Inverse system of a symbolic power III: thin algebras and fat points

A. Iarrobino[1] 9/93

ABSTRACT: The author obtains upper bounds for the Hilbert function H(N,r) of the graded ideals K(P,N) of functions vanishing to given orders N at a general set of points P of projective space $\mathbb{P}^{r-1}$. The bounds arise from translating this problem using Macaulay's inverse systems, to that of finding the number of syzygies of given degrees to a general enough vector space $V = L^J$ spanned by powers of linear forms.

A thin algebra is the quotient of a polynomial ring $\mathfrak{R}$ in r variables defined by an ideal I generated by a general enough s-dimensional vector space V of forms having specified degrees J. A thin power-algebra $\mathfrak{R}/L^J$ is the quotient of $\mathfrak{R}$ by ideals generated by specified J powers of a set L of s general linear forms. The Koszul complex gives natural lower bounds on the Hilbert function T(J,r) of thin algebras, which are also lower bounds on the Hilbert function T'(J,r) of thin power-algebras. The author shows that the set of Hilbert functions {H(N,r), for all N} can be determined from the set {T'(J,r), for all j}. The lower bounds on T'(J,r) in turn give new upper bounds on the Hilbert functions H(N,r). The error in the bounds is the defect def(J,r) = T'(J,r)-T(j,r), modulus the usual conjecture concerning the Hilbert function of thin algebras. The author determines $H(a^s,r)$ in many cases using the symbolic algebra program "Macaulay". He gives new examples where def(J,r) $\neq$ 0. Calculations for r = 4 suggest that the set of pairs (J,r) for which def(J,r) $\neq$ 0 and $|J| \leq a$ is finite and small.

AMS 1991 Subject Classification: Primary 13D40, Secondary 13P10,14B05, 14C05,14Q05.

Keywords: apolar, defect, forms, generic, Hilbert function, ideal, inverse system, points, postulation, powers of linear forms, power sum, punctual Hilbert scheme, relations, secant variety, symbolic power, thin algebra, vanishing ideal.

[1] Supported in part by the National Science Foundation, and the Bulgarian Foundation "Scientific Research" under the US-Bulgarian project "Algebra and Algebraic Geometry".

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Table of Contents:

1. Introduction. 1
1A. Thin algebras. 1
1B. Vanishing ideals at a set of points. 4
2. The Hilbert function of vanishing ideals. 13
2A. Upper bounds for the Hilbert function of R/K(P,N). 13
2B. Koszul regions for the Hilbert function of $R/K(P,a^s)$. 18
2C. Lower bound for regularity. 25
3. Thin algebras and fat points. 27
3A. Consequences for the thin algebra problem when $u = a-1 > 1$ .... .27
3B. From power algebras to vanishing ideals. 30
3C. Calculating defects. 31
3D. Patterns in the defects. 32
3E. Sharpness of the upper bounds. 39
3F. Further defects, r = 4,5. 40
References 43

1. Introduction.

1A. Thin algebras. We let $\mathcal{R} = k[X_1,\dots,X_r]$ denote the polynomial ring, and $\mathcal{D}$ denote the divided power ring in r-variables X over a field k. Given a subspace V of $\mathcal{R}_j$, if $u \geq 0$, we denote by

$$\mathcal{R}_uV = \langle\{X^K v,\ |K| = u,\ v \in V\}\rangle$$

the span of all products fv of elements v of V by elements f of $\mathcal{R}_u$. Thus, $\mathcal{R}_uV$ is the image of the multiplication homomorphism,

$$\mathcal{R}_u \otimes V \rightarrow \mathcal{R}_{u+j}.$$

The Hilbert function $H(M,Z)$ of a graded $\mathcal{R}$ module $M = \oplus M_i$ is the sum $H(M) = \Sigma \dim_k M_i Z^i$. If V is a vector subspace of $\mathcal{R}_j$, we denote by (V) the ideal it generates, and by $T(V,Z) = \Sigma_{i\geq 0} t_i(V) Z^i$ the Hilbert function of the quotient algebra $\mathcal{R}/(V)$. Here,

$$t_i(V) = \dim_k(\mathcal{R}_i/\mathcal{R}_{i-j}V),$$

and we set $\mathcal{R}_uV = 0$ if $u < 0$. We will denote the sequence of coefficients of the Hilbert function $T(V,Z)$ by $T(V)$, and call

$T(V)$ also the "Hilbert function" of V. Any inequality between Hilbert functions is termwise: $T \le T'$ if $t_i \le t'_i$ for each i.

The subspaces V of $\mathfrak{R}_j$ having dimension s are parametrized by the points z_V of the Grassmann variety $\mathrm{Grass}(\mathfrak{R}_j, s)$.

Definition. THIN ALGEBRA, CASE $V \subset \mathfrak{R}_j$. We will say that the s-dimensional vector space $V \subset \mathfrak{R}_j$ is *generically chosen* or *general* for property $\mathfrak{P}$ if there exists a dense open set $\mathbb{U}_{\mathfrak{P}}$ of $\mathrm{Grass}(\mathfrak{R}_j, s)$ such that $z_V \in \mathbb{U}_{\mathfrak{P}}$ implies property $\mathfrak{P}$. We say $R/(V)$ is a *thin algebra* if the Hilbert function $H(\mathfrak{R}/(V)$ is the minimum possible: $H(\mathfrak{R}/(V)_i = T(s,j,r)_i$, the sequence defined by

$$T(s,j,r)_i =_{\mathrm{def}} \min\{H(\mathfrak{R}/(W))_i, W \subset \mathfrak{R}_i, \dim_k W = s\}.$$

We now show that the termwise minimum $T(s,j,r)$ actually occurs as $H(\mathfrak{R}/(V))$ for generically chosen subspaces V of $\mathfrak{R}_j$.

Lemma 1.1. A GENERICALLY CHOSEN V DETERMINES A THIN ALGEBRA. There is a unique Hilbert function $T(s,j,r,Z)$ and an open dense subset $\mathbb{U}(s,j,r) \subset \mathrm{Grass}(\mathfrak{R}_j, s)$ such that a point $z_V \in \mathbb{U}(s,j,r)$ iff $H(\mathfrak{R}/(V)) = T(s,j,r,Z)$. If V is any s-dimensional subspace of $\mathfrak{R}_j$, then $H(\mathfrak{R}/(V)) \ge T(s,j,r,Z)$.

Proof. It is well known that there are only a finite number of Hilbert functions possible for vector spaces parametrized by $\mathrm{Grass}(s,j,r)$ (see [**Be**]). We let

$$T(s,j,r)_i = \min(t_i(V) \mid V \subset \mathfrak{R}_j \text{ and } \dim_k V = s. \tag{1.1}$$

We let $t_i(V) = \dim_k(\mathfrak{R}_i/\mathfrak{R}_{i-j}V)$. The condition

$$t_i(V) < T(s,j,r)_i + 1 \tag{1.2}$$

defines an open dense subset of the irreducible variety $\mathrm{Grass}(\mathfrak{R}_j, s)$. The intersection of the conditions (1.2) for a finite set of integers i also determines an open dense subset of $\mathrm{Grass}(\mathfrak{R}_j, s)$. Since there a finite number of possible Hilbert functions $H(R/(V))$, given s,j,r, the intersection of a finite number of the conditions (1.2) completely determines a unique Hilbert function $T(s,j,r)$, which is termwise minimal, and which occurs for V in a dense open subset of $\mathrm{Grass}(\mathfrak{R}_j, s)$.

Question A_0. NUMBER OF RELATIONS FOR A GENERIC VECTOR SPACE V OF FORMS. Given integers s,j, and r, what is the minimum possible Hilbert function $T(s,j,r)$ of $\mathfrak{R}/(V)$, where (V) is the ideal generated by an s-dimensional subspace V of $\mathfrak{R}_j$? In particular, given (s,j,u,r) what is the maximum possible size of $\mathfrak{R}_u V$?

Definition. THIN ALGEBRA. More generally, we suppose that $J = (j_1,\ldots,j_s)$, $j_1 \le \ldots \le j_s$ is a sequence of positive integers, the *degrees*; and that $f_1,\ldots,f_s$ are homogeneous elements of degrees J in $\mathfrak{R}$. We let $V = \langle f_1,\ldots,f_s\rangle$, set $V_i = V\cap\mathfrak{R}_i$, set $s_i = \dim_k V_i$, and term J the *degrees* of V. We say that V is *generically chosen* for property $\mathfrak{P}$ if there is an open dense set $\mathfrak{U}_{\mathfrak{P}}$ of $\prod \mathrm{Grass}(\mathfrak{R}_i,s_i)$ such that $V \in \mathfrak{U}_{\mathfrak{P}}$ implies property $\mathfrak{P}$. As before $(V)_i$ denotes the degree-i piece of the ideal (V). We call $\mathfrak{R}/(V)$ a *thin algebra* if $H(\mathfrak{R}/(V)) = T(J,r)$ where

$$T(J,r) = \min_{W\mid \deg(W)=J} H(R/(W)).$$

Question A for (J,i,r). HILBERT FUNCTION OF THIN ALGEBRAS. What is the minimum possible Hilbert function $T(J,r)$ of $H(\mathfrak{R}/(V))$? In particular, when the f_u are generically chosen, what is

$$T(J,r)_i = \dim_k \mathfrak{R}_i/(V)_i?$$

See the Remark after Conjecture 2.0, for the status of Question A.

We next assume that each f_i is a power of an element in a set of general enough linear forms.

Definition. THIN POWER ALGEBRA. Suppose that $J = (j_1,\ldots,j_s)$, $j_1 \le \ldots \le j_s$ is a sequence of positive integers, and that $L = [L_1,\ldots,L_s]$ is a sequence of s elements of $\mathfrak{R}_1$, no two linearly dependent. We let L^J denote the vector space $L^J = \langle L_1^{j_1},\ldots,L_s^{j_s}\rangle$, spanned by the j_u-th powers of the elements L_u. We term J the *degrees* or *powers* of L^J, and we let (L^J) denote the ideal generated by L^J. We say that the sequence L is *general* for property $\mathfrak{P}$, or is *generically* chosen, if there exists a dense open set $\mathfrak{U}_{\mathfrak{P}}$ in $\mathbb{P}(r) = \mathbb{P}^{r-1}\times\ldots\times\mathbb{P}^{r-1}$ such that $(L_1,\ldots,L_s) \in \mathfrak{U}_{\mathfrak{P}}$ implies property $\mathfrak{P}$. Using a similar technique to that in the proof of Lemma 1.1, it is easy to see that there is a dense open subset $\mathfrak{U}(J,r)$ of $\mathbb{P}(r)$ such that $H(\mathfrak{R}/(L^J))$ is minimum if L is parametrized by a point of $\mathfrak{U}(J,r)$. Given (r,J) and a set B of linear forms we say that $\mathfrak{R}/(B^J)$ is a *thin power algebra* if its Hilbert function $H(R/B^J)$ is the termwise minimum $T'(J;r) = \min_L(H(R/L^j))$.

Question B for (J,i,r). HILBERT FUNCTION OF THIN POWER ALGEBRAS. What is the minimum possible Hilbert function $T'(J;r)$ of $\mathfrak{R}/(L^J)$? In particular, when L is a general set of linear forms, what is $T'(J,r)_i = \dim_k(\mathfrak{R}_i/(L^J)_i)$?

By semicontinuity, the Hilbert functions $T'(J,r,Z) = \Sigma T'(J,r)_i Z^i$ occurring for these more special spaces L^J must be at least as large as those occurring for a general set of degree-J forms:

$$T'(J,r)_i \geq T(J,r)_i. \qquad (1.3)$$

Examples 1.6A,B below show that there may be strict inequality in (1.3). This was previously known in a different guise (see Example 2.16).

Definition 1.2. We denote by $def(J,r)_i$ the defect

$$def(J,r)_i = T'(J;r)_i - T(J;r)_i. \qquad (1.4)$$

When $J = j^s$ we denote def(J,r) by def(s,j,r).

We now rephrase Question B. Let $U = (u_1,\ldots,u_s)$, $u_1 \geq \ldots \geq u_s \geq 0$ be a sequence of integers, and let $(L^{\underline{i}-U})$ be the ideal

$$(L^{\underline{i}-U}) = (L_1{}^{i-u_1},\ldots,L_s{}^{i-u_s}).$$

Question B′ for (U,i,r). NUMBER OF RELATIONS FOR POWERS OF GENERIC LINEAR FORMS. What is the dimension d(U,i;r) of the vector space $(L^{\underline{i}-U})\cap\mathfrak{R}_i$ in $\mathfrak{R}_i$, when L is chosen sufficiently general?

The dimension of the space $\mathfrak{F}_i(L,i)$ of degree-i relations

$$\mathfrak{F}_i(L,i) = \langle\{f_1,\ldots,f_s) \mid \Sigma_j f_j L_j{}^{i-u_j} = 0,\ f_j \in \mathfrak{R}_{u_j}\}\rangle$$

among the powers $L_j{}^{i-u_j}$ satisfies

$$\dim_k \mathfrak{F}_i(L,i) = \Sigma_j \dim_k \mathfrak{R}_{u_j} - d(U,i;r). \qquad (1.5)$$

We say that two questions are *equivalent* when the answers to each determine the answers to the other. Evidently, Questions B and B′ are equivalent when J and U are related by $J = (\underline{i}-U) = (i-u_1,\ldots,i-u_s)$, since

$$d(U,i;r) = \dim_k \mathfrak{R}_i - T'(\underline{i}-U;r)_i. \qquad (1.6)$$

<u>1B. Vanishing ideals at a set of points.</u> We suppose now that $P = (p_1,\ldots,p_s)$ is a set of s points of $\mathbb{P}^{r-1}$, that $N = (n_1,\ldots,n_s)$, $n_1 \geq \ldots \geq n_s$ is a sequence of nonnegative integers, called the *orders of vanishing* or *multiplicities* at P.

Definition. ORDER N VANISHING IDEAL AT s POINTS P. We denote by $K(P,N)$ the graded ideal of functions in the polynomial ring $R = k[x_1,\ldots,x_r]$ vanishing to order at least n_i at each P_i. Thus

$$K(P,N) = m_{p(1)}{}^{n_1}\cap\ldots\cap m_{p(s)}{}^{n_s}, \tag{1.7}$$

where m_p denotes the graded ideal in R of functions zero at p. If $N = (a,\ldots,a)$ s times we denote $K(P,N)$ by $K(P,\underline{a})$, $K(P,a)$, or $K(P,a^s)$.

A. Geramita, A. Hirschowitz and J. Alexander, M.V. Catalisano, A. Gimigliano, N.V. Trung and G. Valla, and others have studied the regularity and Hilbert functions of the ideals $K(P,N)$. See **[A1]**, **[AH1]**, **[CTV]**, **[G]**, **[GGH]**, **[GH]**, **[Gi1]**, **[Gi2]**, **[GO1]**, **[GO2]**, **[GM]**, **[H1]**, **[H2]**, and **[TV]**.

Definition. GENERAL SET OF POINTS. We say that an ordered set P of s points in $\mathbb{P}^{r-1}$ is *general* or is *generically chosen* for property $\mathfrak{P}$, if there is a dense open subset $U_{\mathfrak{P}}$ of $(\mathbb{P}^{r-1})^s$ such that $P \in U_{\mathfrak{P}}$ implies $\mathfrak{P}$. When the multiplicities N are the same, we consider unordered sets P, and use the symmetric product $\mathrm{Sym}^s(\mathbb{P}^{r-1})$ in place of the product.

Question C. HILBERT FUNCTION OF AN ORDER N VANISHING IDEAL AT GENERAL POINTS OF $\mathbb{P}^{r-1}$. What is the maximum possible Hilbert function $H_N(r,Z) = \max_P H(R/K(P,N))$ for the quotient ring?

Question C′ for (N,i,r). In particular, when the points P are generically chosen in $\mathbb{P}^{r-1}$, what is $H_N(r)_i = \dim_k(R_i/K(P,N)_i)$? When $N = (\underline{a}) = (a,\ldots,a) = (a^s)$ of length s, and the points P are generically chosen, what is $H(s,a,r)_i = H_N(r)_i$?

Definition. INVERSE SYSTEM OF AN IDEAL. Recall that if char k = 0, or char k > j, there is an exact pairing from $R_j\times\mathfrak{R}_j \to k$ given by the partial differential operator action of R on $\mathfrak{R}$ (see **[EI]**). If I is an ideal of R, its inverse system I^{-1} in $\mathfrak{R}$ is the R-submodule of $\mathfrak{R}$ satisfying

$$\begin{aligned}[I^{-1}]_i &= Ann(I)\cap\mathfrak{R}_i \\ &= \langle I_i\rangle^{\perp}.\end{aligned}$$

The above pairing induces an exact pairing $\mathfrak{R}_i/[I^{-1}]_i\times I_i \to k$; and $\mathfrak{R}_i/[I^{-1}]_i$ is the dual vector space to I_i.

Definition. SPECIAL INVERSE SYSTEMS. If $P = (P_1,\ldots,P_s)$, with $P_i = (p_{i1},\ldots,p_{ir})$ we let $L_i = \sum p_{iu}X_u$ up to k^*-multiple in $\mathfrak{R}$. We denote by $\mathfrak{J}(P,N,i)$ the ideal

$$\mathfrak{I}(P,N,i) = ((L_1)^{i+1-n_1}, \ldots, (L_s)^{i+1-n_s}) \tag{1.8}$$

and we denote by $\mathfrak{V}(P,N,i)$ the vector space of degree-i forms in $\mathfrak{I}(P,N,i)$,

$$\begin{aligned} \mathfrak{V}(P,N,i) &= \mathfrak{I}(P,N,i) \cap \mathfrak{R}_i \\ &= \mathfrak{R}_{n_1-1}(L_1)^{i+1-n_1} + \ldots + \mathfrak{R}_{n_s-1}(L_s)^{i+1-n_s} . \end{aligned} \tag{1.9}$$

If $i+1-n_i$ is negative for any i, then we take $V_i(P,N) = \mathfrak{R}_i$.

We let $(\underline{a})$ or (a^s) denote the sequence $(a,\ldots,a)$ of length s. We showed in Theorem I of [**EI**] that

Lemma. The inverse system $K^{-1}(P,N)$ of $K(P,N)$ satisfies

$$[K^{-1}(P,N)]_i = \mathfrak{V}(P,N,i). \tag{1.10}$$

If $N = (\underline{a})$ then

$$[K^{-1}(P,\underline{a})]_i = \mathfrak{R}_{a-1}L(P)^{i+1-a}. \tag{1.11}$$

As corollary of the Lemma we have

Proposition 1.3A. Fix integers s,a,i and any set of points P in $\mathbb{P}^{r-1}$. Let $j = i+1-a$, and $\underline{a}$ denote $a^s = (a,\ldots,a)$. The sequence of vector spaces

$$K(P,\underline{a})_i, K(P,\underline{a+1})_{i+1}, \ldots\ldots, K(P,\underline{a+u})_{i+u}, \ldots \tag{1.12}$$

are the dual vector spaces to the sequence

$$(\mathfrak{R}_i/\mathfrak{R}_{a-1}L(P)^j), (\mathfrak{R}_{i+1}/\mathfrak{R}_a L(P)^j), \ldots, (\mathfrak{R}_{i+u}/\mathfrak{R}_{a+u}L(P)^j), \ldots . \tag{1.12a}$$

The sequence of integers

$$dim_k(K(P,\underline{a})_i), dim_k(K(P,\underline{a+1})_{i+1}), \ldots, dim_k(K(P,\underline{a+u})_{i+u}), \ldots \tag{1.12b}$$

are the portion in degrees at least i of the Hilbert function of the quotient algebra $(\mathfrak{R}_i/\mathfrak{R}_{a-1}L(P)^j)$ of $\mathfrak{R}$.

Proposition 1.3B. With the same notation, the sequence of vector spaces

$$\mathfrak{R}_{a-1}L(P)^j, \mathfrak{R}_{a-1}L(P)^{j+1}, \ldots, \mathfrak{R}_{a-1}L(P)^{j+u}, \ldots \tag{1.12c}$$

are the dual vector spaces to the sequence

$$R_i / K(P,\underline{a})_i, R_{i+1} / K(P,\underline{a})_{i+1}, \ldots, R_{i+u} / K(P,\underline{a})_{i+u}, \ldots . \tag{1.12d}$$

The sequence of integers

$$dim_k(\mathfrak{R}_{a-1} L(P)^j), dim_k(\mathfrak{R}_{a-1} L(P)^{j+1}), \ldots, dim_k(\mathfrak{R}_{a-1} L(P)^{j+u}), \ldots \tag{1.12e}$$

is the portion in degrees at least i of the Hilbert function $H(R/K(P,a))$.

Theorem 1.4. The Hilbert function $H(s,a,r)$ of $K(P,\underline{a},r)$ for a general set of s points P of $\mathbb{P}^{r-1}$ satisfies

$$\begin{aligned} H(s,a,r)_i &= \dim_k((L(P)^{i+1-a}) \cap \mathfrak{R}_i) = d(\underline{a-1},i,r) \\ &= \dim_k \mathfrak{R}_i - T'(\underline{i+1-a},r)_i. \\ &= \dim_k \mathfrak{R}_i - T(\underline{i+1-a},r)_i - \mathrm{def}(\underline{i+1-a},r)_i. \end{aligned} \tag{1.13}$$

Furthermore, we have for P general,

$$H_N(r)_i = \dim_k \mathfrak{R}_i - T'(\underline{i+1}-N;r)_i, \quad \text{and} \tag{1.13a}$$

$$\dim_k(K(P,N,r)_i) = T'(\underline{i+1}-N;r)_i. \tag{1.13b}$$

We have thus shown,

EQUIVALENCE PRINCIPLE: GIVEN (r,s), THE PROBLEM OF DETERMINING THE SET OF HILBERT FUNCTIONS $T'(r,J)$ OF THIN POWER ALGEBRA QUOTIENTS OF $\mathfrak{R}$ FOR ALL s-TUPLES J, AND THE PROBLEM OF DETERMINING THE HILBERT FUNCTIONS OF VANISHING IDEALS OF ALL ORDERS N AT s GENERAL POINTS OF $\mathbb{P}^{r-1}$ ARE EQUIVALENT!

Given (r,s), there is a similar equivalence between i and ii:

i. Determining the set of all Hilbert functions $T'(j^s,r)$ of thin power algebras R/L^j where L^j is the j-th powers of s general linear forms (whose choice may depend on j).

ii. Determining the set of all Hilbert functions $H(s,a,r) = H(R/K(P,a^s))$ of equal order vanishing ideals $K(P,a^s)$ at a set of general enough points P of $\mathbb{P}^{r-1}$ (whose choice may depend on a).

In particular, equation (1.13a) shows that Question B′ for (U,i,r) with $U = \underline{a-1} = (a-1)^s$ is equivalent to Question C′ for (N,i,r) with $N = (\underline{a}) = (a^s)$.

Of course, the two sets of Hilbert functions are not the same! The Koszul relations for the ideal L^J give lower bounds for

$T(J,r)$, which translate into more complex upper bounds for the Hilbert function $H_N(r) = H(R/K(P,N))$ of vanishing ideals.

First Goal of Paper: Using the inequalities on $T(J,r)$ arising from the Koszul relations for the ideal L^J, the non-negativity of $T(J,r)$, formula (1.3) and Theorem 1.4, we will obtain upper bounds for $H_N(r)$ (Theorem 2.2). We also recover in this manner a lower bound for the regularity of the ideals $K(P,N)$ (Proposition 2.15).

We conjecture that when $|N| = \Sigma n_i$ is bounded by n_0, then there are only a finite number of exceptional triples (r,s,N), $|N| \le n_0$, where the upper bounds of Theorem 2.2 are not attained for a general set of s points in $\mathbb{P}^{r-1}$ (Conjecture 2.2.0, and §3E).

Our upper bounds give nothing unexpected for the plane $\mathbb{P}^2$: If $r = 3$ and $s \le 9$ then $H_N(3)$ is known (see [**H2**]). If $s \ge 5$ in the plane, and $N = (a^s)$, our upper bounds are that of points in n-generic position, where n is the degree of $K(P,N)$ (see Definition 2.4 and Lemma 2.5). An explanation is found in Lemma 1.7: for large a only if $s < 2^{r-1}$ do our upper bounds for $H(a,s,r)$ differ from the obvious bound

$$H(a,s,r)_i \le \min(\dim_k R_i, \mathrm{degree}(K(P,a^s))). \tag{1.14}$$

A. Hirschowitz has given a sharper set of inequalities for $H_N(3)$ in the plane case, predicting the defect for general P ([**H2**]).

For $\mathbb{P}^3$, when $s \le 5$, then $H_N(4)$ is known by Proposition 2.3. We now give the first example - the vanishing ideal of order 10 at six points of $\mathbb{P}^3$, where our upper bounds imply that their Hilbert function $H(10,6,4)$ cannot be the Hilbert function of points in n-generic position on $\mathbb{P}^3$.

Example 1.5. A VANISHING IDEAL AT SIX GENERAL POINTS OF $\mathbb{P}^3$ IN NON n-GENERIC POSITION. If $(r,s) = (4,6)$ and $N = (10^s)$, $a = 10$, then $n = \mathrm{mult}(K(P,N)) = 6\dim_k R_9 = 1320$. In degree $i = 18$, $\dim_k \mathcal{R}_{18} = 1330$. The upper bound (2.11) for $H(R/K(P,\underline{10}))_i$ is

$$T(\underline{18+1-10},4)_{18} = T(\underline{9},4)_{18} \le 1305$$

which is less than $\min(n,\dim_k \mathcal{R}_{18}) = 1320$, so $K(P,N)$ is neither 18-regular, nor in n-generic position. In fact $T(\underline{9},4)_{18} = 1270$, smaller than our bound (see the entry $j = 9$, $v = 2$ of Table 7).

Remark. There are more examples of non n-generic behavior in $\mathbb{P}^3$ when s is 6 or 7, and the order of vanishing is large. As r increases, our upper bounds for $H(R/K(P,N))$ become more salient, and sharply restrict when n-genericity occurs (See Lemma 1.7, Examples 2.8A,B, Examples 2.12-2.14, and Proposition 2.15).

If $T(J;r)_i = T'(J,r)_i$ and we assume the standard Conjecture 2.0 below concerning the Hilbert function of thin algebras, then the upper bound we give in (2.11) for $H_N(r)_i$ would be an equality for the corresponding triple $(N,i,r) = (\underline{i+1}-J,i,r)$. However, the following examples show that the defect $\text{def}(J,r)_i = T'(J;r)_i - T(J;r)_i$ can be nonzero, and even affect the regularity. For further examples of nonzero defect see §3B-§3D.

Example 1.6A. NONZERO DEFECT. When $(r,s) = (3,5)$, $\mathfrak{R} = k[X,Y,Z]$, and $V = \langle X^8, Y^8, Z^8, (X+Y+Z)^8, (X+13Y+7Z)^8\rangle$,
the Hilbert function of $\mathfrak{R}/(V)$, as calculated by the algebra program "Macaulay" [**BSE**] satisfies

$$H = (1,3,6,10,15,21,28,36,40,40,36,28,16,\mathbf{6},\mathbf{1}).$$

Varying the coefficients of the last form in examples leads to the same Hilbert function; we conclude experimentally that $T'(5,8,3) = H$, so $\mathfrak{R}_5V$ satisfies

$$\begin{aligned}\dim_k \mathfrak{R}_5 V &= \dim_k \mathfrak{R}_{13} - H_{13}\\ &= 105-6 = 99.\end{aligned}$$

By D. Anick's result [**A**] for $r = 3$, thin algebras defined by vector spaces V of dimension 5 and degree 8 have Hilbert function $T(5,8,3)$ ending in $(40,40,36,28,16,0)$, hence $\dim_k \mathfrak{R}_5V = 105$. There is a nonzero defect

$$\text{def}(5,8,3)_{13} = T'(5,8,3)_{13} - T(5,8,3)_{13} = 105-99 = 6$$

in degree 13, and a defect of one in degree 14. The translation of this example to vanishing ideals was known (Example 2.16).

Example 1.6B. NONZERO DEFECT AFFECTING REGULARITY. When $(r,s) = (4,6)$, $\mathfrak{R} = k[W,X,Y,Z]$ and $L^4 = \langle W^4, X^4, Y^4, Z^4, (X+Y+Z+W)^4, (X+2Y+3Z+4W)^4\rangle$ "Macaulay" gives

$$H(\mathfrak{R}/(L^4)) = (1,4,10,20,29,32,24,4).$$

After varying the last form, and obtaining the same Hilbert function, we conclude experimentally that $H(\mathfrak{R}/(L^4)) = T'(6,4,4)$. Another calculation in "Macaulay" verifies that for thin algebras, $T(6,4,4) = (1,4,10,20,29,32,24,0)$, the conjectured value. Thus, there is a nonzero defect, $\text{def}(6,4,4)_7 = 4$, affecting the regularity of $K(P,4^6)$ in $\mathbb{P}^3$: the Hilbert function $H(R/K(P,4^6)) = (1,4,10,20,35,56,84,116,120,120,\ldots)$ is regular in degree 8, not 7.

Second Goal of Paper: In §3 we experimentally study the defects $\text{def}(s,j,r)$ and the mysterious patterns that occur as j varies. We give examples, tables, and conjectures in the special cases $(r,s) = (5,7),(4,6)$ and as well as other cases. Our hope is that this data and our viewpoint will be useful to others.

We now estimate the range of values (s,r) for which there exist integers "a" such that the "Koszul bounds" of Section 2 prevent the ideal $K(P,a^s)$ in r-variables from being in n-generic position. The *socle degree* of an Artin algebra A is the largest i such that $A_i \neq 0$.

Lemma 1.7. We assume that the usual Conjecture 2.0 below concerning $T(j^s,r)$ is true, and we suppose that (r,s) with $r \geq 2$ and a constant b satisfying $1 < b < 2$ is given. Then if j is large and

$$s \geq \left(\frac{b}{b-1}\right)^{r-1} \tag{1.15}$$

the socle degree SOCDEG(j,r,s) of a thin algebra A generated by s degree-j forms in r variables satisfies

$$\mathrm{SOCDEG}(j,r,s) \leq bj. \tag{1.16}$$

Equality in (1.15) for a value $b < 2$ implies asymptotic equality in (1.16) up to an error $O(j)$. When $s \geq 2^{r-1}$, the "Koszul bounds" of Theorem 2.2 for $H(s,a,r)_i$ for arbitrary a and large i are the same as the n-generic position bounds (1.14).

Proof. We want $\mathrm{SOCDEG}(L^j) \leq bj-1$. Since $bj-1 < 2j-1$ it suffices to show that $s(\dim_k R_{[(bj]-1-j}) \geq \dim_k R_{[bj]}$,

$$s\binom{(\lfloor bj\rfloor - j + r - 1}{r-1} \geq \binom{\lfloor bj\rfloor + r - 1}{r-1}. \tag{1.17}$$

Let $f(x) = (xj+r-1)_{r-1} \cdot j^{-r}$. By expanding f in powers of x, it is easy to see that if s satisfies (1.15) then $s \cdot f(b-1) \geq f(b)$, which implies (1.17) when j is large. For $b < 2$ and j large, equality in (1.17) is approximated by equaltiy in the top degree terms of $s \cdot f(b-1) = f(b)$, namely $s(b-1)^{r-1} \geq b^{r-1}$, or equality in (1.15).

It is easy to verify directly that if j is larger than r, $\mathrm{SOCDEG}(j,r,2^{r-1}) < 2j$; consequently, if $s > 2^{r-1}$, $\mathrm{SOCDEG}(j,r,s) < 2j$. It follows that if $s > 2^{r-1}$, and $j > r$ the "Koszul upper bound" for $H(R/K(P,a^s))_i$ is the same as the n-generic position bound (1.14). When $s \geq 2^{r-1}$, and j is small, and if i is larger than $r(j-1)$ then $T(j^s,r)_i = 0$ so (1.3) and (1.13b) yield only the empty condition $\dim_k K(P,a^s)i \geq 0$. This proves the Lemma.

Remark. By Stanley's result, Proposition 2.3A, the size of $K(P,a^s)_i$ for general P is known when $s \leq r+1$. Thus, the interesting values of s for which the "Koszul bound" of Section 2 may be nontrivial and not previously known for given a and large i, satisfy

$$r+2 \le s \le 2^{r-1}. \tag{1.18}$$

Thus, Theorem 2.2 gives no new information when $r = 3$ and i is large. When $r = 4$, the interesting values of s for the "Koszul bounds" are $s = 7,8$; for $r = 5$, they are $7 \le s \le 16$. As r increases, we expect a greater number of "Koszul regions" to intervene: taking $b = 3$ in we find that for $s \le (3/2)^{r-1}$ we use two terms of the Koszul resolution for large i (see §2B).

Plan of Paper. In Theorem 2.2 of Section 2A we give the "Koszul" upper bounds for $H_N(r) = H(R/K(P,N))$. In Section 2B we give more explicit bounds for $H_N(r)$ in the case of equal weights $N = (\underline{a})$. The degrees i fall into regions $S_1,\ldots,S_a,S_\infty$ whose size depends on the steps of the Koszul resolution of L^{i+1-a}. In each region S_u, $u \le a$, the bounding Hilbert function $G(a^s,r)$ is a polynomial of degree u in s (Theorem 2.10). In Section 2C we give a lower bound for the regularity of $K(P,N)$.

In Section 3 we compare our bounds with experiment and with previous work. Let $VER(j,r)$ denote the Veronese embedding of $\mathbb{P}^{r-1}$ into $\mathbb{P}(\mathfrak{R}_j)$, via j-th powers. When $J = (j^s)$, and $u = 1$, Question B relates to the tangent variety $TAN(1,s,j,r)$ of the s-secant variety $SEC(s,j,r)$ of $VER(j,r)$: this is the Terracini-Bronowski approach to the Waring problem for forms. We have shown that both Question B′ for $u = 1$, and the Waring problem for degree-d forms when k is an algebraically closed field of char $k = 0$, or char $k > d$ are solved by the results of A. Alexander and A. Hirschowitz on the Hilbert functions of order-2 vanishing ideals at general points of $\mathbb{P}^{r-1}$ (see [**I3**]). It is natural to ask, what is known for $u > 1$?

When $u > 1$, Question B′ relates to the higher osculating u-varieties $TAN(u,s,j,r)$ to the s-secant variety. In Section 3A we survey the consequences for Question B′ when $u > 1$ of previous work on vanishing ideals. We translate results from work of A. Geramita, A. Gimigliano, P. Maroscia, N.V. Trung, G. Valla, and others on the postulation of a general set of fat points. We then compare these results with experiment.

In two short Sections 3B and 3C we show how to obtain the postulation of $K(P,N)$ from knowlege of the Hilbert function of thin power algebras (see Table 5 and Examples 3.7A,B). Our approach here is algorithmic and our calculations use the "Macaulay" algebra program [**BSE**].

In Section 3D, we summarize extensive calculations, to make some conjectures about the patterns in the defect $def(s,a,r)$. This section is intended to illustrate our suggestion, that the patterns are more visible in the context of *thin power algebras*, than in the context of vanishing ideals. Section 3E gives evidence for our main conjectures.

Overview: We use Macaulay's inverse systems, which are R-submodules of $\mathfrak{R}$. The map taking ideals I of R to their inverse systems I^{-1} in $\mathfrak{R}$ takes intersections to sums. Macaulay used them to give a theory of primary decomposition, dimension, and other properties of ideals. The inverse system I^{-1} is *not* itself an ideal of $\mathfrak{R}$. The a-th order vanishing ideal K(P,a) is the a-th symbolic power of the order-one vanishing ideal K(P,1). Suitably chosen pieces of the inverse systems $[K(P,a)]^{-1}$ will themselves form an ideal of $\mathfrak{R}$, here the ideal $L(P)^J$ determined by the J-th powers of a set L of s linear forms in $\mathfrak{R}$. Our main results come from translating the Koszul and other relations for the ideal $L(P)^J$ to information on vanishing ideals.

I warmly thank J. Emsalem and V. Kanev, whose collaborations with the author in [**EI**] and [**IK**] led directly to this work. I thank also I. Dolgachev, D. Eisenbud, A. Galligo, A. Hirschowitz, M. Johnson, and B. Reznick, for helpful comments, and A. Geramita, M. Coppo, and G. Valla who answered questions.

The series began from discussions between J. Emsalem and the author, and a visit to V. Kanev at the Bulgaria Academy of Sciences. I thank the Algebra Section of the Bulgaria Academy of Sciences, and the Mathematics Department of the University of Nice, in particular the Nice geometry group of J. Briançon, A. Galligo, A. Hirschowitz, P. Maisonobe, M. Merle, and J. Yameogo, for their hospitality and discussions during the preparation of most this work.

2. The Hilbert function of vanishing ideals.

2A. Upper bounds for the Hilbert function of R/K(P,N).

We first recall the lower bound $F_{\mathbf{d}}(Z)$ for the Hilbert function $H_{A(f)}(Z)$ of algebras $A(f) = \mathfrak{R}/(f)$ where the ideal (f) is generated by a set f of s forms f_i of degree j_i, as described by D. Anick [**A**]. Suppose that $f_1,\ldots,f_s$ are homogeneous forms of $\mathfrak{R}$, and let $J = (j_1,\ldots,j_s)$, where j_i = degree(f_i). We call the sequence $\mathbf{d} = (r,s;j_1,\ldots,j_s)$ the *degree vector* for the graded algebra $A(f) = R/(f_1,\ldots,f_s)$. Let $F'_{\mathbf{d}}(Z) = (1-Z)^{-r}\prod_{1\le u\le s}(1-Z^{j_u})$, and set

$$F_{\mathbf{d}}(Z) = |(1-Z)^{-r}\prod_{1\le u\le s}(1-Z^{j_u})|, \tag{2.1}$$

where the notation $|\Sigma a_iZ^i|$ denotes Σb_iZ^i where

$$\begin{aligned} b_i &= a_i \text{ if } a_k \ge 0 \text{ for all } k \le i \\ &= 0 \quad \text{if some } a_k < 0,\ k \le i. \end{aligned} \tag{2.2}$$

That the Hilbert function of a thin algebra is bounded below by $F'_{\mathbf{d}}(Z)$ comes from the Koszul complex for A(f); the emendation to a bound by $F_{\mathbf{d}}(Z)$ in (2.3) of the Lemma below comes from noting that $I_k = \mathfrak{R}_k$ implies $I \supset m^k$. Although the Koszul complex for a thin algebra cannot be exact except in the CI case $s \le r$, we nevertheless call the lower bound (2.3) and the related upper bound (2.8) for H(R/K(P,N)) "Koszul" bounds.

Lemma (See [**A**].) If $f = (f_1,\ldots,f_s)$ are any forms in $\mathfrak{R}$ having degree vector **d**, then the Hilbert function $H_{A(f)}(Z)$ of A(f) satisfies

$$H_{A(f)}(Z) \ge F_{\mathbf{d}}(Z). \tag{2.3}$$

When the f are chosen generically, then $H_{A(f)}(Z) = T(J,r,Z)$, and $T(J,r,Z) \ge F_{\mathbf{d}}(Z)$. When $J = (\underline{j})$, we obtain

$$T(s,j,r,Z) \ge |(1-Z)^{-r}(1-Z^j)^s|. \tag{2.4}$$

We let $F_{\mathbf{d}}(Z)_i$ denote the coefficient of Z^i in $F_{\mathbf{d}}(Z)$, and we let the "error" $err(J,r)_i$ denote the difference

$$err(J,r)_i = T(J,r)_i - F_{\mathbf{d}}(Z)_i. \tag{2.5}$$

Conjecture 2.0. When the f are chosen generically of degrees J, there is equality in (2.3), so err(J,r) is always zero.

Remark. The Conjecture has been shown for $r \le 3$ (see [**I1**] Proposition 4.2A for $r = 2$, and D. Anick's [**A**] for $r = 3$), and for $s \le r+1$ by R. Stanley (p.367 [**I1**]). Also, M. Hochster and D. Laksov have shown equality in (2.4) for $i = j+1$ (see [**HL**] and also [**I3**]). The Conjecture states that a set f of general enough forms of given degrees tries to have the Koszul resolution, and succeeds up until the smallest degree $i = \sigma+1$, where, for dimension reasons, $(A_f)_i$ must be zero. One might further conjecture that A_f has the simplest resolution (Betti numbers and degrees) consistent with $\mathrm{err}(J,r) = 0$:

In view of Conjecture 2.0, we will henceforth denote by $\Delta(J,r)_i$ (or when $J = j^s$, $\Delta(s,j,r)_i$) the difference

$$\Delta(J,r)_i = T'(J,r)_i - F_{\mathbf{d}}(Z)_i$$

$$= \mathrm{def}(J,r)_i + \mathrm{err}(J,r)_i$$

and will also call it the *defect* - from the Koszul bounds.

We apply the Lemma to the ideal $\mathfrak{J}(P,N,i)$ generating the vector space $\mathcal{V}(P,N,i)$. The degree vector $\mathbf{d}(N,i)$ of $\mathfrak{J}(P,N,i)$ satisfies

$$\mathbf{d}(N,i) = (r,s;i+1-n_1,\ldots,i+1-n_s). \qquad (2.6)$$

Definition 2.1. We let $G(N,r,Z) = \Sigma c_i(N,r)Z^i$, denote the series defined by

$$\begin{aligned} c_i(N,r) &= \dim_k R_i - F_{\mathbf{d}(N,i)}(Z)_i \\ &= \text{the coefficient of } Z^i \text{ in } [(1-Z)^{-r} - F_{\mathbf{d}(N,i)}(Z)]. \qquad (2.7)\end{aligned}$$

A subtlety of this definition is that we choose a *different* series $F_{\mathbf{d}(N,i)}$ to define $c_i(N,r)$ for each degree i. When $N = a^s$ we denote the sequence $G(N,r)$ by $G(s,a,r)$ or $G(a^s,r)$.

Theorem 2.2. KOSZUL UPPER BOUND ON THE HILBERT FUNCTION OF VANISHING IDEALS. If P is any set of points of $\mathbb{P}^{r-1}$, the algebra $R/K(P,N)$ has Hilbert function satisfying

$$H(R/K(P,N)) \le G(N,r,Z). \qquad (2.8)$$

For P generic, we have $H_N(r,Z) = \max_{P\in\mathbb{P}^{r-1}}(H(R/K(P,N),Z)$ and

$$H_N(r,Z) \le G(N,r,Z), \qquad (2.9)$$

For P generic, we have in degree i

$$H_N(r)_i = d(N-\underline{1},i,r)$$

$$= c_i(N,r) - \mathrm{def}(i+1-N,r)_i - \mathrm{err}(i+1-N,r)_i$$

$$= c_i(N,r) - \Delta(i+1-N,r)_i \quad (2.10)$$

$$\leq c_i(N,r). \quad (2.11)$$

Proof. The formula (2.8) is a consequence of the definition preceding (1.8), (1.10) and (2.3). Let $f_u = L_u^{i+1-n_u}$, where $L = L_1,\ldots,L_s$ are defined from P as in (1.8). Then we have

$$\begin{aligned}\dim_k\langle R_i/K(P,N)_i\rangle &= \dim_k V_i(P,N)\\ &= \dim_k \mathfrak{R}_i - H_{A(f),i} \quad \text{by (1.10)},\\ &\leq \dim_k \mathfrak{R}_i - F_{\mathbf{d}(N,i)}(Z)_i = c_i(N,r) \quad \text{by (2.3)}.\end{aligned}$$

Here the degree vector $\mathbf{d}(N,i) = (r,s;\underline{i+1}-N)$, and $H_{A(f),i}$ and $F_{\mathbf{d}(N,i),i}$ are the coefficients of Z^i in $H_{A(f)}(Z)$ and $F_{\mathbf{d}(N,i)}$, respectively.

The equality $H_N(r,Z) = \max_{P\in\mathbb{P}^{r-1}}(H(R/K(P,N),Z)$, and (2.9) follows from (1.10). The formula (2.10) follows from the definitions of the functions d(U,i,r), def, and err (see Question B′, Definition 1.2, and (2.5), respectively).

It is natural to ask when there is equality in (2.9).

Question E. When s points P are generically chosen in $\mathbb{P}^{r-1}$, for which sequences N of s nonnegative integers is the Hilbert function of R/K(P,N) equal to G(N,r,Z)?

Conjecture 2.2.0.. There are only a finite number $\tau(a)$ of exceptional triples (N,i,r) satisfying, $n_u \leq a$ for each u, and for which $H_N(r)_i \neq c_i(N,r)$.

Remark. J. Alexander and A. Hirschowitz have shown this Conjecture with four exceptional triples when a = 2 (See [**A1**],[**AH1**],[**AH2**], and [**I3**]). See §3E for a discussion of experimental evidence for the Conjecture when a > 2. Equality in Question E for some pair (N,r) is equivalent to $\mathrm{err}(\underline{i+1}-N,i,r) = \mathrm{def}(\underline{i+1}-N,i,r) = 0$ for all i.

Proposition 2.3A. (R. Stanley). Questions A and B have the same answer, and Conjecture 2.0 is satisfied whenever $s \leq r+1$.

Proposition 2.3B. There is equality in (2.9) whenever $s \leq r+1$ and the s points P are generically chosen in $\mathbb{P}^{r-1}$.

Proof. The known Hilbert function of complete intersections $(L_1^{j_1},\ldots,L_s^{j_s})$ handles the case $s \leq r$. When $s = r+1$ R. Stanley's proof concerning thin algebras, quoted p.367 of [**I2**], applies also to thin power algebras. Recall that he used the strong Lefschetz theorem on the cohomology ring $B = H^*(\mathbb{P}) = \mathfrak{R}/(X_1^{j_1},\ldots,X_r^{j_r})$, of a product $\mathbb{P}$ of projective spaces, to show that the Hilbert function of the Artin algebra $A = B/(L_{r+1}{}^{j_{r+1}})$ verifies Conjecture 2.0. This proves Proposition 2.3A, and with (2.10) shows Proposition 2.3B.

Note. Because of Proposition 2.3B, we will rarely consider the case $s \leq r+1$ further, even though it readily yields examples where the upper bounds of Theorem 2.2 prevent n-genericity (see Example 2.8A). The case $s \leq r$ was known to specialists, but the knowledge of $H(s,a,r)$ when $s = r+1$ appears to be new.

Definition 2.4. (After A. Geramita, P. Maroscia [**GM**]). The length-n subscheme Z in $\mathbb{P}^{r-1}$ with defining ideal $I(Z)$ in R, is in "n generic position" if its Hilbert function satisfies $H(R/I(Z),\mathcal{Z}) = HGP(n,r,\mathcal{Z})$, where

$$HGP(n,r,Z) = \Sigma(\min(\dim_k R_i, n)\mathcal{Z}^i. \qquad (2.12)$$

That is, in each degree i for which $n \leq \dim_k R_i$ the scheme Z cuts out n linearly independent conditions on degree-i functions; and if $n \geq \dim_k R_i$ there are no functions of degree i vanishing on Z.

Lemma 2.5. When Z is $\mathrm{Spech}(R/K(P,N))$ then the *length* of Z, or *degree* of $K(P,N)$ satisfies

$$n = \deg K(N,P) = \Sigma_u \dim_k R_{n_u-1}. \qquad (2.13)$$

Proof. This follows from the case $s = 1$, $P = p$, where $\deg K(N,p) = \dim_k(R_p/m_p{}^{n_u}) = \dim_k R_{n_u-1}$.

When $r = 2$, the points P belong to $\mathbb{P}^1$ and $R \cong k[x,y]$. We use inverse systems to show that any power algebra is thin when $r = 2$, and thus reprove the classical Jordan Lemma for binary forms.

Proposition 2.6. POWER ALGEBRAS ARE THIN WHEN $r = 2$. For any set of distinct points $P = (p_1,\ldots,p_s)$ in $\mathbb{P}^1$ and for any sequence of multiplicities $N = (n_1,\ldots,n_s)$, the principle ideal $K(P,N)$ is in n-generic position. If $L = (L_1,\ldots,L_s)$ with $L_k = a_kX+b_kY \in \mathfrak{R}$ are arbitrary linear forms, and J is an arbitrary sequence of positive integers then $\mathfrak{R}/L^J$ is a thin algebra. We have $\mathrm{def}(J,2)_i = \mathrm{err}(J,2)_i = \Delta(J,2)_i = 0$ for all i.

Proof. We have $K(P,N) = (g_{P,N})$, of degree $n = \Sigma n_k$, of Hilbert function

$$H(R/K(P,N)) = (1,2,\ldots,n-1,n,n,\ldots),$$

satisfying $H(P,N)_i = \min(i+1,n)$. Thus, the scheme $Z = \Sigma n_k p_k$ is in n-generic position.

If J is a sequence of s positive integers, $i \geq j_{max}$, and N = $\underline{i+1}$-J, so $n_k = i+1-j_k$, then by (1.10), $(L^J) \cap \mathcal{R}_i = K^{-1}(P,N)_i$, whose dimension is

$$\min(i+1,n) = \min(i+1,\Sigma_u(i+1-j_u))$$

by the n-genericity of K(P,N). This shows that when r = 2, any power algebra $\mathcal{R}/(L^J)$ is a thin algebra, and that def(J,2) = err(J,2) = Δ(J,2) = 0. We have shown

Claim. If $n = \Sigma n_i$ satisfies $n \leq i+1 = \dim_k \mathcal{R}_i$, and if $j_k = i+1-n_k$, then the subspace $\mathcal{R}_{n_1-1}L_1^{j_1} \oplus \ldots \oplus \mathcal{R}_{n_s-1}L_s^{j_s}$ of $\mathcal{R}_i$ is a direct sum.

The Claim is an avatar of the classical Jordan lemma:

Lemma (Appendix III of [**GY**]). If $L = (L_1,\ldots,L_s)$ are linear forms in $\mathcal{R} = \mathbb{C}[X,Y]$, no two of which are dependent, and U = $u_1,\ldots,u_s$ are s positive integers satisfying $\Sigma u_i = j-s+1$, then it is impossible to find binary forms $F_1,\ldots,F_s$ of degrees $u_1,\ldots,u_s$ such that $L_1^{j-u_1}F_1 + \ldots + L_s^{j-u_s}F_s = 0$.

Proposition 2.7. CONDITION FOR $G(N,r,\mathcal{Z})$ TO BE IN n-GENERIC POSITION. Assume that $N = a^s$, $a \geq 2$, and $2 \leq s \leq \dim_k \mathcal{R}_{a-1}$, and let deg K denote deg(K(P,N)). Then TFAE:

i. $G(N,r,\mathcal{Z})$ = HGP(deg K, r),

ii. $c_i(N,r) \geq \dim_k \mathcal{R}_i$ for each $i \leq 2a-2$. (2.14)

This can occur only if

$$(\deg K - \binom{s}{2}) \geq \dim_k R_{2a-2}. \qquad (2.15)$$

Proof. If $N = a^s$, $a \geq 2$, $2 \leq s \leq \dim_k \mathcal{R}_{a-1}$, then by Lemma 2.5 deg K = $(\dim_k R_{a-1})s$. By (1.9) and (1.11) $K^{-1}(P,a^s)_i = \mathcal{R}_{a-1}(L(P)^{i+1-a}$. By (2.3) when i = 2a-2, the dimension of $\mathcal{R}_{a-1}(L_1^{a-1},\ldots,L_s^{a-1})$ satisfies the inequality

$$\dim_k \mathcal{R}_{a-1}L(P)^{a-1} \leq (\dim_k \mathcal{R}_{a-1})s - \binom{s}{2} = (\deg K - \binom{s}{2}). \qquad (2.16)$$

Thus $c_{2a-2}(a^s,r) < \deg K$ unless (2.15) is satisfied. Similarly, if $i < 2a-2$, $c_i(N,r)$ is strictly less than deg K, while for i > 2a-

2, $c_i(N,r) = \min(\deg K, \dim_k \mathfrak{R}_i)$. The Corollary now follows from Definition 2.4.

Example 2.8A. TWO FAT POINTS NOT IN n-GENERIC POSITION, BUT DEFECT ZERO. Two fat points are rarely in n-generic position. When $(r,s) = (3,2)$, and $N = (3,3)$, then $\deg(K) = 2(\dim_k \mathfrak{R}_2) = 12$, and

$$\deg(K) - \binom{2}{2} = 11 < \dim_k \mathfrak{R}_3.$$

Thus, when $P = (p_1,p_2)$ are arbitrary in $\mathbb{P}^2$, we have

$$H(R/K(P,N)) = G(N,3,\mathbb{Z}) = (1,3,6,9,11,12,12,\ldots).$$

Here the choice $P = ((1,0,0),(0,1,0))$ is general enough, so $K = (y,z)^2 \cap (x,z)^2$. Then $I_3(P,N) = (X,Y)$ and $V_3(P,N) = (X,Y) \cap \mathfrak{R}_3$ has codimension 1 and dimension 9 in $\mathfrak{R}_3$. Likewise, $I_4(P,N) = (X^2,Y^2)$, so $V_4(P,N) = (X^2,Y^2) \cap \mathfrak{R}_4$ has codimension 4 and dimension 11 in $\mathfrak{R}_4$, and is the annihilator $K^{-1}(P,N)_4$ of $K(P,N)_4 = (xz^3, yz^3, xyz^2, z^4)$ in R_4.

Example 2.8B. TWENTY-FOUR FAT POINTS NOT IN n-GENERIC POSITION, DEFECT ZERO. We consider $(r,s) = (10,24)$, $N = (\underline{4})$, $i = 6$, and $J = (\underline{6+1-4}) = (\underline{3})$. A calculation in "Macaulay"[2] shows that $T'(J,10)_6 = T(J,10)_6 = 1$, so $\operatorname{def}(J,10)_6 = \operatorname{err}(J,10)_6 = 0$. It follows from (1.10) that

$$H_N(10)_6 = c_6(N,10) = \dim_k \mathfrak{R}_6 - T'(J,10)_6 = 5004,$$

so $K(P,N)_6$ has dimension one, and codimension 5004 in $\mathfrak{R}_6$. By (2.13) the degree of $K(P,N)$ is $n = (24)(\dim_k R_3) = 24(220) = 5280$. Thus, $5004 < \min(n, \dim_k \mathfrak{R}_6)$ and $K(P,N)$ cannot be in n-generic position. What is striking about the example is that the upper bound is the exact value. See also §3E.

<u>2B. Koszul regions for the Hilbert function of</u> $K(P,a^s)$.

We elaborate the Koszul upper bound for the Hilbert function of vanishing ideals $K(P,a^s)$ at s points of $\mathbb{P}^{r-1}$ when all the multiplicities are equal. The following material is complex and, at times technical. The complexity illustrates one of our themes: problems involving the Hilbert functions of vanishing ideals at s general points P of $\mathbb{P}^{r-1}$ can be simpler when viewed in the context of thin algebras - so are more complicated when stated in

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[2] This calculation, done in characteristic 17, used 4.5MB and took over 12 hours on an accelerated Macintosh SE-30. A try for $J = (\underline{4})$ ran out of space after 26MB. See §3C below for the "Macaulay" program used.

terms of $K(P,a^s)$! We will show that the integers i fall into "Koszul regions" S_u which depend only on the order of vanishing a, and not on r or s. For values i in each region S_u the upper bound $G(a^s,r)_i$ is governed by min(s,u) terms of the Koszul resolution for the corresponding thin algebra R/L^{i+1-a} (Theorem 2.10).

We illustrate the Koszul region concept by the cases a = 3, a = 4, and a = 7 (Examples 2.12-2.14). The reader may find it helpful to look at Example 2.12A, before reading Theorem 2.10. There is a summarizing Remark after Corollary 2.11.

Definition 2.9. KOSZUL REGIONS. Given the positive integer a, we decompose the positive integers into no more than a+1 disjoint intervals $S_1 > S_2 > \ldots > S_a > S_\infty$, some of which may be empty:

$$\begin{aligned} &S_1: 2(a-1) < i. \\ &S_u: \left(\frac{u+1}{u}\right)(a-1) < i \leq \left(\frac{u}{u-1}\right)(a-1). \\ &S_\infty: i \leq a-1. \end{aligned} \tag{2.17}$$

These regions correspond to the steps in the Koszul resolution of $L^{i+1-\underline{a}}$.

If i is in S_u, $u \leq a$, then $(u-1)i \leq u(a-1)$, and $i \geq u(i-(a-1))$, so we may write

$$i = u(i-(a-1)) + e_{i,u}, \tag{2,18}$$

with $e_{i,u} \geq 0$. We define $e_{i,u}$ for all (i,u) by (2.18).

If $J = (j_1,\ldots,j_s)$ we let $F'_{\mathbf{d},i}$ denote the coefficient of $F'_{\mathbf{d}}(Z)$ on Z^i, where $\mathbf{d} = (r,s,J)$ and $F'_{\mathbf{d}}(Z)$ is the series inside the brackets of (2.1). We set

$$\tau(J,r) = \begin{cases} \min(\{i \mid F'_{\mathbf{d},i} < 0\}), \text{ or} \\ +\infty \text{ if } F'_{\mathbf{d},i} \geq 0 \text{ for all } i. \end{cases}$$

In stating the following result, we wish to evade regions where $F_{\mathbf{d}}(Z) \neq F'_{\mathbf{d}}(Z)$.[3] Thus, given a, we omit certain very small i from the decomposition below, paradoxically, by requiring $i < \tau(\underline{i+1-a},r)$. We suppose that the s points P are chosen generically in $\mathbb{P}^{r-1}$, and consider the ideal $K(P,\underline{a}) = m_{p(1)}{}^a \cap \ldots \cap m_{p(s)}{}^a$. Recall that $\underline{a} = (a,\ldots,a)$ denotes the pair [s;a] where s is the length of the sequence.

[3] An example of the inequality $F_{\mathbf{d}} \neq F'_{\mathbf{d}}$ is (r,s) = (3,5), j = 2: the $F'_{\mathbf{d}}$ series begins (1,3,1,-5,1,...), to be replaced by $F_{\mathbf{d}}$ = (1,3,1,0,0...).

Theorem 2.10. KOSZUL UPPER BOUND ON THE HILBERT FUNCTION OF a-TH ORDER VANISHING IDEALS. Suppose $a \geq 2$, $i \in S_u$, $2 \leq s \leq \dim_k R_{i-(a-1)}$, and $i < \tau(\underline{i+1-a}, r)$. Then the codimension of $K(P,\underline{a})_i$ in R_i satisfies

$$H(P,\underline{a})_i \leq c_i(\underline{a},r), \tag{2.19}$$

where

$$c_i(\underline{a},r) = \sum_{1\leq t\leq min(u,s)} c_i(\underline{a},t,r), \text{ with}$$

$$c_i(\underline{a},t,r) = \begin{cases} (-1)^{t+1}(dim_k R_{e_{i,t}})\binom{s}{t}, & \text{if } 1 \leq \mu \leq s, \text{ and } e_{i,t} \geq 0 \\ 0, & \text{if } e_{i,t} < 0. \end{cases} \tag{2.20}$$

If $s \geq \dim_k R_i$, or if $i \in S_\infty$, then we have $K(P,\underline{a})_i = 0$ and $c_i(\underline{a},r) = \dim_k R_i$. If $i \geq \tau(\underline{i+1-a})$ then we also have $c_i(\underline{a},r) = \dim_k R_i$.

Proof. The first statement and (2.19),(2.20) follow immediately from Theorem 2.2. The statement concerning $s \geq \dim_k R_i$ follows from the result of Geramita-Maroscia and Orecchia that a set of s general points in $\mathbb{P}^{r-1}$ are in s-generic position (see **[GM]**), which in turn is a consequence of (1.10) and the well known fact that the vector space $\mathfrak{R}_i$ is spanned by powers of linear forms.

We let the series $G_t(\underline{a},r,Z) = \Sigma c_i(\underline{a},t,r)Z^i$ for $t \geq 1$. We have

Corollary 2.11. For $i \geq \tau(\underline{i+1-a},r)$, the series $G(\underline{a},r,Z)$ satisfies

$$G(\underline{a},r,Z)_{\geq\tau} = \max(0, \Sigma_{1\leq t\leq s} G_t(\underline{a},r,Z)_{\geq\tau}).$$

When i is in S_u and $t \leq u$, then $G_t(\underline{a},r,Z)$ has degree t in s; if i is in S_u and $t > u$ then $G_t(\underline{a},r,Z)_i$ is zero.

Summary. We have now decomposed $G(\underline{a},r,Z)_{\geq\tau}$ into a sum whose t-th term $G_t(\underline{a},r)_i$ is for each i a fixed polynomial of degree t in s. The dependence of the t-th term on i is more complicated: we have $e_{i,u} = e_{i+1,u} + (u-1)$, and i enters into $G_t(\underline{a},r)_i$ in the factor $\dim_k \mathfrak{R}_{e_{i,u}}$. In each Koszul region S_u, there are $b = \min(s,u)$ possibly nonzero terms $G_1(\underline{a},r,Z),\ldots,G_b(\underline{a},r,Z)$ in the expansion of $G(\underline{a},r,Z)$. When a and i are fixed, $i \in S_u$, and $s \geq u$, the bound $c_i(\underline{a},r) = G(\underline{a},r)_i$ for $\dim_k(R_i/K(P,\underline{a})_i$ is a polynomial of degree u in s, the number of points.

If Conjecture 2.2.0 is true, then given the integer a, there are only a finite number of exceptional triples (s,r,i) for which the bound (2.19) is not an equality for a general set P of s points in $\mathbb{P}^{r-1}$. By (2.10), such a failure of equality in (2.19) requires either a nonzero *defect* def($\underline{i+1-a}$,r) or nonzero *error*, thus occurs when $\Delta(\underline{i+1-a},r) \neq 0$.

Example 2.12A. KOSZUL REGIONS. When a = 3, the regions are

S_1: 2(2) < i: 4 < i, where

$$h(s,3,r)_i \leq c_i(\underline{3},r) = \min((\dim_k R_2)s, \dim_k R_i);$$

S_2: (3/2)2 < i ≤ 2(2):   i = 4, where

$$h(s,3,r)_4 \leq c_4(\underline{3},r) = \min\left((\dim_k R_2)s-\binom{s}{2}, \dim_k R_4\right);$$

S_3: (4/3)2 < i ≤ (3/2)(2):   i = 3, where

$$c_3(\underline{3},r) = \min\left(\left[(\dim_k R_2)s-\dim_k R_1\binom{s}{2}+\binom{s}{3}\right], \dim_k R_3\right),$$
$$=\binom{r+2}{3}-\binom{r-s+2}{3} \text{ if } s \leq r, \text{ and } \binom{r+2}{3} \text{ otherwise..}$$

S_∞: i ≤ 2. $c_i(\underline{3},r) = \dim_k R_i$.

Example 2.12B. KOSZUL REGIONS AND n-GENERIC POSITION FOR H(s,3,4), s SMALL. When s ≤ r+1 H(s,a,r) = G(a^s,r) (Proposition 2.3B). In Table 1 below we give G(a^s,4) for 2 ≤ s ≤ 5.

For s ≤ 3, the scheme Spec(R/K(P,$\underline{3}$)) becomes regular only in degree 5. For s = 4, the scheme is not in 40-generic position, because there is at least one quartic vanishing on it. For s = 5, the ideal K(P,$\underline{3}$) has degree n = 50, and is in 50-generic position. Note that as s increases, with r,a fixed, the scheme approaches n-generic position. The defect Δ = 0 throughout.

s	The sequence G($\underline{3}$,4)							Comment
2	1	4	10	16	19	20	20	Regularity i = 5.
3	1	4	10	19	27	30	30	Regularity i = 5.
4	1	4	10	20	34	40	40	Note t_4 = 34
5	1	4	10	20	35	50	50	50-generic position

Table 1. Values for H(s,3,4) = G($\underline{3}$,4), when r = 4, 2 ≤ s ≤ 5. (See Example 2.12B.)

Example 2.12C. KOSZUL REGIONS AND n-GENERIC POSITION FOR H(s,3,r), s LARGE. When a = 3, r ≤ 7 and s > r+1, G($\underline{3}$,r) = HGP(n,r), the Hilbert function of points in n-generic position, n = ($\dim_k R_2$)s: we do not predict surprising behavior.

However, if we fix b, set $s = r+b$, and increase r, we soon find a contradiction to n-generic behavior for $i = 4$ in the S_2 region. The degree n of $R/K(P,3,r)$ is bounded by a constant multiple of the cube of r, but $\dim_k R_4$ is approximately $r^4/4$; when r is large enough the scheme $\mathrm{Spec}(R/K(P,3^s,r))$ cannot be in n-generic position, by Proposition 2.7.

When $(r,s) = (8,10)$, ten points on $\mathbb{P}^7$, we have

$$H(10,3,8)_4 \leq (36)(10)-45 = 315,$$

which is less than $\dim_k R_4 = 330$, so $K(P,3^{10})$ is *not* in n-generic position in $\mathbb{P}^7$.

Likewise, when $(r,s) = (9,11)$, for eleven points on $\mathbb{P}^8$ we have $H(11,3,9)_4 \leq (45)(11)-55 = 445$, which is less than the degree $n = 495 = 11(\dim_k R_2)$, again preventing n-genericity. Here the regularity is also affected by the Koszul bound (2.11), as $\dim_k R_4 = 495$, equal to the degree, but the ideal $K(P,3^{11},9)$ is not 4-regular.

For $(r,s) = (9,12)$, twelve points on $\mathbb{P}^8$, we have $H(12,3,9)_4 \leq (45)(12)-66 = 485$, so $K(P,3^{12},9)$ is not n-generic.

For $a = (\underline{3})$ and more than 12 points, we must take $r > 9$ to obtain non n-generic behavior from (2.11).

Example 2.13. KOSZUL REGIONS AND n-GENERICITY FOR $a = 4$. When $N = 4^s$, the Koszul regions of Theorem 2.10 are

$$S_1,\ i \geq 7;\ S_2,\ i = 5,6;\ S_3,\ i = 4;\ \text{and}\ S_\infty,\ i < 4.$$

If $s \geq r+1$ and we take $i = 6$ then $(r,s) = (6,8)$ is the example with lowest embedding dimension r where the Koszul bound requires non n-generic behavior for $K(P,4^s,r)$. When $i = 5$ the first such example is $(r,s) = (10,12)$.

Notation for Table 2. *The $i = 6$ column* of Table 2 below lists first the Koszul upper bound $G(\underline{4},r)_i$ for $H(s,4,r)_i = H(R/K(P,4^s,r))$ satisfying, since $i = 6$ is in the S_2 region,

$$G(\underline{4},r)_i = \min\left(\deg K(P,4^s,r) - \binom{s}{2},\ \dim_k \mathfrak{R}_i\right)$$

We list the bound in boldface, when it is smaller than $\dim_k \mathfrak{R}_i$ so by Proposition 2.7 prevents n-genericity. We then list the codimension $\mathrm{cod} = \dim_k \mathfrak{R}_6 - G(\underline{4},r)_6$ which is a lower bound for $\dim_k(K(P,4^s,r))$. We next list the *difference* of $G(4^s,r)_6$ from n-generic position,

$$\mathrm{diff} = \min(\mathrm{degree}\ K(P_s,\underline{4},r),\ \dim_k R_6) - G(\underline{4},r)_6. \tag{2.21}$$

We finally list in boldface the defect

$$\Delta = G(\underline{4},r)_6 - H(s,4,r)_6 \qquad (2.22)$$

$$= \Delta(J,r),\ J = \underline{i+1-a} = 3^s$$

between the actual value of $H(s,4,r)_6$ as calculated in "Macaulay", and the Koszul bound.

The $i = 7$ *column* of Table 2 lists $G(\underline{4},r)_7 = \deg(K(P_s,\underline{4},r))$. *Arrangement of rows in Table 2:* For each r, $6 \le r \le 10$, we begin with $s = r+2$, and end with the highest value of s, for which the difference of (2.21) is nonzero in degree 6. Thus, for $r = 9$, $K(P_s,\underline{4},9)_6$ has diff $\neq 0$ for $11 \le s \le 19$, but diff = 0 for $s \ge 20$. A striking aspect of Table 2 is the accuracy of the Koszul upper bounds: Δ is nonzero only once!

r;s \ i	5 dim/cod/Δ	6 dim/cod/diff/Δ[4]	7 dim=n
6; 8	252/0/ **0**	**420**/42/28 / **1**	448
7; 9	462/0/ **0**	**720**/204/36/ **0**	756
7;10	462/"	**795**/129/45/ **0**	840
7;11	462/"	**869**/55/55/ **0**	924
8;10	792/0/ **0**	**1155**/561/45/ **0**	1200
8;15	792/"	**1695**/21/21/ **0**	1800
9;11	1287/0/**0**	**1760**/1243/55/**0**	1815
9;19	1287/"	**2964**/39/39/ **0**	3135
10;12	**1980**/22/**0**	**2574**/2431/66/**0**	2640
10;13	2002/0/**0**	**2782**/2123/78/**0**	2860
10;24	2002/"	**5004**/1/1/ **0**	5280

Table 2. Koszul upper bounds $G(s^4,r)$ for $H(s,4,r)$ in the S_2 region $i = 5,6$. The bound for $i = 6$ prevents n-generic position. (See Example 2.13). The defect $\Delta_i = G(s^4,r)_i - H(s,4,r)_i$.

Example 2.14A. KOSZUL REGIONS. When $a = 7$, the Koszul regions are

$$S_1,\ 12 < i;\ S_2,\ 10 \le i \le 12\ ;\ S_3,\ i = 9;\ S_4,\ i = 8;\ S_7,\ i = 7;\ S_\infty,\ i < 7.$$

Example 2.14B. KOSZUL REGIONS AND n GENERICITY, $a = 7$, s SMALL. We suppose that $r = 4$, and $N = \underline{7}$. Table 3 lists the Koszul upper bounds $G(7^s,4)_i = c_i(7^s,4)$ of Theorem 2.10 for the Hilbert function $H(s,7,4)_i = H(R/K(P,7^s))_i$ for $s = 2,\dots,6$. In each case, $G(7^s,4)$ is regular by degree $i = 14$. A value is listed in

[4] The value $\Delta_6 = 0$ (or 1 when $(r,s) = (6,8)$) was checked by calculation in "Macaulay" for the highest s value for each $r \ge 7$, and implies $\Delta = 0$ for lower s. See Example 2.8B for $(r,s) = (10,24)$. Finding Δ_7 in Table 2 is out of the effective range of the computer available, however Δ_7 is certainly zero because the codimension is so large in each case.

boldface when it prevents $G(7^s,4)$ from being in n-generic position.

When $s \leq 5$, $H(s,7,4)_i = G(7^s,4)_i$ by Proposition 2.3. For $s \geq 6$, $G(7^s,4) = HGP(n,4)$, the Hilbert function of an ideal in n-generic position, $n = 84s$. The defect $\Delta = G(7^s,4) - H(s,7,4)_i$ is nonzero when $s = 6$, $i = 12,13$, or $s = 7$, $i = 13$, *and is otherwise zero* in Table 3.

s \ i	6	7	8	9	10	11	12/Δ	13/Δ	14
2	84	**112**	**133**	**148**	**158**	**164**	**167**	168	168
3	84	**119**	**157**	**193**	**222**	**240**	**249**	252	252
4	84	120	165	220	**276**	**312**	**330**	336	336
5	84	120	165	220	286	364	**410**	420	420
6	84	120	165	220	286	364	455/**1**	504/**4**	504
7	84	120	165	220	286	364	455	560/**1**	588

Table 3. Upper bounds for $H(s,7,4)$ when $r = 4$, $2 \leq s \leq 7$. (See Example 2.14B.). The three nonzero values of defect are in bold.

Example 2.14C. KOSZUL REGIONS AND n GENERICITY, $a = 7$, $r = 9,10$, S_3 REGION. If $N = 7^s$, $s \geq r+2$, the case $(r,s) = (9,11)$ is the smallest value of r for which there is impact of Theorem 2.10 on the S_3 region, $i = 9$. See Table 4.

Notation for Table 4. We follow the notation of Table 2. In degrees $i = 10-12$ of the S_2 region we give the predicted difference, usually $(\dim_k R_{12-i})(\mathrm{Bin}(s,2))$, from the n-generic-position value $HGP(n,r)$ of (2.12). The entries are in boldface when the upper bound prevents n-generic position. The four entries Δ in bold when $i = 9$ are the actual defects from the Koszul upper bound, based on calculation in "Macaulay".

Remark. Proposition 2.7 requires non n-generic position in $H(R/K(P,N))$ only for sufficiently large r. When $N = \underline{7}$, and $s \geq r+2$, one must go to $r = 5$ variables to have non n-generic behavior: there, when $r = 5$ only $s = 7$ and 8 must be non n-generic because of the Koszul bounds (2.11), and then only in the largest possible degree, $i = 12$, the top of the S_2 region. Again when $N = \underline{7}$, and $s \geq r+2$, for Proposition 2.7 to imply non n-generic behavior in degree $i = 9$, the S_3 region, we need at least 9 variables.

r;s \ i	9 dim/cod/Δ[5]	10 dim/cod/dif	11 dif	12 dif	degree
9;10	**22725**/1585	**28005**/15753/45·45	9·45	45	30030
9;11	**24123**/187/**154**	**31558**/12200/45·55	9·55	55	33033
9;12	24310/0/**0**	**34066**/9692/45·66	9·66	66	36036
9;16	" /0/**0**	**42648**/1110/1110	9·120	120	48048
9;17	" /0/**0**	43758/0/0	9·136	136	51051
10;12	**45760**/2860	**56430**/35948/55·66	10·66	66	60060
10;13	**48191**/429	**60775**/31603/55·78	10·78	78	65065
10;20	48620/0	**89650**/2728/2728	10·190	190	100100
10;21	" /0	92378/0/0	10·210	210	105105

Table 4. Koszul upper bounds $G(7^s,r)$ for $H(P,\underline{7},r)$, in the S_3 and S_2 regions, $r = 9,10$. Values also of cod, a lower bound for $\dim_k(K(P,7^s,r)_i)$, difference from n-generic position, and some defects Δ (See Example 2.14C.).

2C. Lower bound for regularity.

We now give a lower bound for the regularity of $K(P,N)$. We also give an example with nonzero defect Δ, showing that the lower bound is not always sharp.

Proposition 2.15. LOWER BOUND FOR REGULARITY. Suppose that r is fixed, that $N = (n_1,\dots,n_s)$, $n_1 \geq \dots \geq n_s$, and that $K(P,N) = m_{p(1)}^{n_1}\cap\dots\cap m_{p(s)}^{n_s}$, where $P = \{p(1),\dots,p(s)\}$ are s general points of $\mathbb{P}^{r-1}$. Let $t = t(s,N,r)$ denote the smallest integer such that

$$\deg(K(P,N)) \leq \dim_k R_t.$$

Then the regularity $\gamma(K(P,N))$ satisfies

$$\gamma(K(P,N)) \geq \max(n_1+n_2-1,t). \tag{2.22}$$

Proof. Immediate from Theorems 2.2 and 2.10, since when $i = n_1+n_2-2$ there are relations in degree i between the two lowest degree generators of $K^{-1}(P,N)$, and the dimension of $R_i/K(P,N)_i$ cannot attain $n = (\Sigma_{1\leq v\leq s}\dim_k R_{n_v-1})$, the multiplicity of $R/K(P,N)$.

[5] The values of Δ for $i > 9$ and those not listed when $i = 9$ have not been checked by "Macaulay", as they are out of the effective range of the available computer. Technically, the values listed should be considered upper bounds for Δ - it is conceivable that the sets of linear forms we used were not general enough - but we believe they are accurate. The value $\operatorname{def}(3^{11},9)_9 = 154$ included for $r = 9$, $s = 11$, $i = 10$ was calculated in characteristic 997, and took 15 hours and 5 MB space on an accelerated Macintosh SE-30; the calculation of $\operatorname{def}(3^{12},9)_9 = 0$ took over 40 hours.It is often simpler to check higher values of s, where the socle degree is lower: finding $\operatorname{def}(3^{19},9)_6 = 0$ for $(r,s) = (9,19)$ in Table 2 took only one hour.

Remark. When $r = 3$, A. Hirschowitz gives a set of inequalities satisfied by the regularity $\gamma(K(P,N))$ in [**H2**]; the formula (2.21) is the simplest of the set. We now give an example where the defect is nonzero, and his bounds are sharper.

Example 2.16. LOWER BOUND NOT OPTIMAL, POSITIVE DEFECT. By Example 1.6A and Theorem 2.2, if $(r,s) = (3,5)$, $N = (\underline{6})$, so P consists of 5 general enough points of $\mathbb{P}^2$, then $K(P,\underline{6})$ has degree $(21)(5) = 105$. We have calculated $H(5,6,3)$ to be

$$H(5,6,3) = (1,3,6,10,15,21,28,36,45,55,66,78,\underline{90},\underline{99},\underline{104},105,\dots)$$

while $G(5,6,3) = (1,3,\dots,78,91,105,105,105)$. Thus, $K(P,\underline{6})$ is regular only in degree 15, not 13, the bound of (2.21).
A. Hirschowitz states that the Hilbert function of $R/K(P,N)$ is known for $r = 3$, $s \le 9$ points, and any weights N, and he explains this kind of example in [**H2**]. Here the five points P lie on a conic Y, and $\dim_K(\Gamma(Y,\mathcal{O}(13))) = 105-78 = 27$. The condition that a form of degree 13 on Y vanish to order 6 at each of the points would tend to impose 30 conditions, but there are only 27 available: three don't count. This explains a defect, but more is needed to explain $\Delta_{13} = 6$. See §1-4 of [**H2**], and also [**Ha**],[**G**].

Remark. PATTERN IN THE DEFECT. To calculate $H(5,6,3)$ above, we found the size of R_5L^j, $L^j = \langle x^j,y^j,z^j,(x+y+z)^j,(x+2y+3z)^j\rangle$, for $6 \le j \le 10$, and checked this when the result was surprising, for $j = 7,8,9$, by altering the last linear form. The difference of the sides in (2.11) is $Z^{12}+6Z^{13}+Z^{14}$. The key part of the Hilbert function $H(5,6,3)$, can be written as

$$(\underline{90},\underline{99},\underline{104}) = (91,104,120)-(1,6,16),$$

where (91,104,120) is a portion of $H(R)$ and (1,6,16) is related to stability in the last portion of the set of Hilbert functions $\{H(R/(L^j))\}$, with socle $\sigma = 2j-2$, namely

$$H(R/(L^j)) = (1,3,\dots\dots,\mathit{16},\mathit{6},\mathit{1}).$$

By the degree $j = 20$, the stable ending sequence of $H(R/(L^j))$ has grown to $(\dots,\mathit{106},\mathit{76},\mathit{51},\mathit{31},\mathit{16},\mathit{6},\mathit{1})$ with 1 in degree $\sigma = 38$. We study patterns in defects further in §3D below.

3. Thin algebras and fat points.

When a = 2, A. Alexander and A. Hirschowitz have determined the Hilbert function H(2,s,r) = H(R/K(P,2)); we have discussed the consequences in **[I3]**. In Section 3A we consider what is known for $a \geq 3$. In Section 3B we show how to calculate the Hilbert functions of vanishing ideals from those of power algebras. In Section 3C we give a symbolic algebra program for finding the Hilbert functions of vanishing ideals in this way. Section 3D gives examples of patterns in the defects, and Section 3E gives evidence for our Conjecture 2.2.0.

3A. Consequences for the thin algebra problem when u = a-1 > 1.

We first compare the lower bounds previously obtained by others for $H(R/K(P,\underline{a}))$ with the upper bounds of Theorem 2.2 and Theorem 2.10, as well as with experimental evidence.

N.V. Trung and G. Valla as well as A. Gimigliano have studied the question of regularity of $K(P,\underline{a})$ for points in generic position. We give a result of N.V. Trung and G. Valla (Theorem 2.6 from [**TV**]). For simplicity we assume char k = 0 here. If $N = (n_1,\ldots,n_s)$, $n_1 \geq \ldots \geq n_s$ we let v = v(N) be the least integer such that

$$\sum_{2\leq i\leq s}\binom{n_i+r-2}{r-1} < \binom{r-1+v}{r-1}. \qquad (3.1)$$

Equivalently, v is the smallest integer for which

$$\dim_k R_v > \deg K(P,N) - \deg(n_1P_1). \qquad (3.2)$$

Lemma. (See [**TV**]). If $P = (P_1,\ldots,P_s)$ are generically chosen points of $\mathbb{P}^{r-1}$, then the regularity $\gamma(K(P,N))$ satisfies,

$$\gamma(K(P,N)) \leq (v+n_1-1). \qquad (3.3)$$

If $N = \underline{a}$ then u = a-1 is the difference in degree from the generators $\mathcal{J}(P,N,i)$ of $\mathcal{V}(P,N,i)$ to the degree i. Thus (3.3) implies $\gamma(K(P,N)) \leq (v+a-1) = v+u$. Applying (1.11), we have

Corollary 3.2. Suppose that j,u,s satisfy

$$\dim_k\mathfrak{R}_j > (\dim_k\mathfrak{R}_u)(s-1). \qquad (3.4)$$

If L is a sufficiently general set of s linear forms, then the vector space $L^j = \langle L_1{}^j,\ldots,L_s{}^j\rangle$ satisfies,

$$\dim_k\langle\mathfrak{R}_uL^j\rangle = (\dim_k\mathfrak{R}_u)s. \qquad (3.5)$$

When r = 3, then A. Gimigliano [**Gi1**] has shown

Lemma. Let d be the regularity of $K(P,\underline{1};3)$. If $N = (n_1,\ldots,n_s)$, with $n_1 \geq \ldots \geq n_s$, and P is generically chosen in $\mathbb{P}^2$, then $\gamma(K(P,N)) \leq n_1+\ldots+n_d$.

N.V. Trung and G. Valla have weakened the hypothesis on P to uniform position property in most cases (Corollary 3.7 of [**TV**]). Since for P general, $K(P,\underline{1};3)$ is in s-generic position by [**GM**] or [**GO2**], d is the smallest integer such that $s \leq \dim_k R_d$. When $N = \underline{a}$ and $u = a-1$, we obtain from Gimigliano's result, and (1.10),

Corollary 3.3. Suppose that $r = 3$, $s \leq \dim_k R_d$, $u = a-1$, and $i = j+u$ satisfies

$$i \geq ad. \tag{3.6}$$

If L is a sufficiently general set of s forms in $\mathcal{R}_j$, then the dimension of $\mathcal{R}_u L^j$ is $(\dim_k \mathcal{R}_u)s$. Also,

$$def(j^s,3)_{j+u} = err(j^s,3)_{j+u} = 0, \text{ when } j > (u+1)(\sqrt{2s}-1). \tag{3.7}$$

We apply these results for $s \geq r+2$, $r = 3,4,5$.

Example 3.4A. CONSEQUENCES IN THE PLANE. If we take $(r,s) = (3,15)$ then degree $K(P,\underline{3};3) = (\dim_k \mathcal{R}_2)(15) = 90$, one less than $\dim_k \mathcal{R}_{12}$. We have $d = 4$, so that taking $u = 2$ in (3.6) requires $j \geq 10$. Hence for $L_1,\ldots,L_{15}$ generically chosen,

$$\dim_k \mathcal{R}_2\langle L_1^{10},\ldots,L_{15}^{10}\rangle = 90$$

in the space $\mathcal{R}_{12}$ of dimension 91, a sharp result - when we consider degrees j less than 10 we find relations in degree j+2.

Likewise, if we take $r = 3$, $s = 9$, then degree $K(P,\underline{4};3) = (\dim_k \mathcal{R}_3)(9) = 90$, of codimension one in $\mathcal{R}_{12}$. We have $d = 3$, so that taking $u = 3$ in (3.6), requires $j \geq 9$. Thus, if L is general,

$$\dim_k \mathcal{R}_3\langle L_1^{9},\ldots,L_{9}^{9}\rangle = 90$$

in the space $\mathcal{R}_{12}$ of dimension 91, also a sharp result.

Finally, taking $r = 3$, $s = 12$, $u = 3$, then $d = 4$ and (3.6) requires $j \geq 13$. Hence Corollary 3.3 gives for L general

$$\dim_k \mathcal{R}_3\langle L_1^{13},\ldots,L_{12}^{13}\rangle = 120$$

in the space $\mathcal{R}_{16}$ of dimension 153. A computer calculation in "Macaulay" shows that in fact we may replace the degree 13 here by degree 11: the space $\mathcal{R}_3\langle L_1^{11},\ldots,L_{12}^{11}\rangle$ has dimension 120 inside a space of dimension 120.

Example 3.4B. CONSEQUENCES IN $\mathbb{P}^3$. If we take $(r,s) = (4,7)$, then $(\dim_k \mathfrak{R}_4)(6) = 210$, which is less than $\dim_k \mathfrak{R}_9 = 220$. Corollary 3.2 states that if $L_1, \ldots L_7$ are generically chosen, then

$$\dim_k \mathfrak{R}_4\langle L_1{}^9, \ldots, L_7{}^9\rangle = 245$$

inside $\mathfrak{R}_{13}$, a space of dimension 560. A calculation shows $K(P,5^7;4)$ is in 245-generic position, and we may take $j = 6$ above in place of 9: $\mathfrak{R}_4 L^6$ has dimension 245 and codimension 41 in $\mathfrak{R}_{10}$.

Example 3.4C. HIGHER DIMENSIONS. When $(r,s) = (5,8)$ then $(\dim_k \mathfrak{R}_4)(8-1) = 490$, less than $\dim_k \mathfrak{R}_8 = 495$. Corollary 3.2 states that if $L_1, \ldots, L_8$ are generically chosen, then

$$\dim_k \mathfrak{R}_4\langle L_1{}^8, \ldots, L_8{}^8\rangle = 560$$

inside $\mathfrak{R}_{12}$, a space of dimension 1820.

Calculation shows that for $s = 8$, $K(P,5^8,5)$ is in 560-generic position, regular in degree 9, so $\dim_k \mathfrak{R}_4\langle L_1{}^5, \ldots, L_8{}^5\rangle = 560$ in $\mathfrak{R}_9$, a space of dimension 715.

When $s = 9,10$ there is a defect $\mathrm{def}(\underline{5},5)_9$ of 5 and 7, respectively, so $H(9,4,5)_9 = 625$ and $H(10,4,5) = 692$, preventing regularity of $K(P,4^9,5)$ and $K(P,4^{10},5)$ in degree 9.

Remark. The consequences for the thin power algebra problem, $u > 1$ of the existing regularity results for $K(P,N)$ are most striking for $\mathbb{P}^2$; there is room for substantially stronger results when $r \geq 4$. Even when $r = 3$, there are cases not covered by the existing regularity results.

It is natural to believe that higher powers of a sequence of forms are more independent - have less relations - than lower powers. We show this as a Corollary of a result of Geramita and Maroscia, based on $R/K(P,N)$ being Cohen-Macaulay of dimension one. Proposition 3.5B is perhaps classical, but we do not have a reference; the case $N = (\underline{1})$ is in [**DK**].

Proposition 3.5A (A. Geramita and P. Maroscia, [**GM**]) Suppose $Z = \mathrm{Spech}(R/K(P,N))$ is the subscheme of $\mathbb{P}^{r-1}$ defined by the intersection $K(P,N) = m_{p(1)}{}^{n_1} \cap \ldots \cap m_{p(s)}{}^{n_s}$. Then the Hilbert function $H(R/K(P,N)) = (\ldots, h_i = \dim_k(R_i/K(P,N)_i), \ldots)$ is nondecreasing, and attains degree $K(P,N)$ for i large.

Proposition 3.5B. If $L = \langle L_1, \ldots, L_s\rangle$ are s linear forms in $\mathfrak{R}$, and u is a natural number, then we have

$$\dim_k \mathfrak{R}_u L^j \leq \dim_k \mathfrak{R}_u L^{j+1}, \tag{3.8}$$

and more generally,

$$\dim_k[(L^J)\cap \mathfrak{R}_i] \le \dim_k[(L^{J+1})\cap \mathfrak{R}_{i+1}]. \qquad (3.9)$$

Proof. A. Geramita and P. Maroscia point out that Proposition 3.5A is a consequence of R/K(P,N) being Cohen-Macaulay of dimension one (Proposition 1.4 of [**GM**], but see also [**GO**]). The formula (1.10) shows that if $J = \underline{i+1}-N$, then the vector spaces $(L^J)\cap\mathfrak{R}_i$ and $(L^{J+1})\cap\mathfrak{R}_{i+1}$ in (3.9) are the duals of the spaces $R_i/K(P,N)_i$ and $R_{i+1}/K(P,N)_{i+1}$ (Proposition 1.3B is a special case). Thus, (3.9) follows from Proposition 3.5A.

3B. From power algebras to vanishing ideals.

We study the case r = 4, s = 6, where the weights $N = \underline{a} = \underline{u}+1$ are equal. We use computer calculation to ascertain the defects

$$\text{def}(6,\underline{i+1-a},4 = T'(6,\underline{i+1-a},4) - T(6,\underline{i+1-a},4),$$

and thus determine the postulation of $K(P,\underline{a})$. Here $L = \langle L_1,\ldots,L_6\rangle = \langle W,X,Y,Z, W+X+Y+Z, W+2X+3Y+4Z\rangle$ and (L^j) is the ideal generated by $L^j = \langle L_1{}^j,\ldots,L_6{}^j\rangle$. The (j,u) entry of Table 5 is the size of $\mathfrak{R}_u L^j$ computed in the "Macaulay" symbolic algebra program [**BSE**], followed by the defect $\Delta = \text{def}(J,4)_{u+j}$ from the Koszul bound when Δ is nonzero. We have omitted some entries where $\mathfrak{R}_u L^j = \mathfrak{R}_{u+j}$.

The striking aspect of Table 5 is the existence of limit values of the defects from the thin algebra values: the value 4 and the value 35, the first values of f(v) in Table 7. We discuss the limit behavior further in Example 3.11 below.

j \ u/Δ	1	2	3/Δ	4	5	6	7	8/Δ
2	20	35	56					
3	24	56	84					
4	24	60	116/**4**	165				
5	24	60	120	206/**4**	286/**35**			
6	24	60	120	210	332/**4**	454/**35**		
7	24	60	120	210	336	500/**4**	670/**35**	
8	24	60	120	210	336	504	716/**4**	940/**35**
9	24	60	120	210	336	504	720	986/**4**

Table 5. Dimension of $\mathfrak{R}_u L^j$ and defects Δ from thin algebra values, when r = 4 and s = 6. The degree i = j+u.

Observation 3.6. When (r,s) = (4,6), then when $5 \le j \le 8$, the defects of Table 5 in the j row are zero except for two values: $\dim_k \mathfrak{R}_{j-1}L^j$ which has defect $\Delta = \text{def}(j^6,4)_{2j-1} = 4$ from the Koszul bound when $j \geq 4$, and $\dim_k \mathfrak{R}_j L^j$ which has defect $\Delta = \mathrm{def}(j^6,4)_{2j} = 35$ when $j \geq 5$. The length of the sequence of nonzero defects in the j row increases with j (see Table 7 and Example 3.11).

The points $P = P_1,\ldots,P_6 \subset \mathbb{P}^3$, corresponding to the forms L are

$$(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(1,1,1,1),(1,2,3,4).$$

By (1.12e), the Hilbert function of $R/K(P,\underline{a};4)$ is obtained by reading the $u = a-1$ column of Table 5 (including the values not shown). We explore $(r,s) = (4,6)$ further in Example 3.11.

Example 3.7A. POSTULATION OF $K(P,4^6;4)$. From the $u = 3$ column of Table 5, we have

$$H(R/K(P,4^6;4)) = (1,4,10,20,35,56,84,116,120,120,\ldots),$$

with regularity in degree 10, and defect 4 in degree 7.

Example 3.7B. POSTULATION OF $K(P,7^6;4)$. From the $u = 6$ column of Table 5 we have

$$H(R/K(P,7^6;4)) = (1,4,\ldots,286,364,454,500,504,504,\ldots),$$

regular in degree 14, with defects 35 and 4 in degrees 12 and 13.

3C. Calculating defects, using a symbolic algebra program.

We here give the program in "Macaulay" that we used to calculate Table 5 with comments in italics to right.[6] Other examples and tables in the article were calculated similarly.

```
<ring 4 w-z R            Or set up a ring in small characteristic
<ideal L w x y z w+x+y+x  w+13x+3y+17z
<pow_entry L j I          Script raises entries to j-th power
std I I
hilb I                    Gives Hilbert function of ℜ/I
```

A second script

```
<pow_hilb L s j1 j2
```

repeats the above process for $j1 \leq j \leq j2$, using the first s entries of L.

When working close to the limits of the machine, we first set up a ring S of small characteristic, as 17 in Example 2.8B, where $r = 10$ variables. There for $s = 24$ we constructed L via

```
random 10 14 M
mult S M L                Gives a 1×14 matrix L of linear forms
```

---

6 The "Macualay" scripts used are available from the author.

concat L S *Adds the variables of S to L*

This gives a 1X24 matrix L of linear forms, containing the variables of S. One then proceeds as above.

The Hilbert function "hilbI" of R/I, is hopefully $T'(J,r)$ with $J = (j^s)$, if L has been general enough. We subtract it from the Hilbert function of R to find the entries of Table 5,

$$\dim_k \mathfrak{R}_u L^j = \dim_k \mathfrak{R}_{u+j} - H(\mathfrak{R}/(L^j))_{u+j} = \dim_k (R/K(P,(u+1)^s))_{u+j}.$$

Example 3.8. When $(r,s) = (5,10)$, we used the scripts

```
<ring 5 v-z R
<ideal L v w x y z v+w+x+y+z ...... ten linear forms,
<pow_hilb L 10 2 8
```

The resulting list of Hilbert functions $T'(10,j,5)$, $2 \leq j \leq 8$, shows that for $2 \leq j \leq 4$, the socle degree of R/L^j is $2j-2$; for $5 \leq j \leq 8$ it is $2j-1$. Also $T'(10,j,5) = T(10,j,5)$, and $\Delta(j^{10},5) = err(j^{10},5) = 0$, for $j \leq 8$, except for the values

$$\Delta(j^{10},5)_{2j-1} = j+2 \quad \text{when } 5 \leq j \leq 8. \tag{3.10}$$

Thus, for $i+1-a \leq 8$ we have by experiment,

$$H(10,a,5)_i = G(a^{10},5)_i \text{ for } i \leq 2a-2,$$

$$H(10,a,5)_{2a-1} = G(N,5)_{2a-1} - (a+2). \tag{3.11}$$

When $i = 2j-1$ then $j = i+1-a = a$ so (3.10) implies (3.11).

<u>3D. Patterns in the defects.</u>

We suggest that determining the Hilbert functions of order a vanishing ideals at s general points of $\mathbb{P}^{r-1}$ is best approached experimentally by determining the Hilbert functions $T'(j^s,r)$ of thin power algebras R/L^j. Both the Koszul bounds F_d estimating the Hilbert functions $T(j^s,r)$ of thin algebras, and the patterns in the defects $\Delta(j^s,r) = T'(j^s,r) - F_d$ are simpler than their translations to vanishing ideals. We give data and suggest some patterns in the defects $\Delta(j^s,r)$ when $(r,s) = (5,10)$, $(5,7)$, $(4,6)$, and $(4,9)$. In the last two cases, we give conjectures that fit the data, would determine $T'(j^s,r)$ for all j, and that we feel are compelling. The conjectures concern the postulation of any equal-order-a vanishing ideal at 6 and 9 general points of $\mathbb{P}^3$ (Examples 3.11ff and 3.12ff). We have chosen $r = 4$, so $\mathbb{P}^3$, as it is the lowest r where the Koszul bounds give new data about vanishing ideals. . The pair $(r,s) = (4,6)$ is special since six general points of $\mathbb{P}^3$ lie on a unique complete intersection of quadrics. The pair (4,9) is special as nine general points on $\mathbb{P}^3$ lie on a unique quadric.

Definition 3.9. If J is a nondecreasing sequence of nonnegative integers, we denote by $\sigma(J,r)$ the degree of the socle of $R/(L^J)$, for L a generically chosen set of linear forms. We let the *adjusted defect* be the following polynomial in Z,

$$AD(J,r,Z) = def(J,r,Z^{-1}) \cdot Z^{\sigma(J,r)}.$$

Conjecture 3.9.0. For appropriate pairs (r,s), given a nondecreasing sequences $J = (j_1,\ldots,j_s)$ with $j_1 = 0$, there is a fixed polynomial SD(J,r,t,Z), in t and Z, the *stable defect,* and an integer b(J,r) such that if $t \geq b(J,r)$, then

$$SD(J,r,t,Z) = AD(J+\underline{t},r,Z).$$

Example 3.10A. If (r,s) = (5,10), the date in Example 3.8 for $j \leq 8$ gives $AD(J,5,Z) = (j+2)$ there. This data is consistent with the guess, $SD(0^{10},5,t,Z) = t+2$. However, we don't think so! In Example 3.16 we show that the socle degree of R/L^j is at least 2.29j for large j. When $j \leq 8$ the socle degree is less than 2j. Our data does not include high enough values of j to determine a limit $SD(j^{10},5,t,Z)$.

j \ i	2j-1	2j	2j+1	2j+2	2j+3	2j+4	2j+5
2	1						
3		1					
4		5	1				
5		1	35	1			
6			15[7]	36	1		
7			5	70	36	1	
8			1	35	210	36	1

Table 6. Case (r,s) = (5,7), defects $\Delta = def(j^7,5)_i$, $2 \leq j \leq 8$. Zero values are omitted. (See Example 3.10C).

Example 3.10B. When (r,s) = (4,6), from Table 5 we have

$$AD(J,4,Z) = 35+4Z,$$

for $J = j^6$, $5 \leq j \leq 8$. We conjecture below that $\Sigma f(v) Z^v = 35+4Z+\ldots = \lim_{j\to\infty} Z^{-(2j-2)} def(j^6,4,Z)$ (see Example 3.11).

[7] If $\dim_k \mathcal{R}_6$ is not considered, then $def(3^7)_6$ would be 15, equal to $def(6^7)_{13}$.

Example 3.10C. Calculation of other cases suggest that for certain J, the Conjecture 3.9.0 needs modification. In Table 6, where (r,s) = (5,7) we have indicated in boldface values of the defect $\Delta_i = \mathrm{def}(j^7,5)_i$ (See Table 6). There appears to be both a stable defect (...,36,1), and a pattern depending on j mod 3. For example, the start (1,35,...) for j = 5 is repeated for j = 8.

Example 3.11. When (r,s) = (4,6), we calculated the Hilbert function $T'(j^6,4)$ for $1 \le j \le 15$. It has a pattern, depending on j mod 5 and detailed in Observation 3.11A. We suppose j = 5t±1 or 5t+2, 5t+3, and let

$$\begin{aligned} t &= \lfloor (j+1)/5 \rfloor, \\ k &= \begin{Bmatrix} 2t & \text{if } j \equiv 1,2, \text{ or } 3 \bmod 5 \\ 2t-1 & \text{if } j \equiv 0 \text{ or } 4 \bmod 5 \end{Bmatrix}, \\ \sigma &= 2j-2+k, \\ f(v) &= \begin{Bmatrix} \frac{1}{8}\left(35v^3+6v^2-12v\right) \text{ if } v \text{ is even} \\ \frac{1}{8}\left(35v^3+9v^2-27v+15\right) \text{ if } v \text{ is odd} \end{Bmatrix}. \end{aligned} \tag{3.12}$$

v	1	2	3	4	5	6	7	8	9	10
f(v)	**4**	**35**	**120**	**286**	**560**	**963**	**1534**	**2276**	**3252**	**4435**

We let $\Delta' = \mathrm{def}'(j^s,r)_i$ denote the deficiency of the actual value of $\dim_k \mathcal{R}_i(L^j)$, $L = (L_1,\ldots,L_s)$ general, from the Koszul value, *without* considering the size of $\dim_k \mathcal{R}_i$. In the region $j \le i < 3j$ we have, for general L,

$$\dim_k \mathcal{R}_{i-j}L^j = (\dim_k \mathcal{R}_{i-j})(s) - \dim_k \mathcal{R}_{i-2j}\binom{s}{2} - \mathrm{def}'(j^s,r)_i. \tag{3.13}$$

By definition, if $\mathrm{cod}(\mathcal{R}_{i-j}L^j)$ is the codimension in $\mathcal{R}_{i+j}$, we have

$$\mathrm{def}(j^s,r)_i = \min(\mathrm{cod}(\mathcal{R}_{i-j}L^j), \mathrm{def}'(j^s,r)_i). \tag{3.14}$$

Observation 3.11A. For (r,s) = (4,6) and $j \le 15$, we have

a. The socle degree of $\mathcal{R}/L^j$ is σ.

b. $\mathrm{def}(j^6,4)_i = \mathrm{def}'(j^6,4)_i = 0$ if $i \le 2j-2 = \sigma-k$.

c. $\mathrm{def}'(j^6,4)_{2j-2+v} = f(v)$ if $1 \le v \le k$.

d. $T'(j^6,4)_\sigma = \begin{Bmatrix} 1 \text{ if } j \equiv 1 \bmod 5 \\ 4 \text{ if } j \equiv -1 \bmod 5 \end{Bmatrix}$

Note: We calculated f(v) in (3.12) using Observation d. Here we need to modify Conjecture 3.9.0 as follows. We first define

$$AD'(j^6,4,Z) = def'(j^6,4,Z)\cdot Z^{-(\sigma+1-k)}.$$

The following conjecture would completely determine $T'(j^6,4)$, hence also $H(K(P,a^6,4))$ for all a and general P in $\mathbb{P}^3$.

Conjecture 3.11B. $AD'(j^6,4,Z)$ satisfies $AD'(j^6,4,Z) = \Sigma_{v\le k} f(v)Z^v$ and approaches $F = \Sigma f(v)Z^v$ as $j \to \infty$.

j \ v/Δ'	v=1/Δ'	2	3	4	5
9	154/**4**	60/**35**	**4**/120	0	
10	224/**4**	105/**35**	**20**/120	0	
11	312/**4**	166/**35**	**50**/120	**1**/286	0
12	420/**4**	245/**35**	**96**/120	**10**/286	0
13	550/**4**	344/**35**	160/**120**	**35**/286	0
14	704/**4**	465/**35**	244/**120**	**78**/286	**4**/560
15	884/**4**	610/**35**	350/**120**	**141**/286	**20**/560

Table 7. Case (r,s) = (4,6). Calculated values of $H(R/(L^j))_{2j-2+v}$, and of Δ'_{2j-2+v}, $9 \le j \le 15$, $1 \le v \le 5$. The defect $\Delta = \min(\Delta',H)$ is in boldface.

Observation 3.11C: Translation to Vanishing ideals $K(P,a^6,4)$. We translate to the Hilbert function $H(a,6,4) = H(R/K(P,a^6,4))$ determined by the vanishing ideals $K(P,a^6,4)$. We let $j = i+1-a$, and define σ,t,k as in (3.12). Then for a general set of points P in $\mathbb{P}^3$, and $i+1-a \le 15$, we have

a. $K(P,a^6)_i = 0$ if $i < 2a-k$.

b. $H(a,6,4)_i = c_i(a^6,4)$ if $i \ge 2a$. In particular, $K(P,a^6,4)$ is regular in degree 2a.

c. $H(a,6,4)_{2a-v} = c_{2a-v}(a^6,4) - f(v)$, for $1 \le v \le k$.

d. $\dim_k(K(P,a^6)_{2a-k}) = \begin{Bmatrix} 1 \text{ if } j \equiv 1 \bmod 5 \\ 4 \text{ if } j \equiv -1 \bmod 5 \end{Bmatrix}.$

Conjecture 3.11B asserts that Observation 3.11A is valid for all pairs j,i, or that Observation 3.11C is valid for all pairs a,i.

Example 3.12. When (r,s) = (4,9), then observation of Table 8 shows the socle degree σ of R/L^j is 2j-2. The first and second difference columns - with respect to j - of the columns of Table 8 are highly regular (see Table 8). In the k = 1 column of Table 8 the boldface values 11,30,40,56 are symmetrically positioned with respect to the pattern shown in Table 9. Likewise, in the k = 2 column of Table 8, the bolface values 26, 213,250, and the predicted value 416 for j = 17 are symmetrically located. We define

$$g(k) = \begin{cases} \binom{17k+1}{3} - 9\binom{8k+1}{3} & \text{if } k \text{ is odd,} \\ \binom{17k+3}{3} - 9\binom{8k+2}{3} & \text{if } k \text{ is even.} \end{cases} \tag{3.15}$$

k	0	1	2	3	4	5	6
g(k)	1	60	426	1400	3299	6400	11060

j \ i	σ	σ-1	σ-2	σ-3	σ-4	σ-5	σ-6	σ-7	σ-8	σ-9	σ-10
2	**1**	4	1								
3	1	**11**	10	4	1						
4	1	20	**26**	20	10	4	1				
5	1	**30**	48	**47**	35	20	10	4	1		
6	1	**40**	75	84	**75**	56	35	20	10	4	1
7	1	49	106	130	129	**111**	84	56	35	20	10
8	1	**56**	140	184	196	184	**156**	120	84	56	35
9	1	60	176	245	275	274	250	**211**	165	120	84
10	1	60	**213**	312	365	380	365	328	**277**	220	165
11	1	60	**250**	384	465	501	500	470	419	**355**	286
12	1	60	286	460	574	636	654	636	590	524	**446**
13	1	60	320	539	691	784	826	825	789	726	644
lim	1	60	426	1400	limit g(k), k = 0,...						

Table 8. Case (r,s) = (4,9). Calculated values of $H(R/(L^j)_i$, i = σ-k, where σ = 2j-2 is the socle degree of $R/(L^j)$: we write the Hilbert function backwards. The values for i = j are in boldface, as well as certain others - see Example 3.12.[8]

The value g(k) is just the "Koszul" estimate of $H(R/(L^j))_i$ for (j_0,i) = (9k, 2(9k)-2-k) = $(9k,\sigma_{j_0}-k)$ when k is odd, and the Koszul estimate for (j_1,i) = (9k+1,2(9k+1)-2-k) when k is even.

[8] These values were obtained for L = (w,x,y,z,w+x+y+z,w+3z+7y+23z, w-5x+11y-17z,13w-2x+9y-z,2w+29x-5y+19z).

Conjecture 3.12A. If $r = 4$, $s = 9$, the socle degree of $R/(L^j)$ is $\sigma = 2j-2$. If $k \geq 0$, and $j \geq 9k$ for odd k, or $j \geq 9k+1$ for even k, we have

$$H(R/(L^j))_{2j-2-k} = g(k). \tag{3.16}$$

For $j \leq 9k$, $H(R/(L^j)$ has the "Koszul" value

$$H(R/L^j)_{2j-2-k} = \dim_k R_{2j-2-k} - 9\dim_k R_{j-2-k}. \tag{3.17}$$

The defect satisfies

$$\Delta(j^9,4)_{2j-2-k} = \begin{cases} 0 \text{ if } j \leq 9k, \text{ or } k < 0. \\ g(k) - max[(dim_k R_{2j-2-k} - 9 dim_k R_{j-2-k}),\ 0] \text{ if } j > 9k. \end{cases}$$

As j increases the defect $\Delta(j^9,4)_{2j-2-k}$ approaches $g(k)$, so $SD(0^9,4,t,Z) = \Sigma g(k)Z^k$.

j\k	0 S	1,S	1,δS	2,S	2,δS	2,δ²	3,S	3,δS	4,S	4,δS
2	1	4	4	1	1	1	0	0	0	0
3	1	11	7	10	9	8	4	4	1	1
4	1	20	9	26	16	7	20	16	10	9
5	1	30	10	48	22	6	47	27	35	25
6	1	40	10	75	27	5	84	37	75	40
7	1	49	9	106	31	4	130	46	129	54
8	1	56	7	140	34	3	184	54	196	67
9	1	60	4	176	36	2	245	61	275	79
10	1	60	0	213	37	1	312	67	365	90
11	1	60	0	250	37	0	384	72	465	100
12	1	60	0	286	36	-1	460	76	574	109
13	1	60	0	320	34	-2	539	79	691	117
lim	1	60	-	426	-	-	1400	-	3299	-

Table 9. Calculated values of $S(j,k) = H(R/(L^j))_{2j-2-k}$, for $(r,s) = (4,9)$ and of the first differences $\delta S(j,k) = S(j+1,k)-S(j,k)$, $1 \leq k \leq 4$, as well as the second difference δ^2 when $k = 2$. The shaded columns are the same as the first 4 columns of Table 8. (See Example 3.12 and Remark 3.12B.)

Remark 3.12B What makes this prediction compelling is a comparison of the first differences $\delta^1(S(j,k))$ with respect to j of the sequence $S(\cdot,k)$,

$$S(j,k) = H(R/(L^j))_{2j-2-k} = H(6,j-1-k,4)_{2j-2-k} \tag{3.18}$$

The sequence $S(j,k)$ for k fixed has a beginning portion $j \leq k+1$ satisfying $S(j,k) = H(R)_{2j-k-2}$. The middle portion $k+2 \leq j \leq 8k-2$ has first differences δS that are symmetric about $[9k/2]\pm\varepsilon$, and second differences -1. The last portion $j \geq 8k-1$ has first differences the same in size and opposite in sign to the first portion, and ends as $S(j,k)$ achieves the stable value $g(k)$. From the symmetry of the first differences we predicted the degree of stability $S(j,k) = g(k)$ to be $9k$ for k odd or $9k+1$ for k even. We assumed that, as for $j \leq 13$, the defect Δ_{2j-k} is nonzero only when $j \geq 9k$ (or $9k+1$) where stability occurs: the formula (3.15) for $g(k)$ follows. In Table 9 we show the observed values of $S(j,k)$ and the first differences δS for $(j,k) \leq (13,3)$ as well as the second differences $\delta^2 S$ for $k = 2$. Nine general points in $\mathbb{P}^3$ lie on a unique quadric surface, and this behavior may be related to the geometry of the surface obtained by blowing up the points. Note that since the socle degree of $R/(L^j)$ is $2j-2$, less than $2j$, any nonzero defect Δ_i must arise from extra relations among the generators of the ideal (L^j), that are non-Koszul.

Conjecture 3.12C. Translation to Vanishing ideals $K(P,a^9,4)$. We translate to the Hilbert functions $H(a,9,4) = H(R/K(P,a^9,4))$ determined by the vanishing ideals $K(P,a^9,4)$. Then $j = i+1-a$, $\sigma = 2j-2$, $k = \sigma-i = i-2a$, so $i \leq \sigma$ is the condition $i \geq 2a$, and $j \geq 9k+\alpha$ is the condition $i \leq (17a+1-\alpha)/8$. We have conjecturally,

$$H(a,9,4)_i = \begin{cases} \dim_k R_i & \text{if } i < 2a, \\ \dim_k R_i - g(i-2a) & \text{if } 2a \leq i \leq (17a+\kappa)/8, \\ & \quad \text{where } \kappa = 0 \text{ if } i-2a \text{ is odd, and } \kappa = 1 \text{ if } i-2a \text{ is even,} \\ 9\dim_k R_{a-i} & \text{if } i \geq (17a+\kappa)/8. \end{cases}$$

The conjecture has been verified experimentally when $j \leq 13$, so for all pairs (i,a) for which $i \leq 12+a$. The Koszul upper bound is $H(a,9,4)_i \leq \min(\dim_k R_i, 9\dim_k R_{a-1})$; according to the conjecture there is nonzero defect when $2a \leq i < (17a+\kappa)/8$.

Remark. BETTI NUMBERS FOR R/L^J. It is possible that the patterns in the Betti numbers/degrees, which we denote $\mathrm{Betti}(R/L^j)$, of the minimal resolution of $R/(L^j)$, would be simpler than the patterns in the Hilbert functions. Does R/L^j have the "earliest" resolution possible consistent with the observed defects? Sometimes one can conclude by degree that the observed defects must come from extra relations among the generators L^j, so $\mathrm{Betti}(R/(L^j) \equiv H(R/L^j)$. Unfortunately, finding $\mathrm{Betti}(R/L^j)$ in "Macaulay" even for $(r,s) = (4,6)$, $j = 4$ was beyond our equipment range, requiring more than 26 MB active memory.

<u>3E. Sharpness of the upper bounds</u>. Conjecture 2.2.0 states that there would be a finite number $\tau(a)$ of triples (N,i,r) satisfying $n_u \leq a$ for each u, and for which $H_N(r) \neq c_i(N,r)$. The data above suggest that nonzero defects def(J,r) occur very close to the socle degree of $\mathcal{R}/L^J$, and when the expected dimension of $(\mathcal{R}/L^J)_i$ is small. It is natural to believe that when $(L^J)_i$ has large expected codimension in $\mathcal{R}_i$, the condition for it to be of smaller than expected rank should greatly restrict L.

If $N = (a^s)$, then we take $J = (j^s)$ and $u = a-1$; in order to have $H(s,a,r) \neq c_i(a^s,r)$ we must have nonzero defect

$$\operatorname{def}(j^s,r)_{j+u} \neq 0.$$

Suppose for the moment that r, a are fixed and that any nonzero defect for (s,a,r) occurs in the degree σ of the socle of $(\mathcal{R}/V)_\sigma$, where $V = L^{\sigma+1-a}$. The larger the codimension of $\mathcal{R}_{a-1}V$ in $\mathcal{R}_\sigma$, the more restrictive is the condition on V for $\mathcal{R}_{a-1}V$ to drop rank. So we expect defects only when this codimension is small enough. Can we find pairs (s,j) with j large so that for a general set L of s forms,

i. $\mathcal{R}_u L^j$ has small codimension in $\mathcal{R}_{u+j}$? (3.19)

The distribution of the remainders $\dim_k \mathcal{R}_{u+j}/\dim_k \mathcal{R}_u V$ should be random, so given r, there are indeed examples of pairs (s,j) with $s = \lfloor \dim_k \mathcal{R}_{u+j}/\dim_k \mathcal{R}_u V \rfloor$ and j arbitrarily large, where the codimension of (3.19) is small. However, when u = 1 the work of A. Alexander and A. Hirschowitz shows there are only four exceptional values [**I3**].The Example 2.8B where (s,a,r) = (24,4,10), $\Delta_6 = 0$ and the codimension of $\mathcal{R}_3 L^3$ in $\mathcal{R}_6$ is 1, the Koszul value, suggest that nonzero defects are rare. How rare?

Question F. Is there a uniform bound $\tau(a,r)$ for regularity for K(P,N) where r is fixed, N satisfies $n_u \leq a$, and s is arbitrary?

Comment on Question F. The Koszul regions for $N = a^s$ depend only on a, so for given a and j > 2a, one is always in the S_1 region where

$$c_i(N,r) = \deg K(P,N).$$

Thus, Conjecture 2.2.0 restricted to a fixed r is equivalent to there being such a uniform bound for regularity of K(P,N,r), independent of s (Question F). Corollary 3.2 does not answer Question F: when j is large, Corollary 3.2 requires very high codimension of $K(P,N)_i$ in R_i. Corollary 3.3 of Gimigliano's result in the case r = 3, is closer, but since (3.7) depends on s, it is not the uniform bound needed to answer Question F affirmatively for r = 3.

<u>3F. Further defects</u>, $r = 4,5$. We include the results of computer calculation of $\Delta(j^s,4) = T(j^s,4) - T'(j^s,4)$ assuming Conjecture 2 for the value of $T(j^s,4)$, for $r = 4$ and $s = 7,8,11,12,16,19$, and for $j \leq 12$ or so. The values were calculated using the "Macaulay" computer algebra program on an accelerated SE-30 with 26MB of RAM (i.e. on a souped up antique Model T Ford). We used generators $L = (w,x,y,z,w+x+y+z,L_6,\ldots,L_s)$ where $L_6,\ldots L_s$ were chosen using "random" values, and worked in characteristic $p = 31{,}997$. The calculations passed some rudimentary checks on accuracy: they are consistent with Propositions 3.5A,B, which compares the values for different j, although the computation for each value j is made independently. And the defects were non-negative![9]

Example 3.13. CASE $r = 4$. We found $\Delta(s,a,4)_i = 0$ for $s = 8,10,16,19$, and all pairs (a,i) for which $j = i+1-a$ was less or equal to 12. If $\Delta(s,a,4)_i = 0$ except for $s \leq 7$ and $s = 9$, and our Conjectures 3.11B, 3.12A, and 3.13B concerning the cases $(r,s) = (4,6),(4,9)$, and $(4,7)$ are correct, then Conjecture F would be true for $r = 4$.

Example 3.13A. CASE $(r,s) = (4,7)$. We found that for $2 \leq j < 8$ the socle degree of R/L^j is $2j-2$ and $\Delta(j^7,4)_i = 0$ there. When $8 \leq j \leq 13$, the socle degree of R/L^j is $2j-1$, and $\Delta(j^7,4)_i = \Delta'(j^7,4)_i = 0$ except for

$$\Delta'(j^7,4)_{2j-1} = 28 \qquad (8 \leq j \leq 13)$$

When $j = 14$ or 15 the socle degree of R/L^j is $2j$ and $\Delta(j^7,4)_i = 0$ except for

$$\Delta(j^7,4)_{2j-1} = 28,\ \Delta'(j^7,4)_{2j} = 245 \qquad (j = 14,15).$$

Conjecture 3.13B. CASE $(r,s) = (4,7)$. The defect $\Delta(j^7,4)_i = 0$ for $i < 2j-1$, and $\lim_{j\to\infty}\Delta'(j^7,4)_{2j-1+a} = 7\dim_k R_{3a+1}$.

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[9] The data thrown out: strange phenomena occurred when <to_div_powers was inserted after finding L^j before finding $H(R/L^j)$.

i. We found an error in "Macaulay" function "diff $x_i{}^j$ L^j" for $j \geq 13$.

ii. When $j \leq 12$, the Hilbert functions $H(R/\mathrm{div}(L^j))$ were in general *larger* than $H(R/L^j)$, and much more sensitive to the randomness of L when s was large. This result was counterintuitive, as the ideal $\mathrm{div}(L^j)$ is then calculated in the usual polynomial ring, where the divided powers ought to be *less* related or more general than the usual powers, so yield a lower Hilbert function.

We have not included the strange calculations here; they did pass the same inner consistency tests just mentioned in the text for $H(R/L^j)$.

Example 3.13C. CASE $(r,s) = (4,8)$. We found $\Delta(j^8,4) = 0$ for $j \leq 11$. The socle degree values of $T'(j^8,4)$ from $j = 4$ to 11 were $T'(j^8,4)_\sigma = j$, with $\sigma = 2j-2$. It is easyto check that these statements would be true for all j if $\Delta(j^8,4)$ is always zero.

Example 3.13D. CASE $(r,s) = (4,s)$, $s > 9$. In limited exploration of pairs $r = 4, s > 9$ we found $\Delta(j^s,4) = 0$ for all values we tried, including

i. $(r,s) = (4,10)$, $j \leq 10$
ii. $(r,s) = (4,16)$, $j \leq 12$
iii. $(r,s) = (4,19)$, $j \leq 12$.

Example 3.14. CASE $r = 5$. We are expecting $\Delta(j^s,r)$ to be zero for $r \geq 4$ and $2^{r-1} \leq s$, and nonzero for values of s close to r. Here $2^{r-1} = 16$ is the boundary. We tested $s = \dim_k R_2 - r$ and $s = \dim_k R_2 - 1$ to see if patterns in Examples 3.11 and 3.13 carry over to higher embedding dimension than four. The case $(r,s) = (5,7)$ is Example 3.10C and Table 6, and the case $(5,10)$ is in Example 3.10A.

Example 3.14A. CASE $(r,s) = (5,8)$. When $j \leq 8$ we found $\Delta(j^8,5)_i = 0$ except for

$\Delta(5^8,5)_{10} = 8$, $\Delta(7^8,5)_{15} = 40$, both in the socle degrees, and
$\Delta(8^8,5)_{17} = 8$, $\Delta(8^8,5)_{18} = 280$, the latter in the socle degree.

From Proposition 3.15 below the socle degree of R/L^j for large j is greater than 2.5j!

Example 3.14B. CASE $(r,s) = (5,9)$. For $j \leq 7$, $\Delta(j^9,5) = 0$.

Example 3.14C. CASE $(r,s) = (5,14)$. For $4 \leq j \leq 7$ we found socle degrees $2j-2$, and $\Delta(j^{14},5) = 0$ and except for $\Delta(4^{14},5)_4 = 1$. The latter corresponds to $\dim_k(K(P,3^{14})_4) = 1$ for a general set P of 14 points in $\mathbb{P}^4$, in instead of the expected zero; the scheme $\mathrm{Spec}(R/K(P,3^{14}))$ of length 210 is regular in degree five instead of four.

Example 3.14D. CASE $(r,s) = (5,16)$. Here $16 = 2^{r-1}$, so this is the boundary case where socle degree R/L^j is predicted to be just below 2j (see Proposition 3.15ff. below). When $3 \leq j \leq 7$ the socle degree is $2j-2$ and $\Delta(j^{16},5) = 0$.

Example 3.14E. CASE $(r,s) = (5,19)$. For $j \leq 9$, $\Delta(j^{19},5) = 0$

We now determine the socle degree of certain thin algebras:

Proposition 3.15. SOCLE DEGREE OF THIN POWER ALGEBRAS. Suppose that $s \geq 2^{r-1}$. Then *if the defect* $\Delta(j^s,r)_{j'} = 0$ *so does not intervene,* the socle degree $j' = \mathrm{SOCDEG}(j,r,s)$ of a thin power algebra satisfies

$$SOCDEG(j,r,s) \approx bj + O(j), \text{ where } b = 1 + \frac{1}{s^{1/(r-1)} - 1}. \quad (3.20)$$

Proof. Immediate from Lemma 1.7.

Remark. The limit ratio b of (3.20) is rarely a rational number. Thus, SOCDEG(j,r,s) cannot usually be simply expressed in terms of j or of j mod k for a fixed integer k unless the defect enters strongly, as when (r,s) = (4,6) or (4,9). An example of the defect intervening strongly is the case (r,s) = (4,9): if the Conjecture in Example 3.12 is true the SOCDEG(j,4,9) = 2j-2.

When $(r,s) = (4, 8 = 2^{4-1})$, then $\sigma = \mathrm{SOCDEG}(j,r,s) = 2j-2$ for $j \leq 11$. We expect T'(j,s,r) for $s = 2^{r-1}$ to have zero defect, hence it would have predictable patterns resembling Examples 3.13C and 3.14D.

When $s < 2^{r-1}$, the expression for the limit

$$b_{r,s} = \lim_{j \to \infty} SOCDEG(j,r,s)/j$$

is more complicated. If $2 \leq b < 3$, corresponding to roughly, $(3/2)^{r-1} < s \leq 2$ then if the defect $\Delta(j^s,r)_{j'} = 0$, it is easy to see from a refinement of the proof of Lemma 1.7 that $b = b_{r,s}$ satisfies

$$b^{r-1} - s(b-1)^{r-1} + \binom{s}{2}(b-2)^{r-1} = 0. \quad (3.21)$$

Example 3.16. When (r,s) = (5,10) we obtain a limit ratio b = 2.293765553 (See also Example 3.10A). When (r,s) = (5,8) we obtain a limit ratio b = 2.509833693. When (r,s) = (4,7) we obtain b = 2.096961266.[10] The existence of nonzero defects for thin power algebras could only increase the actual limit ratio over these predicted thin algebra values.

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[10] Solutions to (3.21) were calculated using the Maple software.

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Sept.17, 1993

Errata for Inverse Systems III.

p.32 Example 3.8: OMIT. In fact, the value of $\Delta(j^{10},5) = 0$ for $j \leq 9$. The incorrect value given came from a too-special choice of linear forms.

p.33. Example 3.10A: OMIT OLD EXAMPLE - again the case $(r,s) = (5,10)$. Replace with following NEW EXAMPLE:

Example 3.10A. When $(r,s) = (6,8)$, $3 \leq j \leq 5$ we have that the socle degree of $A = R/L^j$ is $3j-3$ for $j = 3$, and is $3j-2$ for $j = 4,5$. We also find $\Delta(j^8,6)_{3j-3} = \dim_k R_{j-2}$; while $\Delta(4^8,6)_{3j-2} = 118$, and $\Delta(5^8,6)_{3j-2} = 150$. Does the polynomial $\Delta(j^8,6)\cdot Z^{-3j+3}$ approach a limiting value?