

A CANONICAL LINEAR SYSTEM ASSOCIATED TO ADJOINT DIVISORS IN CHARACTERISTIC $p > 0$

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ABSTRACT. Suppose that X is a projective variety over an algebraically closed field of characteristic $p > 0$. Further suppose that L is an ample (or more generally in some sense positive) divisor. We study a natural linear system in $H^0(X, \mathcal{O}_X(K_X + L))$. We further generalize this to incorporate a boundary divisor Δ . We show that these subsystems behave like the global sections associated to multiplier ideals, $H^0(X, \mathcal{J}(X, \Delta) \otimes L)$ in characteristic zero. In particular, we show that these systems are in many cases base-point-free. While the original proof utilized Kawamata-Viehweg vanishing and variants of multiplier ideals, our proof uses test ideals.

1. INTRODUCTION

Suppose that X is a normal variety and Δ is an effective $\mathbb{Q}$ -divisor such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. In characteristic zero, to this data one associates an ideal $\mathcal{J}(X, \Delta)$ called the *multiplier ideal*, which reflects subtle properties of both the singularities of X and Δ . The utility of the multiplier ideal for projective varieties lies mostly in its connections with a form of Kawamata-Viehweg vanishing usually called Nadel-vanishing. For example, consider the following application of the aforementioned vanishing and Castelnuovo-Mumford regularity. If M is a globally generated ample Cartier divisor on an n -dimensional normal variety and L is another Cartier divisor such that $L - K_X - \Delta$ is big and nef, then it follows that $\mathcal{J}(X, \Delta) \otimes \mathcal{O}_X(nM + L)$ is globally generated, cf. [Laz04, Proposition 9.4.26].

In characteristic $p > 0$, to the same data (X, Δ) , one can associate an ideal $\tau(X, \Delta)$ called the *test ideal*. The test ideal agrees with the multiplier ideal after reduction to characteristic $p \gg 0$ and it has been shown that $\tau(X, \Delta)$ satisfies many of the same *local* properties as the multiplier ideal. However, global applications of the test ideal have so far been lacking. This is largely because the Nadel-type vanishing theorems are known to be false. In this paper we prove the following result.

Theorem (Corollary 4.5). *Suppose that X is an n -dimensional normal projective variety over an F -finite field and that M is a globally generated ample Cartier divisor. Let Δ be an effective $\mathbb{Q}$ -divisor such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier and suppose that L is a Cartier divisor such that $L - K_X - \Delta$ is big and nef.*

Then

$$\tau(X, \Delta) \otimes \mathcal{O}_X(L + nM)$$

is globally generated. In particular, if (X, Δ) is strongly F -regular, then $\mathcal{O}_X(L + nM)$ is globally generated.

One application of the statement analogous to Corollary 4.5 for multiplier ideals is the construction of effective bounds on the degrees of hypersurfaces passing through points on $\mathbb{P}^n$.

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The corresponding bound in characteristic $p > 0$ has recently been observed by B. Harbourne as a corollary of [HoHu02], see [Har09, Slide 17] and *cf.* [HaHu11, ELS01]. More generally, in characteristic zero, these bounds are a special case of a result originally proved using vanishing theorems and multiplier-ideal-like constructions in [EV83], *cf.* [Wal79]. Another proof of these bounds explicitly using the language of multiplier ideals can be found in [Laz04, Section 10.1], *cf.* [EV92]. This latter proof goes through without change and so we obtain a proof of the following characteristic $p > 0$ theorem that exactly mimics global proofs using multiplier ideals.

Theorem ([Har09], Theorem 4.9). *cf.* [Laz04, Section 10.1].

Fix a reduced closed subscheme $S \subseteq \mathbb{P}_k^n = X$, where k is an algebraically closed field of characteristic $p > 0$, such that each irreducible component $Z \subseteq S$ has codimension $\leq e$ in $\mathbb{P}_k^n$. Suppose that $A \subseteq \mathbb{P}_k^n$ is a hypersurface of degree d such that $\text{mult}_x A \geq l$ for all $x \in S$. Then S lies on a hypersurface of degree $\lfloor \frac{de}{l} \rfloor$.

We actually prove a substantially stronger theorem than Corollary 4.5. We identify a canonical sub-vector space of $H^0(X, \tau(X, \Delta) \otimes \mathcal{O}_X(L + nM))$ which globally generates $\tau(X, \Delta) \otimes \mathcal{O}_X(L + nM)$. It is the study of this linear system that forms the basis for this paper. Let us briefly describe how this linear system arises in a simple case. Suppose that X is a variety over a perfect field k and that $\mathcal{L} = \mathcal{O}_X(L)$ is any line bundle (we are mostly interested in the case when L is positive, *e.g.* ample). Set $F^e : X \rightarrow X$ to be the absolute e -iterated Frobenius. There is a natural $\mathcal{O}_X$ -module map $F_*^e \omega_X \rightarrow \omega_X$ which arises from the Cartier operator if X is smooth, but exists generally as the Grothendieck-trace map. By twisting this map by $\mathcal{L}$ and taking cohomology we obtain maps

$$H^0(X, F_*^e(\omega_X \otimes \mathcal{L}^{p^e})) \rightarrow H^0(X, \omega_X \otimes \mathcal{L}).$$

For $e \gg 0$, the image of this map stabilizes (note $H^0(X, \omega_X \otimes \mathcal{L})$ is a finite dimensional vector space). We denote this stable image by $S^0(X, \omega_X \otimes \mathcal{L})$. We should note that this image is sometimes a proper submodule of $H^0(X, \omega_X \otimes \mathcal{L})$, even if the map $H^0(X, F_*^e \omega_X) \rightarrow H^0(X, \omega_X)$ is bijective, see for example [Tan72].

Suppose now that X is F -rational and $\mathcal{L}$ is a globally generated ample line bundle. In [Smi97b, Har05, Kee08] it is shown that $\omega_X \otimes \mathcal{L}^{\dim X + 1}$ is globally generated, *cf.* [Ara04]. However, essentially using Keeler's proof, we actually conclude that $\omega_X \otimes \mathcal{L}^{\dim X + 1}$ is globally generated by $S^0(X, \omega_X \otimes \mathcal{L}^{\dim X})$. In fact, the same result holds under the weaker hypothesis that X has F -injective singularities (a characteristic $p > 0$ analog of Du Bois singularities).

This naturally leads one to ask whether Fujita's conjecture [Fuj87] (or the bounds in [AS95]) might hold for $S^0(X, \omega_X \otimes \mathcal{L})$.

Question 1.1. Suppose that X is a d -dimensional smooth variety and $\mathcal{L}$ is an ample line bundle. Is it true that for all $n \geq d$ that

- (i) $S^0(X, \omega_X \otimes \mathcal{L}^{n+1})$ is base-point-free, and
- (ii) $S^0(X, \omega_X \otimes \mathcal{L}^{n+2})$ induces an embedding.

Even without the assumption that $\mathcal{L}$ is globally generated, we answer this question in the affirmative for curves in Theorem 3.3.

This special linear system $S^0(X, \omega_X \otimes \mathcal{L})$ also behaves better with respect to restriction than $H^0(X, \omega_X \otimes \mathcal{L})$. Suppose that Z is any F -pure center of a pair (X, Δ) . There is a surjection between a canonical linear system on X and another one on Z , for any line bundle L such that $L - K_X - \Delta$ is ample. This, combined with the previous results implies:

Theorem (Corollary 5.5). *Suppose that X is a normal projective variety and Δ is a $\mathbb{Q}$ -divisor such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier with index not divisible by $p > 0$ and that (X, Δ) is sharply F -pure. Further suppose that M is a Cartier divisor on X such that $M - K_X - \Delta$ is ample.*

Suppose that $Z \subseteq X$ is a normal F -pure center of (X, Δ) . Write $(K_X + \Delta)|_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z$ with Δ_Z as in Definition 5.1. Finally suppose that one of the following holds:

- (i) $\mathcal{O}_Z(M)$ is equal to $\mathcal{O}_Z((\dim Z)A + B)$ for some globally generated ample Cartier divisor A on Z and some Cartier divisor B such that $B - K_Z - \Delta_Z$ is ample, or
- (ii) Z is a minimal F -pure center and $\mathcal{O}_Z(M)$ is equal to $\mathcal{O}_Z((\dim Z)A + B)$ for some globally generated ample Cartier divisor A on Z and some Cartier divisor B such that $B - K_Z - \Delta_Z$ is big and nef, or
- (iii) Z is a smooth curve and $M|_Z - K_Z - \Delta_Z \sim_{\mathbb{Q}} (M - K_X - \Delta)|_Z$ has degree > 1 , or
- (iv) Z is a closed point.

Then $H^0(X, \mathcal{O}_X(M))$ has no base points along Z .

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2. PRELIMINARIES

Throughout this paper, all schemes will be assumed to be separated and of finite type over an F -finite field k of characteristic $p > 0$. Note every perfect field is automatically F -finite. A *variety* will mean a connected reduced equidimensional scheme over k (note that we do not assume varieties are irreducible). For any variety of dimension d , we use ω_X to denote $\mathbf{h}^{-d}\omega_X^\bullet$ (here $\omega_X^\bullet$ is used to denote $\eta^!k$ where $\eta : X \rightarrow k$ is the structural map). We point out also that little is lost by simply restricting to normal irreducible varieties. If X is normal, then a *canonical divisor* on X is any divisor K_X such that $\omega_X \cong \mathcal{O}_X(K_X)$.

We also record the following special case of Castelnuovo-Mumford regularity which we will frequently use.

Theorem 2.1 (Mumford). *cf. [Laz04, Theorem 1.8.5] Suppose that X is a projective variety and $\mathcal{M}$ is a globally generated ample line bundle on X . Additionally suppose that $\mathcal{F}$ is a coherent sheaf on X such that $H^i(X, \mathcal{F} \otimes \mathcal{M}^{-i}) = 0$ for all $i > 0$. Then $\mathcal{F}$ is globally generated.*

The rest of this section is somewhat technical and deals with $\mathbb{Q}$ -divisors and test ideals. Thus we suggest that the reader may wish to read Section 3 below first in order to learn the motivating ideas of this paper.

On a normal variety, a $\mathbb{Q}$ -divisor Δ is a divisor with rational coefficients, $\Delta = \sum a_i D_i$ where the D_i are prime divisors. With such a Δ , we use $\lceil \Delta \rceil$ to denote $\sum \lceil a_i \rceil D_i$. A $\mathbb{Q}$ -divisor Γ is called *$\mathbb{Q}$ -Cartier* if there exists an integer $n > 0$ such that $n\Gamma$ has integer coefficients and as such is a Cartier divisor. In this case, the smallest such $n > 0$ is called the *index* of Γ .

Definition 2.2. A *pair* (X, Δ) is the combined information of a normal integral scheme X and an effective $\mathbb{Q}$ -divisor Δ .

Our next order of business is to recall the definition of the test ideal and parameter test ideal of a pair (X, Δ) , as well as other notions of F -singularities. Consider now the e -iterated absolute Frobenius map $F^e : X \rightarrow X$. The source of F^e is isomorphic to X , although as a variety over k , $\eta \circ F^e : X \rightarrow k$ it is different than X . We notice that

$(F^e)^! \omega_X^\bullet = (F^e)^! \eta^! k = \eta^! (F^e)^! k \cong \eta^! k = \omega_X^\bullet$. Thus we have a trace map $F_*^e \omega_X^\bullet \rightarrow \omega_X^\bullet$. Taking cohomology yields $\Phi^e : F_*^e \omega_X \rightarrow \omega_X$.

Definition 2.3 ([Smi95, Bli04, ST08]). Suppose that X is a variety. The *parameter test submodule* of X , denoted $\tau(\omega_X)$, is the unique smallest $\mathcal{O}_X$ -submodule M of ω_X , non-zero on any component of X , such that $\Phi^1(F_* M) \subseteq M$ where $\Phi^1 : F_* \omega_X \rightarrow \omega_X$ is the canonical dual of Frobenius.

If $X = \text{Spec } R$ is a normal variety and $\Gamma \geq 0$ is a $\mathbb{Q}$ -divisor on X . Then the *parameter test submodule* of (X, Γ) , denoted $\tau(\omega_X, \Gamma)$, is the unique smallest non-zero $\mathcal{O}_X$ -submodule of ω_X such that $\phi(F_* M) \subseteq M$ where ϕ ranges over all $\phi \in \text{Hom}_{\mathcal{O}_X}(F_*^e \omega_X(\lceil (p^e - 1)\Gamma \rceil), \omega_X)$ and all $e > 0$ (notice $\omega_X \subseteq \omega_X(\lceil (p^e - 1)\Gamma \rceil)$).

If Γ is not necessarily effective, then choose A a Cartier divisor such that $\Gamma + A$ is effective (working locally if necessary). We then define $\tau(\omega_X, \Gamma) = \tau(\omega_X, \Gamma + A) \otimes \mathcal{O}_X(A)$. This is independent of the choice of A .

Both $\tau(\omega_X)$ and $\tau(\omega_X; \Delta)$ exist and their formation commutes with localization and so they can be defined on any (not necessarily affine) variety.

Definition 2.4. Suppose that X is a normal variety and Δ is a $\mathbb{Q}$ -divisor on X . Fix a canonical divisor K_X . We define the test ideal $\tau(X, \Delta)$ to be $\tau(\omega_X, K_X + \Delta)$.

Note $\tau(X, \Delta)$ is always an ideal sheaf of X if $\Delta \geq 0$, see [BST11, Lemma 2.29] (otherwise, it is a fractional ideal sheaf). We also note that $\tau(X, \Delta)$ is actually what is typically called the *big test ideal* or *non-finitistic test ideal* and denoted by $\tau_b(X, \Delta)$ or $\tilde{\tau}(X, \Delta)$. However, in the case that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier, the only setting considered in this paper, $\tau_b(X, \Delta)$ and the classically defined $\tau(X, \Delta)$ coincide, [Tak04, BSTZ10].

We also record the following useful facts about (parameter) test ideals:

Proposition 2.5. *cf. [BMS08, BSTZ10, ST08, ST11] Suppose that (X, Δ) is as above and fix A to be a Cartier divisor on X . Then*

- (a) $\tau(X, \Delta + A) = \tau(X, \Delta) \otimes \mathcal{O}_X(-A)$.
- (b) For all $1 \gg \varepsilon > 0$, we have that $\tau(X, \Delta + \varepsilon A) = \tau(X, \Delta)$.
- (c) $\tau(\omega_X, \Delta + A) = \tau(\omega_X, \Delta) \otimes \mathcal{O}_X(-A)$.
- (d) For all $1 \gg \varepsilon > 0$, we have that $\tau(\omega_X, \Delta + \varepsilon A) = \tau(\omega_X, \Delta)$.

Suppose now that $(X, \Delta \geq 0)$ is a pair such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier with index not divisible by $p > 0$. Choose $e > 0$ such that $(p^e - 1)(K_X + \Delta)$ is Cartier. Then associated to Δ and e we obtain a canonically determined (up to multiplication by a unit) map $\phi_\Delta : F_*^e \mathcal{L}_{e, \Delta} \rightarrow \mathcal{O}_X$, where $\mathcal{L}_{e, \Delta} = \mathcal{O}_X((1 - p^e)(K_X + \Delta))$ is a line bundle. Conversely, given a map $\phi : F_*^e \mathcal{L} \rightarrow \mathcal{O}_X$, it uniquely determines an effective $\mathbb{Q}$ -divisor Δ_ϕ such that $\mathcal{O}_X((1 - p^e)(K_X + \Delta)) \cong \mathcal{L}$. See [Sch09a] or [ST11] for further discussion. In this context however, $\tau(X, \Delta)$ is the unique smallest non-zero ideal $J \subseteq \mathcal{O}_X$ such that

$$\phi_\Delta(F_*^e(J \cdot \mathcal{L}_{e, \Delta})) \subseteq J.$$

In fact, it is easy yet important to observe that we have equality:

$$(2.5.1) \quad \phi_\Delta(F_*^e(J \cdot \mathcal{L}_{e, \Delta})) = J.$$

We also record the following fact for future use. First recall that if $x \in X$ is a smooth point and D is a $\mathbb{Q}$ -divisor on X , then we define the multiplicity of D at x , denoted $\text{mult}_x D$ to be $\frac{1}{n} \text{mult}_x(nD)$ where $n > 0$ is such that nD is Cartier at $x \in X$. See for example [Laz04, Page 163].

Lemma 2.6. *cf. [Laz04, Proposition 9.3.2] Fix X to be a smooth n -dimensional variety and suppose that D is a $\mathbb{Q}$ -divisor with multiplicity $\geq l$ at a (possibly non-closed) point $x \in X$ of codimension l . Then $\tau(X, D) \subseteq \mathfrak{q}_x$ where $\mathfrak{q}_x$ is the prime ideal sheaf corresponding to $x \in X$.*

Proof. Suppose $\pi : Y \rightarrow X$ is the blow up of the point $x \in X$ and set E to be the component of the exceptional divisor dominating x . Observe that E 's coefficient in $K_Y - \pi^*(K_X + D)$ is $\leq l - 1 - l = -1$. Therefore, $\pi_* \mathcal{O}_Y([K_Y - \pi^*(K_X + D)]) \subseteq \mathfrak{q}_x$. It follows from the usual arguments relating test ideals and multiplier ideals that $\tau(X, D) \subseteq \pi_* \mathcal{O}_Y([K_Y - \pi^*(K_X + D)])$, cf. [ST11, HW02], and the result follows. $\square$

Definition 2.7. [FST11, BB09, Bli09] Suppose that $(X, \Delta \geq 0)$ is a pair such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier with index not divisible by p . Fix $\phi_\Delta : F_*^e \mathcal{L}_{e, \Delta} \rightarrow \mathcal{O}_X$ as above. Then we define the *non- F -pure ideal* of (X, Δ) , denoted $\sigma(X, \Delta)$ or $\sigma(X, \phi_\Delta)$, to be the unique largest ideal J of $\mathcal{O}_X$ such that

$$\phi_\Delta(F_*^e(J \cdot \mathcal{L})) = J.$$

For any (possibly non-normal) reduced scheme over k with a given $\phi : F_*^e \mathcal{L} \rightarrow \mathcal{O}_X$, we define $\sigma(\phi)$ similarly.

For an arbitrary variety X , we define $\sigma(\omega_X)$ to be the unique largest submodule M of ω_X such that $\Phi(F_* M) = M$ where $\Phi : F_*^e \omega_X \rightarrow \omega_X$ is as above.

Remark 2.8. While we defined $\tau(X, \Delta)$ for possible non-effective Δ , we only define $\sigma(X, \Delta)$ for $\Delta \geq 0$.

Remark 2.9. Alternately, set $J_0 = \mathcal{O}_X$ and define recursively $J_{n+1} := \phi_\Delta(F_*^e(J_n \cdot \mathcal{L}))$. Then it can be shown that $J_n = J_{n+1} = \dots$ for $n \gg 0$ [Gab04], cf. [BSTZ10, Lyu97, HS77]. It follows immediately that $\sigma(X, \Delta) = J_n$ for $n \gg 0$. Likewise, $\sigma(\omega_X) = \Phi^e(F_*^e \omega_X)$ for $e \gg 0$.

Definition 2.10. With notation as above:

- (i) we say that $(X, \Delta \geq 0)$ is *strongly F -regular*, if $\tau(X, \Delta) = \mathcal{O}_X$
- (ii) we say that X is *F -rational* if $\tau(\omega_X) = \omega_X$.
- (iii) we say that $(X, \Delta \geq 0)$ is *sharply F -pure* if $\sigma(X, \Delta) = \mathcal{O}_X$.
- (iv) we say that X is *F -injective* if $\Phi^i : \mathbf{h}^i(F_*^e \omega_X^\bullet) \rightarrow \mathbf{h}^i(\omega_X^\bullet)$ is surjective for each i .

Remark 2.11. Strongly F -regular pairs are the analog of log terminal pairs in characteristic zero [HW02]. F -rational varieties are the analog of varieties with rational singularities in characteristic zero [Smi97a, Har98, MS97]. F -pure pairs are the analog of log canonical pairs in characteristic zero [HW02]. Finally, F -injective varieties are the analog of varieties with Du Bois singularities in characteristic zero, [Sch09b].

3. MOTIVATIONAL COMPUTATIONS AND EASY CASES

Suppose that X is a smooth (or simply normal) projective algebraic variety over an algebraically closed field k of characteristic $p > 0$. Further suppose that $\mathcal{L}$ is any line bundle on X . Consider the two linear systems:

$$\begin{aligned} T^0(X, \omega_X \otimes \mathcal{L}) &= \bigcap_{f: Y \rightarrow X} \text{Image} \left(H^0(X, f_*(\omega_Y \otimes_{\mathcal{O}_Y} f^* \mathcal{L})) \rightarrow H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) \right) \\ S^0(X, \omega_X \otimes \mathcal{L}) &= \bigcap_{e \geq 0} \text{Image} \left(H^0(X, F_*^e(\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{p^e})) \rightarrow H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) \right) \end{aligned}$$

where the first intersection runs over all finite maps $f : Y \rightarrow X$ of irreducible varieties (note it is harmless to only consider reduced or even normal Y). This first linear system T^0 was explored in [BST11] where it was used to construct transformation rules for test ideals under birational maps.

Notice that we always have

$$T^0(X, \omega_X \otimes \mathcal{L}) \subseteq S^0(X, \omega_X \otimes \mathcal{L})$$

Furthermore, because $H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L})$ is a finite dimensional k -vector space, we see that both intersections actually stabilize. In other words, both intersections are equal to one of their members.

We are primarily interested in the case where $\mathcal{L}$ is ample or in some weaker sense positive. Thus as a first step, consider the following:

Lemma 3.1. *Suppose that X is a smooth projective variety and $\mathcal{L}$ is an ample line bundle. Then there exists an integer n_0 , depending on X and $\mathcal{L}$, such that*

$$T^0(X, \omega_X \otimes \mathcal{L}^n) = S^0(X, \omega_X \otimes \mathcal{L}^n) = H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^n)$$

for all $n \geq n_0$. In other words, these two linear systems are complete linear systems.

Proof. Consider the section ring $S = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{L}^n)$. The canonical module of this ring is $\omega_S = \bigoplus_{n \in \mathbb{Z}} H^0(X, \omega_X \otimes \mathcal{L}^n)$.

We first show that $S^0(X, \omega_X \otimes \mathcal{L}) = H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L})$ since that is conceptually easier. Now the maps $\Phi^e : F_*^e \omega_S \rightarrow \omega_S$ are induced by the maps $\phi^e : H^0(X, F_*^e(\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{p^e})) \rightarrow H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L})$ in the obvious way, sending $H^0(X, F_*^e(\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{mp^e})) \rightarrow H^0(X, F_*^e(\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^m))$ and sending $H^0(X, F_*^e(\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^n))$ to zero for n not divisible by p^e , cf. [SS10, Section 5.3].

Now, the image of Φ^e stabilizes for $e \gg 0$ by [Gab04], cf. [BSTZ10]. In other words, $\Phi^e(F_*^e \omega_S) = \Phi^{e+1}(F_*^{e+1} \omega_S)$ for all sufficiently large e . Furthermore, it is surjective outside of the irrelevant ideal $S_+ = \bigoplus_{n \geq 1} S_n$ because S is smooth on the punctured spectrum. Thus $\bigcap_{e \geq 0} \Phi^e(F_*^e \omega_S)$ is contained within $S_{\geq n_0} \cdot \omega_S$ for some $n_0 \geq 0$. But this implies that $S^0(X, \omega_X \otimes \mathcal{L}^n) = H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^n)$ for all $n > n_0$.

For $T^0(X, \omega_X \otimes \mathcal{L}^n)$ we use a similar idea. However, in this case it follows that

$$\bigoplus_{n \in \mathbb{Z}} T^0(X, \omega_X \otimes \mathcal{L}^n) \supseteq \tau(\omega_S)$$

where $\tau(\omega_S)$ is the parameter-test submodule [BST11]. Again, because S is smooth on the punctured spectrum, $\tau(\omega_S) \supseteq S_{\geq m_0} \cdot \omega_S$ for some $m_0 > 0$ and so $T^0(X, \omega_X \otimes \mathcal{L}^n) = H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^n)$ for all $n > m_0$. $\square$

Remark 3.2. In fact, the above proof shows that the condition that X is smooth can be weakened substantially. In particular, one still has $S^0(X, \omega_X \otimes \mathcal{L}^n) = H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^n)$ for $n \gg 0$ if one assumes that X has F -injective singularities. More generally one has $T^0(X, \omega_X \otimes \mathcal{L}^n) = H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^n)$ for $n \gg 0$ if one assumes that X has F -rational singularities.

3.1. Curves. Suppose that C is a smooth curve of genus g and further suppose that $\mathcal{L}$ is an ample line bundle of degree $d > 0$ on C . The map

$$\phi_{C, \mathcal{L}} : H^0(C, F_*^e(\omega_C \otimes \mathcal{L}^{p^e})) \rightarrow H^0(C, \omega_C \otimes \mathcal{L})$$

need not be surjective in general, even if $H^0(C, F_*^e \omega_C) \rightarrow H^0(C, \omega_C)$ is surjective, see [Tan72]. However, if $d \geq \frac{2g-2}{p}$, then it is surjective by [Tan72, Lemma 10]. Thus $S^0(X, \omega_X \otimes \mathcal{L}) = H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L})$ for all $\mathcal{L}$ of degree at least $\frac{2g-2}{p}$. However, even if it is not surjective one can still ask whether or not $S^0(X, \omega_X \otimes \mathcal{L})$ is base point free or more generally induces an embedding. For these questions, the answer is as good as one could hope for:

Theorem 3.3. *With notation as above, if $\mathcal{L}$ has degree at least 2, then $T^0(C, \omega_C \otimes \mathcal{L}) \subseteq S^0(C, \omega_C \otimes \mathcal{L})$ is base-point-free. If $\mathcal{L}$ has degree at least 3, then $T^0(C, \omega_C \otimes \mathcal{L}) \subseteq S^0(C, \omega_C \otimes \mathcal{L})$ induces an embedding of C into projective space.*

Notice that $\deg(\omega_C \otimes \mathcal{L}) = 2g - 2 + d$ and merely based upon the degree of $\omega_C \otimes \mathcal{L}$, the d in the theorem is the smallest d such that $H^0(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L})$ is base-point free / respectively an embedding. Of course, one might hope for better bounds [Har77, Chapter IV].

Proof. First suppose $\deg \mathcal{L} = 2$. Further, choose any point $Q \in C$. It is sufficient to show that the map α in the diagram below:

$$0 \longrightarrow H^0(C, \omega_C(-Q) \otimes \mathcal{L}) \longrightarrow H^0(C, \omega_C \otimes \mathcal{L}) \xrightarrow{\alpha} H^0(Q, \omega_Q \otimes \mathcal{L}) \longrightarrow 0 = H^1(C, \omega_C(-Q) \otimes \mathcal{L})$$

does not send $T^0(C, \omega_C \otimes \mathcal{L})$ to zero. In other words, this means that $\alpha|_{T^0(C, \omega_C \otimes \mathcal{L})}$ is surjective. Here the final entry $H^1(C, \omega_C(-Q) \otimes \mathcal{L})$ is zero because $\deg \mathcal{L}(-P) > 0$.

Fix $f : D \rightarrow C$ to be any finite cover of C from a normal scheme D such that the image of $H^0(D, \omega_D \otimes_{\mathcal{O}_D} f^* \mathcal{L}) \rightarrow H^0(C, \omega_C \otimes \mathcal{L})$ is equal to $T^0(C, \omega_C \otimes \mathcal{L})$. Consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(C, \omega_C(-Q) \otimes \mathcal{L}) & \longrightarrow & H^0(C, \omega_C \otimes \mathcal{L}) & \xrightarrow{\alpha} & H^0(Q, \omega_Q \otimes \mathcal{L}) \longrightarrow 0 \\ & & \uparrow & & \uparrow \phi & & \uparrow \psi \\ 0 & \longrightarrow & H^0(D, \omega_D(-f^*Q) \otimes_{\mathcal{O}_D} f^* \mathcal{L}) & \longrightarrow & H^0(D, \omega_D \otimes_{\mathcal{O}_D} f^* \mathcal{L}) & \xrightarrow{\beta} & H^0(f^*Q, \omega_{f^*Q} \otimes_{\mathcal{O}_D} f^* \mathcal{L}) \longrightarrow 0 \end{array}$$

noticing the f^*Q need not be reduced or irreducible. Here the top row is exact since $0 = H^1(C, \omega_C(-Q) \otimes \mathcal{L})$ and the bottom row is exact since $0 = H^1(D, \omega_D(-f^*Q) \otimes_{\mathcal{O}_D} f^* \mathcal{L})$. However, Q is a point, and thus non-singular and so the natural map $f_* \omega_{f^*Q} \rightarrow \omega_Q$ is surjective. But this implies ψ is surjective and so $\psi \circ \beta = \alpha \circ \phi$ is surjective as well. It immediately follows that $\alpha|_{\text{Image}(\phi)}$ is still surjective. This completes the proof of the first statement.

For the second part, choose two points $P, Q \in C$ (possibly allowing $P = Q$). We form a diagram as before:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(C, \omega_C(-P-Q) \otimes \mathcal{L}) & \longrightarrow & H^0(C, \omega_C \otimes \mathcal{L}) & \xrightarrow{\alpha} & H^0(C, \omega_{P+Q} \otimes \mathcal{L}) \longrightarrow 0 \\ & & \uparrow & & \uparrow \phi & & \uparrow \psi \\ 0 & \longrightarrow & H^0(D, \omega_D(-f^*(P+Q)) \otimes_{\mathcal{O}_D} f^* \mathcal{L}) & \longrightarrow & H^0(D, \omega_D \otimes_{\mathcal{O}_D} f^* \mathcal{L}) & \xrightarrow{\beta} & H^0(f^*(P+Q), \omega_{f^*(P+Q)} \otimes_{\mathcal{O}_D} f^* \mathcal{L}) \longrightarrow 0 \end{array}$$

As before, the top row is exact since $0 = H^1(C, \omega_C(-P-Q) \otimes \mathcal{L})$ and the bottom row is exact since $0 = H^1(D, \omega_D(-f^*(P+Q)) \otimes_{\mathcal{O}_D} f^* \mathcal{L})$. Again, we want to prove that $\alpha|_{\text{Image}(\phi)}$ is surjective. If $P \neq Q$, then the same argument as before implies the desired surjectivity. Now suppose that $P = Q$. It is sufficient to prove that $f_* \omega_{f^*2Q} \rightarrow \omega_{2Q}$ is surjective. Of course, there are covers of ω_{2Q} where that map is not surjective (for example, the one corresponding to the reduced scheme Q). Consider the map $\mathcal{O}_{C,Q} \rightarrow \mathcal{O}_{D,(f^{-1}Q)_{\text{red}}}$ of semi-local rings. Since $\mathcal{O}_{C,Q}$ is regular, this map is split. It follows that $\mathcal{O}_{2Q} \rightarrow \mathcal{O}_{2f^*Q}$ is also split. The desired surjectivity now immediately follows. $\square$

3.2. Global generation in higher dimensions. Suppose that $\mathcal{L}$ is a globally generated ample line bundle on a smooth (or even F -injective) variety X of dimension d . In [Smi97b, Har05, Kee08], it was shown that $H^0(X, \omega_X \otimes \mathcal{L}^{d+1})$ is globally generated, respectively that $H^0(X, \omega_X \otimes \mathcal{L}^{d+2})$ induces an embedding of X into projective space. Further refinements are also possible (replacing $\mathcal{L}^{d+1}$ by $\mathcal{L}^d \otimes \mathcal{M}$ for some ample divisor $\mathcal{M}$ [Kee08]). However, those previous proofs actually showed something more, they in fact showed that $S^0(X, \omega_X \otimes \mathcal{L}^{d+1})$ was globally generated and that $S^0(X, \omega_X \otimes \mathcal{L}^{d+2})$ induced an embedding into projective space.

In order to motivate our later work, let us briefly prove this result since the proof is quite straightforward.

Theorem 3.4. [Smi97b, Har05, Kee08] *Suppose that X is a projective F -injective variety and that $\mathcal{L}$ is any globally generated ample line-bundle and $\mathcal{M}$ is any other ample line bundle. Then*

- (1) $\omega_X \otimes \mathcal{L}^d \otimes \mathcal{M}$ is globally generated by $S^0(X, \omega_X \otimes \mathcal{L}^d \otimes \mathcal{M})$.
- (2) If X is smooth, then $S^0(X, \omega_X \otimes \mathcal{L}^{d+1} \otimes \mathcal{M})$ induces an embedding of X into projective space.

Taken from [Kee08]. Choose $e \gg 0$ and notice that $H^i(X, \omega_X \otimes \mathcal{M}^{p^e} \otimes \mathcal{L}^{p^e(d-i)}) = 0$ for all $i > 0$ by Serre vanishing. Therefore, $(F_*^e \omega_X) \otimes \mathcal{M} \otimes \mathcal{L}^d$ is globally generated as an $\mathcal{O}_X$ -module by Theorem 2.1. However, $(F_*^e \omega_X) \otimes \mathcal{M} \otimes \mathcal{L}^d$ surjects onto $\omega_X \otimes \mathcal{M} \otimes \mathcal{L}^d$ by the assumption on the singularities of X . In particular, $\omega_X \otimes \mathcal{M} \otimes \mathcal{L}^d$ is globally generated by the linear system

$$\text{Image} \left(H^0(X, (F_*^e \omega_X) \otimes \mathcal{M} \otimes \mathcal{L}^d) \rightarrow H^0(X, \omega_X \otimes \mathcal{M} \otimes \mathcal{L}^d) \right)$$

This linear system equals $S^0(X, \omega_X \otimes \mathcal{L}^d \otimes \mathcal{M})$ since $e \gg 0$.

For the second half, mimicking the proof of [Kee08], cf. [Laz04, Example 1.8.22], we fix a point $y \in X$ and choose $e \gg 0$ such that

$$\mathfrak{m}_y \otimes (F_*^e \omega_X) \otimes \mathcal{L}^{d+1} \otimes \mathcal{M}$$

is globally generated (this was shown in [Kee08]). Consider the following diagram:

$$\begin{array}{ccc} H^0(X, \mathfrak{m}_y \otimes (F_*^e \omega_X) \otimes \mathcal{L}^{d+1} \otimes \mathcal{M}) & \xrightarrow{\alpha} & H^0(X, (F_*^e \omega_X) \otimes \mathcal{L}^{d+1} \otimes \mathcal{M}) \\ \gamma \downarrow & & \downarrow \delta \\ H^0(X, \mathfrak{m}_y \otimes \omega_X \otimes \mathcal{L}^{d+1} \otimes \mathcal{M}) & \xrightarrow{\beta} & H^0(X, \omega_X \otimes \mathcal{L}^{d+1} \otimes \mathcal{M}) \end{array}$$

It immediately follows that $\beta^{-1}(S^0(X, \omega_X \otimes \mathcal{L}^{d+1} \otimes \mathcal{M}))$ globally generates $\mathfrak{m}_y \otimes \omega_X \otimes \mathcal{L}^{d+1} \otimes \mathcal{M}$. Note this holds for each point $y \in X$ so that $S^0(X, \omega_X \otimes \mathcal{L}^{d+1} \otimes \mathcal{M})$ clearly separates points. It also separates tangent vectors since $\beta^{-1}(S^0(X, \omega_X \otimes \mathcal{L}^{d+1} \otimes \mathcal{M}))$ globally generates $\mathfrak{m}_y \otimes \omega_X \otimes \mathcal{L}^{d+1} \otimes \mathcal{M}$ and thus its image also globally generates $(\mathfrak{m}_y / \mathfrak{m}_y^2) \otimes \omega_X \otimes \mathcal{L}^{d+1} \otimes \mathcal{M}$, cf. [Har70, Page 21]. $\square$

4. $\mathbb{Q}$ -DIVISORS AND GLOBAL GENERATION

Definition 4.1. Suppose that X is a normal proper variety over an algebraically closed field of characteristic $p > 0$. Further suppose $\Delta \geq 0$ is a $\mathbb{Q}$ -divisor such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier with index not divisible by p . Finally suppose that M is any Cartier divisor.

Consider the map $\phi_\Delta^e : F_*^e \mathcal{L}_{e,\Delta} \rightarrow \mathcal{O}_X$ as in Section 2. Notice that ϕ_Δ restricts to surjective maps:

$$\begin{aligned} F_*^e(\sigma(X, \Delta) \otimes \mathcal{L}_{e,\Delta}) &\rightarrow \sigma(X, \Delta) \\ F_*^e(\tau(X, \Delta) \otimes \mathcal{L}_{e,\Delta}) &\rightarrow \tau(X, \Delta) \end{aligned}$$

We then define $S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(M))$ as

$$\begin{aligned} &:= \bigcap_{n \geq 0} \text{Image} \left(H^0(X, F_*^{ne} \sigma(X, \Delta) \otimes \mathcal{L}_{ne,\Delta}(p^{ne} M)) \right. \\ &\quad \left. \rightarrow H^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(M)) \right) \\ &= \bigcap_{n \geq 0} \text{Image} \left(H^0(X, F_*^{ne} \mathcal{L}_{ne,\Delta}(p^{ne} M)) \right. \\ &\quad \left. \rightarrow H^0(X, \mathcal{O}_X(M)) \right) \\ &\subseteq H^0(X, \mathcal{O}_X(M)). \end{aligned}$$

Likewise, we define $S^0(X, \tau(X, \Delta) \otimes \mathcal{O}_X(M))$ as

$$\begin{aligned} &:= \bigcap_{n \geq 0} \text{Image} \left(H^0(X, F_*^{ne} \tau(X, \Delta) \otimes \mathcal{L}_{ne, \Delta}(p^{ne} M)) \right. \\ &\quad \left. \rightarrow H^0(X, \tau(X, \Delta) \otimes \mathcal{O}_X(M)) \right) \\ &\subseteq H^0(X, \mathcal{O}_X(M)). \end{aligned}$$

It is straightforward to see that these objects are independent of the choice of e .

Observe the following:

Lemma 4.2. *With notation as in Definition 4.1, it follows that*

$$S^0(X, \tau(X, \Delta) \otimes \mathcal{O}_X(M)) \subseteq S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(M))$$

Furthermore, if $\Delta_1 \geq \Delta_2$ both satisfy $K_X + \Delta_i$ being $\mathbb{Q}$ -Cartier with index not divisible by p , then

$$S^0(X, \tau(X, \Delta_1) \otimes \mathcal{O}_X(M)) \subseteq S^0(X, \tau(X, \Delta_2) \otimes \mathcal{O}_X(M)).$$

The same statement holds for $\sigma(X, \Delta_i)$.

Proof. The first statement is obvious since $\tau(X, \Delta) \subseteq \sigma(X, \Delta)$. For the second statement, choose a large $e = e_1 = e_2$ and then notice that $\phi_{\Delta_1}^e$ factors through $\phi_{\Delta_2}^e$. $\square$

4.1. General global generation statements for S^0 .

Theorem 4.3. *Suppose that X is a d -dimensional variety and that Δ is a $\mathbb{Q}$ -divisor such that $\Gamma = K_X + \Delta$ is $\mathbb{Q}$ -Cartier with index not divisible by $p > 0$. Further suppose that L is a Cartier divisor such that $L - K_X - \Delta$ is ample. Finally suppose that M is a globally generated ample divisor. Then*

$$\sigma(X, \Delta) \otimes \mathcal{O}_X(L + nM)$$

is globally generated for $n \geq d$ by $S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(L + nM))$. Furthermore, the same result holds for $\tau(X, \Delta)$ in place of $\sigma(X, \Delta)$.

In particular, if (X, Δ) is sharply F -pure, then

$$H^0(X, \mathcal{O}_X(L + dM))$$

is base-point-free.

Proof. Choose $e > 0$ such that $(p^e - 1)(K_X + \Delta)$ is Cartier.

We start with the natural map $\phi_\Delta : F_*^e \mathcal{L}_e \rightarrow \mathcal{O}_X$ where $\mathcal{L}_e = \mathcal{O}_X((1 - p^e)(K_X + \Delta))$. For each $n > 0$, we have the natural map $\phi_\Delta^n : F_*^{ne} \mathcal{L}_{ne} \rightarrow \mathcal{O}_X$ where $\mathcal{L}_{ne} = \mathcal{O}_X((1 - p^{ne})(K_X + \Delta))$. We fix $n \gg 0$. Notice that

$$\phi_\Delta^n(F_*^{ne}(\sigma(X, \Delta) \cdot \mathcal{O}_X((1 - p^{ne})(K_X + \Delta)))) = \sigma(X, \Delta)$$

Twisting by $\mathcal{O}_X(L + dM)$ gives us a surjective map

$$F_*^{ne}(\sigma(X, \Delta) \cdot \mathcal{O}_X((1 - p^{ne})(K_X + \Delta) + p^{ne}L + p^{ne}dM)) \rightarrow \sigma(X, \Delta) \otimes \mathcal{O}_X(L + dM).$$

The left-side is equal to $F_*^{ne}(\sigma(X, \Delta) \otimes \mathcal{O}_X(L) \otimes \mathcal{O}_X((p^{ne} - 1)(L - K_X - \Delta) + p^{ne}dM))$. We will show that this sheaf is globally generated as an $\mathcal{O}_X$ -module by using Theorem 2.1, Castelnuovo-Mumford regularity. To see this, simply notice that

$$\begin{aligned} &H^i \left(X, F_*^{ne}(\sigma(X, \Delta) \otimes \mathcal{O}_X(L) \otimes \mathcal{O}_X((p^{ne} - 1)(L - K_X - \Delta) + p^{ne}dM)) \otimes \mathcal{O}_X(-iM) \right) \\ &= H^i \left(X, F_*^{ne}(\sigma(X, \Delta) \otimes \mathcal{O}_X(L) \otimes \mathcal{O}_X((p^{ne} - 1)(L - K_X - \Delta) + p^{ne}(d - i)M)) \right) \\ &= H^i \left(X, F_*^{ne}(\sigma(X, \Delta) \otimes \mathcal{O}_X(L + (d - i)M) \otimes \mathcal{O}_X((p^{ne} - 1)(L - K_X - \Delta) + p^{ne}(d - i)M)) \right) \end{aligned}$$

which vanishes by Serre vanishing for $\mathcal{N} \gg 0$ since for any $i \leq d$, $L - K_X - \Delta + (d - i)M$ is ample. It follows that

$$F_*^{ne}(\sigma(X, \Delta) \otimes \mathcal{O}_X(L) \otimes \mathcal{O}_X((p^{ne} - 1)(L - K_X - \Delta) + p^{ne}dM))$$

is globally generated as an $\mathcal{O}_X$ -module, as claimed. Therefore, the quotient $\sigma(X, \Delta) \otimes \mathcal{O}_X(L + nM)$ is then also globally generated as an $\mathcal{O}_X$ -module

For the statement involving τ , replace σ by τ . For the final statement, notice that (X, Δ) is sharply F -pure if and only if $\sigma(X, \Delta) = \mathcal{O}_X$. $\square$

Remark 4.4. In fact, the same result also holds for the relative (adjoint-like) test ideals of [Tak08, Sch09a] and more generally for many of the ideals considered in [BB09, Bli09].

We state an easy but important corollary.

Corollary 4.5. *cf. [Laz04, Proposition 9.4.26] Suppose that X is a normal projective variety of dimension d and that M is a globally generated ample Cartier divisor. Let Δ be an effective $\mathbb{Q}$ -divisor such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier and suppose that L is a Cartier divisor such that $L - K_X - \Delta$ is big and nef. Then*

$$\tau(X, \Delta) \otimes \mathcal{O}_X(L + nM)$$

is globally generated for $n \geq d$. In particular, if (X, Δ) is strongly F -regular, then $\mathcal{O}_X(L + nM)$ is globally generated.

Proof. Choose an effective Cartier divisor D such that $D + K_X$ is effective. Choose another effective Cartier divisor E such that $L - K_X - \Delta - \varepsilon E$ is ample for all $1 \gg \varepsilon > 0$. Now fix a rational $\varepsilon > 0$ such that $L - K_X - \Delta - \varepsilon E$ is ample and such that $\tau(X, \Delta + \varepsilon E) = \tau(X, \Delta)$. Additionally suppose that $np^{e_0}(K_X + \Delta + \varepsilon E)$ is Cartier for some integer n with $p \nmid n$ and some integer $e_0 \geq 0$. Choose $e \gg e_0$ and consider the effective $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor

$$\Gamma := \frac{D + K_X + \Delta + \varepsilon E}{(p^e - 1)}.$$

Since $e \gg 0$, we know that $\tau(X, \Delta + \varepsilon E + \Gamma) = \tau(X, \Delta + \varepsilon E) = \tau(X, \Delta)$. Additionally, $L - K_X - \Delta - \varepsilon E - \Gamma$ is also ample for the same reason. On the other hand,

$$\begin{aligned} & n(p^e - 1)(K_X + \Delta + \varepsilon E + \Gamma) \\ &= n(p^e - 1)(K_X + \Delta + \varepsilon E) + n(D + K_X + \Delta + \varepsilon E) \\ &= nD + np^e(K_X + \Delta + \varepsilon E) \end{aligned}$$

which is Cartier. Thus the index of $K_X + \Delta + \varepsilon E + \Gamma$ is not divisible by p and the result follows immediately from Theorem 4.3. $\square$

Remark 4.6. We cannot use the same technique for $\sigma(X, \Delta)$ in place of $\tau(X, \Delta)$ in Corollary 4.5. In particular, we cannot guarantee that $\sigma(X, \Delta + \varepsilon E) = \sigma(X, \Delta)$ no matter how small $\varepsilon > 0$ is.

Remark 4.7. If in the previous corollary, we assume that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier with index not divisible by $p > 0$, then we need not introduce the divisor Γ at all. Now observe that

$$S^0(X, \tau(X, \Delta + \varepsilon E) \otimes \mathcal{O}_X(L + nM)) \subseteq S^0(X, \tau(X, \Delta) \otimes \mathcal{O}_X(L + nM))$$

and so it follows that $S^0(X, \tau(X, \Delta) \otimes \mathcal{O}_X(L + nM))$ globally generates $\tau(X, \Delta) \otimes \mathcal{O}_X(L + nM)$. Likewise, even without the assumption on the index of $K_X + \Delta$, for any definition of $S^0(X, \tau(X, \Delta) \otimes \mathcal{O}_X(L + nM))$ satisfying Lemma 4.2, we obtain the global generation by the same argument.

Corollary 4.8. *Suppose that X is an F -rational projective n -dimensional variety, M is a globally generated ample Cartier divisor and N is any big and nef Cartier divisor. Then $\omega_X(nM + N)$ is globally generated.*

Proof. Fix a canonical divisor K_X . Choose any effective Cartier divisor A such that $A - K_X$ is effective. Set $\Delta = A - K_X$ and set $L = A + N$. Then

$$L - K_X - \Delta = A + N - K_X - A + K_X = N$$

is big and nef. It follows that $\tau(X, \Delta) = \tau(\omega_X, K_X + \Delta) = \tau(\omega_X, A) = \tau(\omega_X) \otimes \mathcal{O}_X(-A) = \omega_X(-A)$ where the last equality comes because X is F -rational. Thus $\tau(X, \Delta) \otimes \mathcal{O}_X(L + nM) = \omega_X(-A + L + nM) = \omega_X(N + nM)$ is globally generated. $\square$

The following application is a demonstration of the utility of the above results. We give a new proof of a characteristic $p > 0$ analog of a special case of the main result of [EV83], cf. [Wal79]. Variants of this problem were also studied by Bombieri, Skoda, Demailly, Wüstholz, Chudnovsky and others. See [EV83] and [Laz04, Section 10.1] for a discussion of the history of this problem in characteristic zero. Recently, B. Harbourne observed that the characteristic $p > 0$ result below could be easily obtained via [HoHu02], the simple argument is briefly described in a special case in [Har09, Slide 17] and in the introduction to [HaHu11].

Theorem 4.9 ([Har09], cf. [EV83]). *Fix a reduced closed subscheme $S \subseteq \mathbb{P}_k^n = X$, where k is an algebraically closed field of characteristic $p > 0$, such that each irreducible component $Z \subseteq S$ has codimension $\leq e$ in $\mathbb{P}_k^n$. Suppose that $A \subseteq \mathbb{P}_k^n$ is a hypersurface of degree d such that $\text{mult}_x A \geq l$ for all $x \in S$. Then S lies on a hypersurface of degree $\lfloor \frac{de}{l} \rfloor$.*

Proof. This proof taken from [Laz04, Proposition 10.1.1, Example 10.1.4], cf. [EV83, EV92]. Set $D = \frac{e}{l}A$. Notice that $\text{mult}_x D \geq e$ for all $x \in S$. It follows from Lemma 2.6 that $\tau(X, D) \subseteq \mathcal{I}_S$ where $\mathcal{I}_S$ is the ideal defining S . Set $\delta = \lfloor \frac{de}{l} \rfloor$.

Set H to be the hyperplane divisor on X and consider $mH - K_X - D \sim_{\mathbb{Q}} mH + (n + 1)H - \frac{de}{l}H$ and observe it is ample as long as $m > \frac{de}{l} - (n + 1)$. Since m is an integer, this happens if and only if $m \geq \lfloor \frac{de}{l} \rfloor + 1 - (n + 1) = \delta - n$. Choose $L = (\delta - n)H$ and notice that then

$$\tau(X, D) \otimes \mathcal{O}_X(L + nH) = \tau(X, D) \otimes \mathcal{O}_X(\delta)$$

is globally generated. But $\tau(X, D) \subseteq \mathcal{I}_S$ and so there exists a non-zero section

$$s \in H^0(X, \tau(X, D) \otimes \mathcal{O}_X(\delta)) \subseteq H^0(X, \mathcal{I}_S \otimes \mathcal{O}_X(\delta)) \subseteq H^0(X, \mathcal{O}_X(\delta)).$$

The zero locus of that section has the desired property. $\square$

Remark 4.10. It would be interesting to try to prove the full results of [EV83] in characteristic $p > 0$.

5. F -PURE CENTERS

Suppose that $(X, \Delta \geq 0)$ is a pair such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier with index not divisible by $p > 0$. Choose $e > 0$ such that $(1 - p^e)(K_X + \Delta)$ is Cartier. Set $\mathcal{L}_{e, \Delta} = \mathcal{O}_X((1 - p^e)(K_X + \Delta))$ and consider $\phi_\Delta : F_*^e \mathcal{L}_{e, \Delta} \rightarrow \mathcal{O}_X$ to be a map corresponding to Δ as in Section 2. Recall the following definition.

Definition 5.1. [Sch10, Sch09a] With (X, Δ) as above, a subvariety $Z \subseteq X$ is called an *F -pure center of (X, Δ)* if (X, Δ) is sharply F -pure at the generic point of Z and additionally if

$$\phi_\Delta(F_*^e(\mathcal{I}_Z \cdot \mathcal{L}_{e, \Delta})) \subseteq \mathcal{I}_Z$$

where $\mathcal{I}_Z$ is the ideal sheaf defining $Z \subseteq X$. Thus we induce a non-zero map:

$$\phi_\Delta^Z(F_*^e(\mathcal{L}_{e, \Delta}|_Z)) \rightarrow \mathcal{O}_Z.$$

If Z is normal, then ϕ_Δ^Z corresponds to an effective $\mathbb{Q}$ -divisor Δ_Z on Z such that $(K_X + \Delta)|_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z$.

Remark 5.2. Even more, if Z is a union of F -pure centers, then it is easy to see that we still have a map

$$\phi_\Delta^Z(F_*^e(\mathcal{L}_{e,\Delta}|_Z)) \rightarrow \mathcal{O}_Z.$$

We now demonstrate how F -pure centers can be used to lift sections and to prove that adjoint linear systems are base point free along certain loci.

Proposition 5.3. *Fix X a normal projective variety and suppose that $(X, \Delta \geq 0)$ is a pair such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier with index not divisible by $p > 0$. Suppose that $Z \subseteq X$ is any union of F -pure centers of (X, Δ) . Finally, suppose that M is a Cartier divisor such that $M - K_X - \Delta$ is ample. Then there is a natural surjective map:*

$$S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(M)) \rightarrow S^0(Z, \sigma(Z, \phi_Z^\Delta) \otimes \mathcal{O}_Z(M)).$$

Proof. Choose $e \gg 0$ such that $(1 - p^e)(K_X + \Delta)$ is Cartier and consider the diagram of short exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_*^e(\mathcal{I}_Z \cdot \mathcal{L}_{e,\Delta}) & \longrightarrow & F_*^e \mathcal{L}_{e,\Delta} & \longrightarrow & F_*^e(\mathcal{L}_{e,\Delta} \otimes \mathcal{O}_Z) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{I}_Z & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_Z \longrightarrow 0 \end{array}$$

Twisting by M and taking cohomology, we obtain

$$\begin{array}{ccccccc} 0 \rightarrow H^0(X, F_*^e(\mathcal{I}_Z \cdot \mathcal{L}_{e,\Delta} \otimes \mathcal{O}_X(p^e M))) & \rightarrow & H^0(X, F_*^e(\mathcal{L}_{e,\Delta} \otimes \mathcal{O}_X(p^e M))) & \xrightarrow{\gamma} & H^0(Z, F_*^e(\mathcal{L}_{e,\Delta} \otimes \mathcal{O}_Z(p^e M))) & \rightarrow & 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow \beta \\ 0 \longrightarrow H^0(X, \mathcal{I}_Z \cdot \mathcal{O}_X(M)) & \longrightarrow & H^0(X, \mathcal{O}_X(M)) & \xrightarrow{\delta} & H^0(Z, \mathcal{O}_Z(M)) & & \end{array}$$

where the top sequence is exact since

$$H^1(X, F_*^e(\mathcal{I}_Z \cdot \mathcal{L}_{e,\Delta} \otimes \mathcal{O}_X(p^e M))) = H^1(X, F_*^e(\mathcal{I}_Z \cdot \mathcal{O}_X(p^e M - (p^e - 1)(K_X + \Delta))))$$

which vanishes for $e \gg 0$ by Serre vanishing.

Now, since $e \gg 0$, the image of α is $S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(M))$ and similarly the image of β is $S^0(Z, \sigma(Z, \phi_Z^\Delta) \otimes \mathcal{O}_Z(M))$. Choose $x = \beta(y) \in S^0(Z, \sigma(Z, \phi_Z^\Delta) \otimes \mathcal{O}_Z(M))$. Fix $w \in H^0(X, F_*^e(\mathcal{L}_{e,\Delta} \otimes \mathcal{O}_X(p^e M)))$ such that $\gamma(w) = y$. Thus $\delta(\alpha(w)) = x$ and so the claimed map is surjective. $\square$

Remark 5.4. If in the previous Proposition Z is a finite union of closed points then it is easy to see that $S^0(Z, \sigma(Z, \phi_Z^\Delta) \otimes \mathcal{O}_Z(M)) \cong H^0(Z, \mathcal{O}_Z)$. It follows that $S^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(M)) \subseteq H^0(X, \mathcal{O}_X(M))$ separates the points of Z .

Now we state a corollary. The author expects that parts of this have been known to experts for several years:

Corollary 5.5. *Suppose that (X, Δ) as in Proposition 5.3. Suppose that $Z \subseteq X$ is a normal F -pure center of (X, Δ) and also suppose that M is a Cartier divisor on X such that $M - K_X - \Delta$ is ample. Write $(K_X + \Delta)|_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z$ with Δ_Z as in Definition 5.1 and suppose that (Z, Δ_Z) is sharply F -pure. Finally suppose that one of the following holds:*

- (i) $\mathcal{O}_Z(M)$ is equal to $\mathcal{O}_Z((\dim Z)A + B)$ for some globally generated ample Cartier divisor A on Z and some Cartier divisor B such that $B - K_Z - \Delta_Z$ is ample, or
- (ii) Z is a minimal F -pure center and $\mathcal{O}_Z(M)$ is equal to $\mathcal{O}_Z((\dim Z)A + B)$ for some globally generated ample Cartier divisor A on Z and some Cartier divisor B such that $B - K_Z - \Delta_Z$ is big and nef, or
- (iii) Z is a smooth curve and $M|_Z - K_Z - \Delta_Z \sim_{\mathbb{Q}} (M - K_X - \Delta)|_Z$ has degree > 1 , or

(iv) Z is a closed point.

Then $H^0(X, \mathcal{O}_X(M))$ has no base points along Z .

Proof. In each case, we utilize Proposition 5.3. Therefore, it is sufficient to show that the submodule $S^0(Z, \sigma(Z, \phi_Z^\Delta) \otimes \mathcal{O}_Z(M))$ is base-point-free. Note that $\mathcal{O}_Z = \sigma(Z, \phi_Z^\Delta)$ by the sharp F -purity hypothesis, [Sch09a].

In case (i), we simply apply Theorem 4.3. For case (ii), we notice that (Z, Δ_Z) is strongly F -regular by [Sch09a] and so we apply Corollary 4.5. Case (iv) is trivial. For case (iii), we apply a similar strategy as in Theorem 3.3. Set $L_Z = (1 - p^e)(K_Z + \Delta_Z) \sim_{\mathbb{Q}} (1 - p^e)(K_X + \Delta)|_Z$. We consider the following diagram for $e \gg 0$ and any closed point $Q \in Z$.

$$\begin{array}{ccccc}
 H^0(Z, \mathcal{O}_Z(M|_Z - Q)) & \longrightarrow & H^0(Z, \mathcal{O}_Z(M|_Z)) & \xrightarrow{\alpha} & H^0(Q, \mathcal{O}_Q) \\
 \uparrow & & \uparrow \phi & & \uparrow \psi \\
 H^0(Z, \mathcal{O}_Z(p^e(M|_Z - Q) + L_Z)) & \longrightarrow & H^0(Z, \mathcal{O}_Z(p^e M|_Z + L_Z)) & \xrightarrow{\beta} & H^0(Q, \mathcal{O}_{p^e Q}) \longrightarrow 0
 \end{array}$$

where the bottom row is exact since

$$\begin{aligned}
 & H^1(Z, \mathcal{O}_Z(p^e(M|_Z - Q) + L_Z)) \\
 &= H^0(Z, \omega_Z(-p^e(M|_Z - Q) - L_Z)) \\
 &= H^0(Z, \mathcal{O}_Z(K_Z - M|_Z + Q - (p^e - 1)(M|_Z - K_Z - \Delta_Z - Q))) \\
 &= 0
 \end{aligned}$$

and the last equality follows since $e \gg 0$ and $\deg M|_Z - K_Z - \Delta_Z - Q > 0$. Therefore β is surjective and certainly ψ is as well. Thus α is surjective and the proof is complete. $\square$

Remark 5.6. One can apply the method of the previous theorem even without the assumption that the F -pure center Z is normal. Indeed, one only needs that (Z, ϕ_Z) is sharply F -pure (meaning that ϕ_Z is surjective) and that $S^0(Z, \sigma(Z, \phi_Z) \otimes \mathcal{O}_X(M))$ globally generates $\sigma(Z, \phi_Z) \otimes \mathcal{O}_X(M)$.

6. FURTHER COMMENTS

Indeed, many of the results of this paper can easily be generalized to the context of F -pure Cartier modules in the sense of [BB09, Bli09]. Indeed, on a scheme X , an F -pure Cartier-module is a coherent $\mathcal{O}_X$ -module $\mathcal{F}$ together with a given surjective map

$$\phi : F_*^e(\mathcal{F} \otimes \mathcal{L}) \rightarrow \mathcal{F}$$

for some choice of invertible sheaf $\mathcal{L}$. We note that the Cartier-modules introduced in [BB09] do not include the line bundle $\mathcal{L}$ (or rather, always assumed $\mathcal{L} = \mathcal{O}_X$), but the authors were certainly aware of this generalization.

Let us give an example of this. Suppose that X is any d -dimensional F -injective variety over a field. Since X is in particular S1, it follows that $\dim \text{Supp } \mathbf{h}^i(\omega_X^\bullet) \leq d + i - 1$ for $-d < i < 0$, see [BST11]. Now, notice that the natural map $F_*^e \mathbf{h}^i(\omega_X^\bullet) \rightarrow \mathbf{h}^i(\omega_X^\bullet)$ is surjective since X is F -injective. Therefore, using the same argument as in [Kee08] or Theorem 4.3, we obtain the following:

Proposition 6.1. *Suppose that X is a projective F -injective d -dimensional variety. Further suppose that $\mathcal{M}$ is a globally generated ample line bundle and $\mathcal{L}$ is any other ample line bundle. Then the sheaf*

$$\mathbf{h}^i(\omega_X^\bullet) \otimes \mathcal{M}^{d+i-1} \otimes \mathcal{L}$$

is globally generated for $-d < i < 0$.

Proof. Choosing some $e \gg 0$, we know that $\mathcal{F} := F_*^e(\mathbf{h}^i(\omega_{\dot{X}}) \otimes \mathcal{M}^{p^e(d+i-1)} \otimes \mathcal{L}^{p^e})$ satisfies the hypothesis of Theorem 2.1 (in other words, it is 0-regular with respect to $\mathcal{M}$). Thus $\mathcal{F}$ is globally generated as an $\mathcal{O}_X$ -module. Thus the quotient $\mathbf{h}^i(\omega_{\dot{X}}) \otimes \mathcal{M}^{d+i-1} \otimes \mathcal{L}$ is also globally generated, proving the theorem. $\square$

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