The concept of geproci subsets of $\mathbb{P}^3$

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AMS Special Session

Organizers: Federico Galetto & Kuei-Nuan Lin

Special Session on Hyperplane arrangements and commutative algebra

October 23, 2021, 2pm PDT

Slides will be available at my website:
`https://www.math.unl.edu/~bharbourne1/`

Etymology & Abstract

GEPROCI

GEneral PROjection to a Compete Intersection

ABSTRACT: The occurrence of finite subsets $Z \subset \mathbb{P}^3$ whose general projection to $\mathbb{P}^2$ is a complete intersection was raised in 2011 by F. Polizzi. Such sets are now called geproci sets.

One example: a complete intersection in a plane.

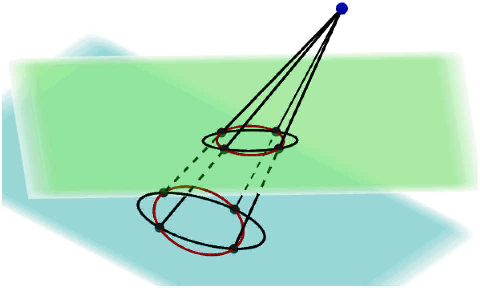
Another example: grids of lines.

Other examples became known only in 2018 as a by-product of work on unexpected surfaces, in turn motivated by work on hyperplane arrangements. I will survey how geproci developed, how it relates to unexpectedness and discuss some recent results.

Genesis: In the beginning...Jun 8, 2011 at 15:09

Mathoverflow Quest. 67265 by Francesco Polizzi: When is a general projection of points in $\mathbb{P}^3$ a complete intersection? Are there nontrivial examples?

Trivial example: A CI of two curves in the same plane projects isomorphically to its image, so is trivially geproci.

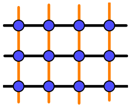


A set $Z \subset \mathbb{P}^3$ of ab points is (a, b)-*geproci* if $a \leq b$ and Z projects to a complete intersection (CI) of plane curves of degrees a and b.

Answer by Dmitri Panov, Jun 8, 2011 at 16:14: Grids!

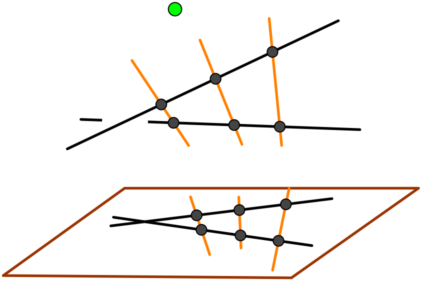
An (a, b)-*grid* is (a, b)-geproci. What is an (a, b)-grid?

It is given by a skew black lines and b skew orange lines, such that each black line meets each orange line in one point. The ab points form the grid. The lines are called *grid lines*.

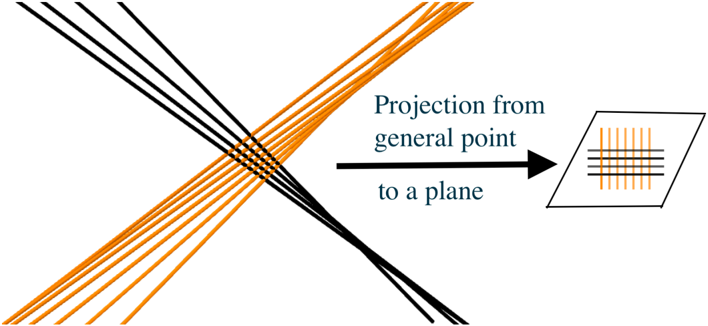


A (3, 4)-grid

Construction is easy when $a = 2$. Here's one with $(a, b) = (2, 3)$.

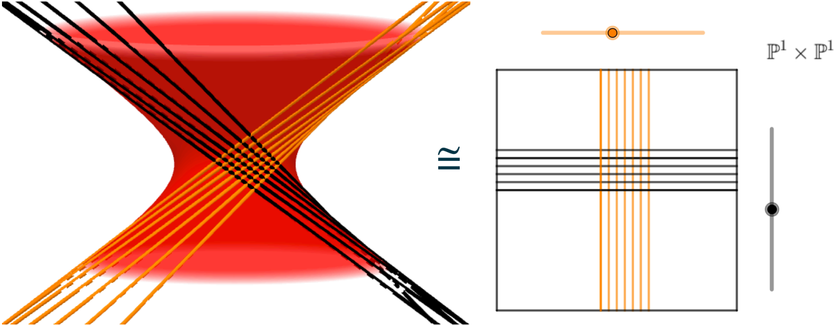


What about $a > 2$?



The graphic shows a $(4, 7)$-grid. It projects to a complete intersection of 4 lines with 7 lines, so it is $(4, 7)$-geproci. But where do such lines come from?

What about $a > 2$?



They come from the two rulings on a smooth quadric! And every grid with $2 < a \leq b$ works this way.

Question edit, Jun 9, 2011 at 6:36

Based on Panov's construction, Polizzi edited his question:

(1) Are there nontrivial nongrid geproci sets?

(2) Can we classify them (at least for small numbers of points)?

Answers: Yes (2018) and yes (on-going)!

(1) In 2018 unexpected surfaces led to nontrivial nongrid examples. Many (maybe all?) nontrivial nongrid geproci (and almost all grids) involve unexpected surfaces; more discussion to follow.

(2a) (Chiantini-Migliore TAMS 2021) All nontrivial nongrid geproci sets have at least 12 points (because (a, b)-geproci sets with $2 = a \leq b$ or $a = b = 3$ are grids).

(2b) (The Geproci Squad: work in progress) There is a unique nontrivial nongrid 12 point geproci set; it is $(3, 4)$-geproci.

[Geproci Squad: Luca Chiantini, Łucja Farnik, Giuseppe Favacchio, Brian Harbourne, Juan Migliore, Tomasz Szemberg, Justyna Szpond.]

Brief remarks on unexpected hypersurfaces.

The concept was introduced first for curves in $\mathbb{P}^2$ in CHMN:
D. Cook, B. Harbourne, J. Migliore and U. Nagel,
Line arrangements and configurations of points with an unexpected geometric property,
Compositio Math. 154:10 (2018) 2150–2194 (arXiv:1602.02300).

A finite set of points in $\mathbb{P}^2$ having an unexpected curve means the line arrangement dual to the points has special properties.

The concept was extended to $\mathbb{P}^n$ in HMNT:
B. Harbourne, J. Migliore, U. Nagel, Z. Teitler,
Unexpected hypersurfaces and where to find them,
Mich. Math. J., 2021 (arXiv:1805.10626).

An unexpected hypersurface for a finite set of points $Z \subset \mathbb{P}^3$ is a surface of some degree t with a general singular point of some multiplicity m. When $m = t$, the surface is a cone.

HMNT and 2018 Levico Terme Workshop

Examples (HMNT): Various root systems R give sets Z_R having unexpected cones of degrees a and b such that $|Z_R| = ab$.

Looking at these examples at Levico led to new geproci sets:

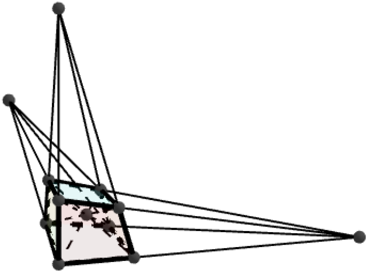
Fact (Workshop working group at Levico Terme, 2018; see Appendix to Chiantini-Migliore TAMS 2021): If $|Z| = ab$ has unexpected cones of degrees a and b, then Z is nontrivial (a, b)-geproci.

Levico also led to the Chiantini-Migliore paper, and to new insights on grids:

Theorem (Chiantini-Migliore TAMS 2021): Any (a, b)-grid with $ab > 4$ has unexpected cones of degrees a and b.

The smallest nontrivial nongrid geproci example: Z_{D_4}

Z_{D_4} has 12 points and unexpected cones of degree 3 and 4. The 12 points come from a cube in 3 point perspective.



This is a half-grid: Its general projections are intersections of a quartic D of 4 lines (coming from 4 skew lines containing Z_{D_4}) and a unique irreducible cubic C (coming from a $(3,3)$-grid whose image defines a pencil of cubics).

Theorem (Geproci Squad): Z_{D_4} is the unique 12 point nontrivial nongrid geproci.

More results and questions

Theorem (P. Pokora, T. Szemberg, J. Szpond, arXiv:2010.08863): A set of 60 points due to Klein is a $(6, 10)$-geproci half-grid.

Theorem (Geproci Squad): Given $4 \leq a \leq b$, there is a nongrid (a, b)-geproci set $Z \subset \mathbb{P}^3$.

Theorem (P. Fraś, M. Zięba, arXiv:2107.08107): Not all nongrid geproci are half-grids: Z_{H_4} is $(6, 10)$-geproci but not a half-grid.

Questions:

• Is Z_{D_4} the only nontrivial nongrid $(3, b)$-geproci Z?

• Which a, b have a unique nontrivial nongrid (a, b)-geproci Z? Is $a = 3, b = 4$ (i.e., Z_{D_4}) the only one?

• Do all nontrivial (a, b)-geproci Z (except the $(2, 2)$-grid) come from unexpected cones of degrees a and b?

First 8 photos show the 2018 Levico Terme working group (those in Geproci Squad, starred; in Squad but not at Levico, double starred)

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Luca Chiantini*

Graham Denham

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Brian Harbourne*

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Tomasz Szemberg*

Justyna Szpond*

Łucja Farnik**

Appendix: What is an unexpected surface?

$Z \subset \mathbb{P}^3$: a finite set of points.

$I(Z) \subset \mathbb{C}[x, y, z, w] = \mathbb{C}[\mathbb{P}^3]$: the ideal of forms vanishing on Z.

$[I(Z)]_t$: vector space span of forms in $I(Z)$ of degree t.

For any point $P \notin Z$ and multiplicity m we have

$$\dim[I(Z) \cap I(P)^m]_t \geq \max(0, \dim[I(Z)]_t - \binom{m+2}{3}).$$

But for P general we typically "expect"

$$\dim[I(Z) \cap I(P)^m]_t = \max(0, \dim[I(Z)]_t - \binom{m+2}{3}).$$

So we say the surfaces defined by $[I(Z) \cap I(P)^m]_t$ are *unexpected* if $\dim[I(Z) \cap I(P)^m]_t > \max(0, \dim[I(Z)]_t - \binom{m+2}{3})$.

We then say Z has unexpected surfaces of degree t with a general point P of multiplicity m. (Note: the surface is a cone if $m = t$.)