (since $I(Z)^m \subseteq I(mZ)$)

Waldschmidt constant: $\widehat{\alpha}(I(Z)) = \lim_{m\to\infty} \frac{\alpha(I(mZ))}{m}$.

Note: $\widehat{\alpha}(I(rZ)) = \lim_{m\to\infty} \frac{\alpha(I(mrZ))}{m} = r \lim_{m\to\infty} \frac{\alpha(I(mrZ))}{rm} = r\widehat{\alpha}(I(Z))$.

Waldschmidt-Skoda bound (1978): $\frac{\alpha(I(Z))}{n} \leq \widehat{\alpha}(I(Z))$.

Chudnovsky Conjecture

Conjecture (Chudnovsky (1980)): $\dfrac{\alpha(I(Z)) + n - 1}{n} \leq \widehat{\alpha}(I(Z))$.

• Chudnovsky posed his conjecture only for reduced Z, but if it holds for them, then it holds for multiples of reduced Z (and so maybe for all Z):

$$\frac{\alpha(I(mZ)) + n - 1}{n} \leq \frac{m\alpha(I(Z)) + n - 1}{n} \leq m\frac{\alpha(I(Z)) + n - 1}{n}$$
$$\leq m\widehat{\alpha}(I(Z)) = \widehat{\alpha}(I(mZ))$$

• If true, the conjecture is sharp: let Z be a star configuration of $r > n$ general hyperplanes. Then:

Bocci-Harbourne (2010): $\widehat{\alpha}(I(Z)) = \dfrac{r}{n}$ and $\alpha(I(Z)) = r - n + 1$ so $\widehat{\alpha}(I(Z)) = \dfrac{\alpha(I(Z)) + n - 1}{n}$.

Some results

Conjecture (Chudnovsky (1980)): $\frac{\alpha(I(Z)) + n - 1}{n} \leq \widehat{\alpha}(I(Z))$.

Theorem (Fouli-Mantero-Xie (2018)): Chudnovsky holds over the complex numbers for Z reduced with sufficiently general points.

Theorem (Esnault-Viehweg (1983)): Over the complex numbers with Z reduced and $n > 1$, then $\frac{\alpha(I(Z)) + 1}{n} \leq \widehat{\alpha}(I(Z))$.

Some evidence for Conjecture 1

Theorem (GHVT): $1 \leq \frac{\alpha(I(Z))}{\widehat{\alpha}(I(Z))} \leq \widehat{\rho}(I(Z)) \leq \rho(I(Z)) \leq n$.

Theorem (HKZ, DD): $\rho(I(Z)) = n$ if and only if $\widehat{\rho}(I(Z)) = n$.

Note: It is sometimes true that $\frac{\alpha(I(Z))}{\widehat{\alpha}(I(Z))} = \widehat{\rho}(I(Z))$, for example for star configurations.

In these cases over the complex numbers with Z reduced (or assuming Chudnovsky's Conjecture) we have:

$$\widehat{\rho}(I(Z)) = \frac{\alpha(I(Z))}{\widehat{\alpha}(I(Z))} \leq \frac{\alpha(I(Z))}{\frac{\alpha(I(Z))+1}{n}} < n.$$

Hence in these cases Conjecture 1 holds.

A new open problem for the other extreme

By GHVT, if $\rho(I(Z)) = 1$, then $\widehat{\rho}(I(Z)) = 1$.

Open Problem 4a: But does $\widehat{\rho}(I(Z)) = 1$ imply $\rho(I(Z)) = 1$?

Theorem (HKZ): Let Z be a fat point subscheme of $\mathbb{P}^n$. If Z is reduced or $n = 1$, then $\widehat{\rho}(I(Z)) = 1$ if and only if $\rho(I(Z)) = 1$.

Idea of proof: If Z is reduced or $n = 1$, then $\widehat{\rho}(I(Z)) = 1$ implies $I(Z)$ is a complete intersection, hence all powers are symbolic so $\rho(I(Z)) = 1$.

Open Problem 4b: What happens if Z is not reduced?

Another open problem: computability

Open Problem 5: Is there an algorithm for $\widehat{\rho}(I(Z))$?

Almost!

Recall: Given an ideal $I \subseteq R = k[\mathbb{P}^n]$, we say $c \in R$ is *integral* over I if there is a d and $a_j \in I^j$, $1 \leq j \leq d$, such that $c^d = \sum_j a_j c^{d-j}$. The *integral closure* $\overline{I}$ of I is the ideal of all c integral over I.

Example: $I(Z) = \overline{I(Z)}$ for all Z.

Theorem (DiPasquale, Francisco, Mermin, Schweig (DFMS), 2019):

$$\widehat{\rho}(I(Z)) = \sup\left\{\frac{m}{r} : I(mZ) \not\subseteq \overline{I(Z)^r}\right\}$$

DFMS shows $\widehat{\rho}(I(Z)) = \max_{v \in R}\left\{\frac{v(I(Z))}{\widehat{v}(I(Z))}\right\}$ where R is the set of *Rees valuations* of $I(Z)$ (it's finite) and $\widehat{v}(I(Z)) = \lim_{m\to\infty} \dfrac{v(I(mZ))}{m}$.

Integral closure resurgence

Definition (HKZ): $\rho_{int}(I(Z)) = \sup\left\{\frac{m}{r} : \overline{I(Z)^m} \not\subseteq I(Z)^r\right\}$

We have $1 \leq \rho_{int}(I(Z)) \leq \rho(I(Z))$.

Open Problem 6: Is there an algorithm for $\rho_{int}(I(Z))$?

Open Problem 7: In fact, is $\rho_{int}(I(Z)) = 1$ for all Z?

If so, this would be a kind of very strong version of the Briançon-Skoda Theorem, as we will see.

Some results

Theorem (HKZ): Let $Z \subset \mathbb{P}^n$, $n \geq 2$. Then

$$1 \leq \rho_{int}(I(Z)) \leq \frac{n}{2},$$

hence $\rho_{int}(I(Z)) = 1$ when $n = 2$.

Proposition (HKZ): $\rho(I(Z)) = 1$ iff $\rho_{int}(I(Z)) = \widehat{\rho}(I(Z)) = 1$.

Corollary (HKZ): If $Z \subset \mathbb{P}^2$, then $\rho(I(Z)) = 1$ iff $\widehat{\rho}(I(Z)) = 1$.

This corollary follows immediately from the Theorem and the Proposition. So at least for $n = 2$, an algorithm for $\widehat{\rho}(I(Z))$ would allow us to compute whether $\rho(I(Z)) = 1$.

Proof of Theorem

Theorem (HKZ): Let $Z \subset \mathbb{P}^n$, $n \geq 2$. Then

$$1 \leq \rho_{int}(I(Z)) \leq \frac{n}{2},$$

hence $\rho_{int}(I(Z)) = 1$ when $n = 2$.

Proof: We need to show $\overline{I(Z)^m} \not\subseteq I(Z)^r$ implies $\frac{m}{r} \leq \frac{n}{2}$. A version of Briançon-Skoda by Aberbach-Huneke (Math Ann, 1993) gives

$$\overline{I(Z)^{n+r-1}} \subseteq I(Z)^r.$$

Thus

$$\overline{I(Z)^m} \not\subseteq I(Z)^r$$

implies

$$r \geq 2 \ \text{ and } \ m \leq n + r - 2$$

so

$$\frac{m}{r} \leq \frac{n+r-2}{r} \leq \frac{n}{2}.$$

Proof of Proposition

Proposition (HKZ): $\rho(I(Z)) = 1$ iff $\rho_{int}(I(Z)) = \widehat{\rho}(I(Z)) = 1$.

Proof: Assume $\rho(I(Z)) = 1$.

Then $\widehat{\rho}(I(Z)) = 1$ since $1 \leq \widehat{\rho}(I(Z)) \leq \rho(I(Z))$

and $\rho_{int}(Z) = 1$ since $1 \leq \rho_{int}(I(Z)) \leq \rho(I(Z))$.

Now assume $\widehat{\rho}(I(Z)) = 1$.

Then (DFMS) $\overline{(I(Z))^m} = I(mZ)$, so clearly $\rho_{int}(I(Z)) = \rho(I(Z))$.

Thus $\rho_{int}(I(Z)) = 1$ now implies $\rho(I(Z)) = 1$.