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THE GEOMETRY OF RATIONAL SURFACES AND HILBERT FUNCTIONS OF POINTS IN THE PLANE

Brian Harbourne

ABSTRACT. We study the structure of the set of numerically effective divisor classes on a rational surface and apply this to study hilbert functions of the homogeneous coordinate rings of 0-cycles on curves of low degree in $\mathbb{P}^2$. For example, for a nonnegative 0-cycle $m_1 p_1 + \cdots + m_n p_n$ of points $p_1, \ldots, p_n$ on an irreducible conic in $\mathbb{P}^2$, we show that the hilbert function depends only on the coefficients m_i, as conjectured by Davis and Geramita [DG]. We also determine for each such 0-cycle the degree at which the hilbert function stabilizes (first equals its hilbert polynomial) and we characterize all such 0-cycles having a generic hilbert function, a generic hilbert function being one which is equal to the hilbert function of the ring of $\mathbb{P}^2$ up to the point at which it stabilizes.

INTRODUCTION. Consider a set of distinct points $p_1, \ldots, p_n$ in $\mathbb{P}^2_k$, k being any algebraically closed field. It is a long studied problem to determine the dimension of the linear system of curves of degree d having an assigned base point at each point p_i of multiplicity at least m_i. Put another way, the problem is to determine the dimension of the vector space of homogeneous forms of degree d contained in $\bigcap_i \mathcal{P}_i^{m_i}$, where $\mathcal{P}_i$ is the ideal of p_i in the homogeneous coordinate ring of $\mathbb{P}^2$.

From the latter point of view one could approach this problem via a study of the hilbert function of the homogeneous coordinate ring of the 0-cycle $\Sigma m_i p_i$ of $\mathbb{P}^2$ [DG]. From the former point of view one could approach this problem via a study of complete linear systems on the blowing-up $X \to \mathbb{P}^2$ of $\mathbb{P}^2$ at the points $p_1, \ldots, p_n$ [H1]. This point of view suggests other questions, the one which we will be concerned with here being to determine the semigroups of divisor classes of X of effective and numerically effective divisors.

Little seems to be known about such questions in general, even if the points $p_1, \ldots, p_n$ are independent generic points of $\mathbb{P}^2$ [H1],[N2]. However, in case the points are smooth points of an irreducible curve Q of degree three or less, these questions can be studied very explicitly [DG], [H1], [H2], [H3], [L].

In this paper we study the geometry of the effective and numerically effective divisors on X under the condition that the points $p_1,\ldots,p_n$ lie on an irreducible curve Q of degree three or less, and as an application we obtain some new results concerning the hilbert function of a 0-dimensional subscheme of $\mathbb{P}^2$.

I am indebted to A. Geramita for suggesting the applications to hilbert functions and for several helpful conversations on the topic and also for sending me his and E. Davis' paper [DG]. It gives, among other things, a complete analysis of the hilbert function of points on a line and it puts forward a number of conjectures concerning the hilbert function of points on a conic, some of which we answer here.

For standard results of algebraic geometry, we refer to [AG], without explicit mention hereafter.

I. GEOMETRY OF CURVES ON RATIONAL SURFACES

I.1 Exceptional configurations. Consider a set $p_1,\ldots,p_n$ of distinct points in $\mathbb{P}^2$. By blowing-up these points we obtain a morphism $X \to \mathbb{P}^2$ of smooth rational surfaces. The divisor classes $E_0, E_1,\ldots,E_n$ on X of the total transforms of, respectively, a line in $\mathbb{P}^2$ and the points $p_1,\ldots,p_n$ give a $\mathbb{Z}$-basis of the divisor class group Pic X. Once having obtained X, there may be other such morphisms $X \to \mathbb{P}^2$ [H3] and any such morphism factors into a sequence of blowings-up at points giving rise, as above, to a basis of Pic X. Such a basis, arising from a morphism $X \to \mathbb{P}^2$, is called an exceptional configuration.

I.2 Riemann-Roch. The motivation for studying the blowing-up $X \to \mathbb{P}^2$ is as follows. The dimension of the linear system of degree d on $\mathbb{P}^2$ having assigned base points $m_i p_i$, $i = 1,\ldots,n$, is also the dimension $h^0(X,F)$ of global sections of the divisor class $F = dE_0 - m_1E_1 - \ldots - m_nE_n$.

A major tool in studying $h^0(X,F)$ is the formula of Riemann-Roch. Now Pic X has an intersection product; in terms of the basis $E_0,\ldots,E_n$, it is given by:

$$E_i \cdot E_j = 0 \;,\; i \neq j \;;\; -E_0^2 = E_i^2 = -1 \;,\; i = 1,2,\ldots,n \;.$$

And Riemann-Roch for a divisor class F on a rational surface X reads:

$$\text{(I.2.1)} \qquad h^0(X,F) - h^1(X,F) + h^2(X,F) = \tfrac{1}{2}(F\cdot F - F\cdot K) + 1 \;,$$

where K is the canonical class of X. In terms of any exceptional configuration, K is $-3E_0 + E_1 + \cdots + E_n$. And the adjunction formula gives the arithmetic genus $p_a(C)$ of a divisor C : $2p_a(C) - 2 = C\cdot C + C\cdot K$.

I.3 Effectivity. It is convenient to refer to a class $F \in \mathrm{Pic}\ X$ as being effective if $h^0(X,F) > 0$ and as being numerically effective if $F \cdot G \geqslant 0$ for every effective class G. In particular, if F is the class of an effective divisor that moves in a linear system without fixed components, then F is both effective and numerically effective.

The following results are standard and allow us to study divisors on X inductively:

LEMMA (I.3.1). Let F lie in $\mathrm{Pic}\ X$ and let X' be the blowing-up $\Pi : X' \to X$ of X at some point p, and Π^* the induced homomorphism $\Pi^* : \mathrm{Pic}\ X \to \mathrm{Pic}\ X'$.

(1) If F is effective then $F \cdot E_0 \geqslant 0$.

(2) If $F \cdot E_0 \geqslant -2$ then $h^2(X,F) = 0$.

(3) The induced map Π^* is an injection that preserves effectivity and numerical effectivity; indeed

$$h^i(X,F) = h^i(X', \Pi^* F) , \; i = 0,1,2 \quad .$$

PROOF. (1) This follows since E_0 is numerically effective.

(2) This follows from (1) by duality.

(3) See (1.3) [H1] and (1.4)[H3].

I.4 A vanishing theorem. Now and hereafter in this paper, we restrict our attention to the case that the points $p_1,\ldots,p_n$ are distinct smooth points of an irreducible reduced curve Q of degree three or less. We denote by Q also its proper transform on X and its divisor class in $\mathrm{Pic}\ X$. For a simultaneous analysis of the cases $\deg Q = 1,2,3$, technical considerations require if $\deg Q = 3$ that we impose the further restriction that the natural map $\mathrm{Pic}\ X \to \mathrm{Pic}\ Q$ have trivial kernel.

Under these conditions we will show (vid. (I.6)) that for any class F of $\mathrm{Pic}\ X$, $h^0(X,F)$ depends only on the coefficients of F written in terms of the exceptional configuration given by blowing up the points $p_1,\ldots,p_n$. In particular, we will give an algorithm for computing $h^0(X,F)$ that is independent of the particular disposition of the points $p_1,\ldots,p_n$. The key result is:

THEOREM (I.4.1). A numerically effective divisor class F on the surface X, given the restrictions of the penultimate paragraph, is effective and has $h^1(X,F) = 0$.

We give the proof following the proof of (I.5.3).

I.5 Numerical effectivity. We give here a description of the numerically effective divisor classes. The case that deg $Q = 3$ is quite complicated and for full details we refer to [H1]. However we have:

PROPOSITION (I.5.1). Suppose deg $Q = 3$, and that F is a numerically effective divisor on X. Then, for some exceptional configuration $E_0',\ldots,E_n'$ on X, F is a nonnegative sum of the classes: E_0', $E_0' - E_1'$, $2E_0' - E_1' - E_2'$, $3E_0' - E_1' - \ldots - E_i'$, $i = 3,4,\ldots,n$. In particular, F is effective.

PROOF. This is (3.2) of [H1].

PROPOSITION (I.5.2). Suppose deg $Q = 1$. For a class F on X the following are equivalent:

(1) F is numerically effective;

(2) $F \cdot E_i \geqslant 0$, $i = 0,\ldots,n$, $F \cdot Q \geqslant 0$;

(3) F is a nonnegative sum of the classes E_0 and $E_0 - E_i$, $i = 1,\ldots,n$.

In particular, a numerically effective class is effective.

PROOF. The proof that (1) implies (2) and (2) implies (3) is easy, while (3) implies (1) follows by the remark in (I.3) since E_0 and $E_0 - E_i$ are the classes of irreducible curves moving in a linear system without fixed components.

We now consider the case deg $Q = 2$:

PROPOSITION (I.5.3). Suppose deg $Q = 2$ and let F be a divisor class of X. Then the following are equivalent:

(1) F is numerically effective;

(2) after reindexing so that $F \cdot E_1 \geqslant \ldots \geqslant F \cdot E_n$, F satisfies the conditions:

$$F \cdot E_0 \geqslant 0\ ,\ F \cdot E_n \geqslant 0\ ,\ F \cdot (E_0 - E_1 - E_2) \geqslant 0\ ,\ F \cdot Q \geqslant 0\ ;$$

(3) F is a nonnegative sum of the classes: E_0, $E_0 - E_i$, $i = 1,\ldots,n$, $2E_0 - E_{i_1} - E_{i_2} - E_{i_3}$, $3E_0 - E_{j_1} - \ldots - E_{j_6}$, $dE_0 - (d-1)E_{\ell_1} - E_{\ell_2} - \ldots - E_{\ell_{d+2}}$, $d \geqslant 2$, where $1 \leqslant i_1,\ldots \leqslant n$ (resp. $j_1,\ldots$; resp. $\ell_1,\ldots$) are distinct indices.

Before we begin the proof, we need the following:

LEMMA (I.5.4). Each of the classes listed in (I.5.3)(3) is the class of an effective irreducible and reduced divisor.

REMARK (I.5.5). Using (I.4.1), (I.2.1) and (I.3.1)(2) it follows from (I.5.4) and (I.5.3) that any numerically effective class is effective without fixed components, in the case that $\deg Q \leq 2$.

PROOF of (I.5.4). This is obviously true for E_0 and $E_0 - E_i$. For the others it is enough to express each as the sum of irreducible and reduced divisors in two different ways having no components in common. The conclusion then follows easily from Bertini's theorem [Z1], [Z2]. For example,

$$dE_0 - (d-1)E_1 - E_2 - \dots - E_{d+2}$$

can be written as

$$Q + E_{d+3} + \dots + E_n + (d-2)(E_0 - E_1)$$

and as

$$(E_0 - E_{d+1} - E_{d+2}) + \sum_{i=2}^{d} (E_0 - E_1 - E_i)$$

PROOF of (I.5.3). Now (1) clearly implies (2) and by the remark in (I.3) using (I.5.4), (3) implies (1). To see that (2) implies (3), by induction on $F \cdot E_0$ it is enough to show for any class $F \neq 0$ satisfying the conditions of (2) that there is a nonzero sum G of the classes of (3) for which $F - G$ satisfies the conditions of (2). In fact, since $A + B$ satisfies (2) if both A and B do, it is enough to find A and B satisfying (2) such that $A + B = F$ and $B - G$ satisfies (2).

So suppose F satisfies (2), and denote $2E_0 - E_1 - \dots - E_i$ by Q_i. It follows from the first three conditions of (2) that after reindexing F is a nonnegative sum of the classes E_0, $E_0 - E_1$ and Q_i, $i \geq 2$. But $Q_2 = (E_0 - E_1) + (E_0 - E_2)$ so indeed F is a sum of $C_0 = E_0$, $C_1 = E_0 - E_1$, $C_2 = E_0 - E_2$ and $C_i = Q_i$, $i \geq 3$; i.e., $F = \sum_{i \geq 0} m_i C_i$, for some nonnegative integers m_i, $i \geq 0$.

If $F \cdot Q \geq 2$, then $m_j > 0$ for some index $j = 0,1,2$ or 3 and, taking $G = C_j$, $F - G$ satisfies (2).

Now assume $F \cdot Q = 1$. If $m_j > 0$ for some index $j = 1,2$ or 3 we take $G = C_j$ and argue as above, so say that $m_1 = m_2 = m_3 = 0$. By distributing the term $m_0 C_0$ over the other summands, we can write F as a sum of terms satisfying (2) of the form: $d_i C_0 + r_i C_i$, $i \geq 4$, i even, $r_i, d_i \geq 0$; $d_j C_0 + C_{j_1} + C_{j_2}$, $d_j \geq 0$, j_1 and j_2 odd; and one term of the form

$d_\ell C_0 + C_\ell$, $\ell > 4$, ℓ odd, and $Q \cdot (d_\ell C_0 + C_\ell) = 1$. But C_4 is one of the classes of (3) and it is easy to see that $d_\ell C_0 + C_\ell - C_4$ satisfies (2), so taking $G = C_4$ we see that $F - G$ satisfies (2).

So now suppose $F \cdot Q = 0$. If $m_4 > 0$ we may take $G = C_4$, so we may assume $m_4 = 0$. By an argument similar to the one above, we may break up F into sums of classes of the form: $mE_0 + Q_i + Q_j$ where i and j are odd, $5 \leq i \leq j$, and $2m + 8 = i + j$; and classes of the form $n_0E_0 + n_1C_1 + n_2C_2 + n_3Q_3 + Q_i$, where $i \geq 4$ and $2n_0 + n_1 + n_2 + n_3 + 4 = i$. Moreover, each such class satisfies (2). If H is a class like the former, the reader can check that $H - Q_4$ satisfies the conditions of (2). Likewise, if H is a class like the latter then $H - G$ satisfies the conditions of (2) for $G = Q_4$ as long as $n_0 = n_1 = n_2 = 0$ or no two of n_0, n_1, n_2 are zero or $n_1 = n_2 = 0$ and $n_0 > 1$. If $n_1 = n_2 = 0$ and $n_0 = 1$, then take G to be $3E_0 - E_1 - \ldots - E_6$ and if $n_0 = n_2 = 0$, $n_1 > 0$, take $G = (n_1 + 2)E_0 - (n_1 + 1)E_1 - E_2 - \ldots - E_{n_1+4}$. The only remaining case is $n_0 = n_1 = 0$, $n_2 > 0$, and this last case is the same as the previous case after transposing the first two indices, ending the proof.

PROOF of (I.4.1). Consider a numerically effective class F. If $\deg Q = 3$ then F is effective by (I.5.1), and $h^1(X,F) = 0$ by (1.1) and (1.2) of [H1].

Suppose now $\deg Q = 1$. Again F is effective, now by (I.5.2). If $F = 0$ then $h^1(X,F) = 0$ since X is a rational surface. Otherwise, we may assume that F is a sum of a numerically effective class F' and $G = E_0$ or, say, $E_0 - E_1$, by (I.5.2). Denote an irreducible section of G by C, and take cohomology of the exact sequence:

$$0 \to F' \to F \to F \otimes \mathcal{O}_C \to 0$$

Now $h^1(X,F') = 0$ by induction on $F \cdot E_0$ and $h^1(C, F \otimes \mathcal{O}_C) = 0$ since $C \simeq \mathbf{P}^1$ and $F \cdot C \geq 0$. Thus $h^1(X,F)$ vanishes.

Lastly, say $\deg Q = 2$. Each of the generators (I.5.3)(3) of the cone of numerically effective divisor classes has arithmetic genus $p_a = 0$ by the adjunction formula, except for $3E_0 - (E_1 + \ldots + E_6)$ which has $p_a = 1$. But by (I.5.4) each of these classes has an irreducible section. Thus, as long as F is not a multiple of $3E_0 - (E_1 + \ldots + E_6)$, we can write F as a sum of numerically effective classes $F' + G$ where G has an irreducible section C isomorphic to $\mathbb{P}^1$. Arguing as above we see $h^1(X,F) = 0$. In case F is a multiple of $3E_0 - (E_1 + \ldots + E_6)$, we can write F as $F' + G$, but now C has $p_a = 1$. Since $C \cdot C > 0$, we have $F \cdot G > 0$ so $h^1(C, F \otimes \mathcal{O}_C) = 0$ and arguing as before we again find $h^1(X,F) = 0$.

I.6 Effectivity. The results of (I.4) and (I.5) allow us to give algorithms for computing $h^0(X,F)$ for any class F of Pic X in case $\deg Q \leq 2$. The algorithms produce a class F' depending on F such that $h^0(X,F) = h^0(X,F')$ and such that either $F' \cdot G < 0$ for some numerically effective class G or F' is itself numerically effective. In the former case numerical effectivity of G implies that $h^0(X,F') = 0$. In the latter case $h^1(X,F') = 0$ and F' is effective, by (I.5.2) or (I.5.3). Thus $F' \cdot E_0 \geq 0$ and so $h^2(X,F') = 0$ by (I.3.1)(2). Now $h^0(X,F') = \frac{1}{2}(F' \cdot F' - F' \cdot K) + 1$ follows from (I.2.1). In particular, this gives $h^0(X,F)$.

It is worth noting that in case $\deg Q = 3$ an algorithm for computing $h^0(X,F)$ is given in [H1] . This algorithm also works by producing from F such a class F', but it is much more complicated than if $\deg Q \leq 2$. We refer the reader to [H1] for details.

We denote by E_0 the class of a line and by $E_1,\dots,E_n$ the classes of the blowings-up of the points $p_1,\dots,p_n$. Let F be any class of Pic X. Since the classes $E_0,\dots,E_n$ give a basis of Pic X we can write $F = m_0E_0 + \cdots + m_nE_n$ for some integers m_i, $i \geq 0$.

As a useful general remark, suppose H is the class of an irreducible and reduced curve such that $F \cdot H < 0$; then $h^0(X,F) = h^0(X,F-H)$. For if $F-H$ is effective then F clearly is, while if F is effective then H is the class of a fixed component of the linear system of effective divisors in the class of F.

Here is the algorithm in case $\deg Q = 1$. The proper transform of Q on X is in the class $E_0 - E_1 - \cdots - E_n$ which we denote by L . If $m_0 = F \cdot E_0 < 0$ then F cannot be effective and taking $F' = F$ we are done. So say m_0 is nonnegative. If, for some $i > 0$, m_i is positive, then $F \cdot E_i < 0$. Thus by our remark above $h^0(X,F) = h^0(X,F - m_iE_i)$. We may therefore reduce to the case that $F \cdot E_i \geq 0$, $i \geq 0$. If now $F \cdot L \geq 0$ then F is numerically effective by (I.5.2) and taking $F' = F$ we are done. If, however, $F \cdot L < 0$, then $h^0(X,F) = h^0(X,F-L)$ and so we replace F by $F-L$ and start over. Since $(F-L) \cdot E_0 < F \cdot E_0$ it is clear that eventually we obtain a class F' as desired, ending the algorithm.

Now say $\deg Q = 2$, and let C denote the class of the proper transform of Q on X; i.e., $C = 2E_0 - E_1 - \cdots - E_n$. As before, we are done if $F \cdot E_0 < 0$, and otherwise, as before, we reduce to the case that $F \cdot E_i \geq 0$, $i \geq 0$. By reindexing we may assume $m_1 \leq \cdots \leq m_n$. Let J denote $E_0 - E_1 - E_2$. If $F \cdot J < 0$ we replace F by $F-J$ and start over. Otherwise we consider $F \cdot C$. If $F \cdot C < 0$ we replace F by $F-C$ and start over. If $F \cdot C \geq 0$ then F is numerically effective by (I.5.3) and we are done. Since both $(F-J) \cdot E_0$ and $(F-C) \cdot E_0$ are less than $F \cdot E_0$, it is clear, as before, that the algorithm terminates.

REMARK (I.6.1). It is worth noting that in both algorithms the coefficients m_i, $i \geq 0$, alone completely determine $h^0(X,F)$. The actual disposition of the points on Q is unimportant. (This is also true of the algorithm given in [H1] for the case that deg Q = 3, under the condition that Pic X $\to$ Pic Q has trivial kernel.) This is proven in case deg Q = 1 and conjectured in case deg Q = 2 in [DG].

REMARK (I.6.2). Our algorithms for finding $h^0(X,F)$ depend principally on identifying the numerically effective classes of Pic X and on being able to compute $h^0(X,G)$ for any numerically effective class G. This leads us to certain conjectures in case the points $p_1,\ldots,p_n$ are sufficiently general (for example, if the coordinates of the points are independent transcendentals over the prime field of the ground field k.)

In particular, let Y be the blowing up of sufficiently general points $p_1,\ldots,p_n$ of $\mathbb{P}^2$. Our first conjecture is that a class F of Pic Y is numerically effective if and only if $F \cdot F \geq 0$ and F is a <u>standard</u> class, i.e., for some exceptional configuration $E'_0,\ldots,E'_n$ of Y F is a nonnegative sum of the classes E'_0, $E'_0 - E'_1$, $2E'_0 - E'_1 - E'_2$ and $3E'_0 - E'_1 - \ldots - E'_i$, $i \geq 3$. Our second conjecture is that if F is standard then $h^0(Y,F)h^1(Y,F) = 0$. Since $h^2(Y,F) = 0$ for a standard class F by (I.3.1)(2), our second conjecture implies that $h^0(Y,F)$ can be determined from (I.2.1). (The techniques of [H1] would allow one to compute $h^0(Y,F)$ for any class F of Pic Y if the second conjecture is true and we think it likely that the second conjecture implies the first.)

We can now prove a proposition that will be of use in applying these results to the study of hilbert functions. Suppose X is a blowing-up of $\mathbb{P}^2$ satisfying the conditions we imposed in the first paragraph of (I.4). For fixed nonnegative integers $m_1,\ldots,m_n$, consider the divisor class $F(d) = dE_0 - m_1E_1 - \ldots - m_nE_n$. Define δ to be the least integer d for which F(d) is numerically effective and define τ to be the least nonnegative integer d for which $h^1(X,F(d)) = 0$.

PROPOSITION (I.6.3). The quantities δ and τ exist and satisfy the inequality $\delta - 1 \leq \tau \leq \delta$.

PROOF. First assume that $m_i > 0$ for some i. By (I.4.1), if δ exists then $\tau \leqslant \delta$. But if $d \geqslant m_1 + \ldots + m_n$ then $F(d)$ can be written as $m_1(E_0 - E_1) + \ldots + m_n(E_0 - E_n) + [d - (m_1 + \ldots + m_n)]\, E_0$, and a check of (I.5.1), (I.5.2) and (I.5.3) verifies that then $F(d)$ is numerically effective. Since in any case d must be positive for $F(d)$ to be numerically effective it follows that δ exists and is positive.

We now show that either $h^1(X,F(\delta - 2)) > 0$ or $h^1(X,F(\delta - 1)) > 0$, from which it follows that τ exists and satisfies $\delta - 1 \leqslant \tau \leqslant \delta$. First, we see $0 = h^2(X,F(\delta - 2)) = h^2(X,F(\delta - 1))$ by (I.3.1). Now suppose for $i = 1$ (and hence $i \geqslant 1$) that $F(\delta - i)$ is not effective. By (I.2.1), $h^1(X,F(\delta - i))$ is therefore equal to $-[\frac{1}{2}(F(\delta - i) \cdot F(\delta - i) - F(\delta - i) \cdot K) + 1]$, for $i = 1,2$. Now the expression in the brackets, considered as a quadratic function of i, attains its maximum at $\delta + \frac{3}{2} > 2$. Therefore $h^1(X,F(\delta - 1)) < h^1(X,F(\delta - 2))$ and thus the latter is positive.

Let us denote $F(\delta - i)$, for $i = 0,1$ and 2, by F, F' and F'' respectively and now assume F' is effective. By definition of δ, we see that there is an irreducible curve C such that $F' \cdot C < 0$, and obviously $C \cdot E_0 > 0$; otherwise, $F' \cdot C = F \cdot C \geqslant 0$. If either $F' \cdot C \leqslant -2$ or the arithmetic genus p_a of C is one, then $h^1(C,F' \otimes \mathcal{O}_C) > 0$. Since $F' - C$ is effective, we see $(F' - C) \cdot E_0 \geqslant 0$ so $h^2(X,F' - C) = 0$ by (I.3.1). Taking cohomology of the exact sequence

$$0 \to F' - C \to F' \to F' \otimes \mathcal{O}_C \to 0$$

we see $h^1(X,F') > 0$ as desired.

It may be that F' is effective but $F' \cdot C = -1$ and $p_a(C)$ is zero. Then we may write $F' = G + C$ where G is effective, and so as before $h^2(X,G - E_0) = 0$. Also, $F'' \cdot C < F' \cdot C$, since $C \cdot E_0 > 0$, so we see $h^1(C,F'' \otimes \mathcal{O}_C) > 0$. Now keeping in mind that $G - E_0 = F'' - C$, and taking cohomology of

$$0 \to F'' - C \to F'' \to F'' \otimes \mathcal{O}_C \to 0$$

we see that $h^1(X,F'') > 0$.

To finish the proof, now consider the case $m_1 = \ldots = m_n = 0$. Then $\delta = 0$ and from (I.2.1), (I.3.1) and (I.4.1) we see $h^1(X,F(d)) = 0$ for any $d \in \mathbf{Z}$. Thus $\tau = 0$ by definition.

II. HILBERT FUNCTIONS

II.1 Preliminaries. In this chapter we will apply the results of Chapter I to study hilbert functions of certain 0-cycles of $\mathbb{P}^2$. The questions we will consider here are: for which 0-cycles is the hilbert function generic and, for a given 0-cycle, when does the hilbert function stabilize? We first recall the pertinent notions.

Let $p_1,\dots,p_n$ be distinct points of $\mathbb{P}^2$, and $m_1,\dots,m_n$ nonnegative integers. The hilbert function $H(d,Z)$, $d \geqslant 0$, of a 0-cycle $Z = m_1p_1 + \dots + m_np_n$ is the k-dimension of the homogeneous component $\overline{R}_d$ of degree d of the homogeneous coordinate ring $\overline{R}$ of Z. As usual, $\overline{R}$ is R/I where R is the homogeneous coordinate ring of $\mathbb{P}^2_k$ and I is the homogeneous ideal of Z.

By blowing-up $X \to \mathbb{P}^2$ the points $p_1,\dots,p_n$ we can identify I_d with the global sections of the divisor class $dE_0 - m_1E_1 - \dots - m_nE_n$ which we will denote by $Z(d)$. In particular, the dimension of I_d is $h^0(X,Z(d))$. We will denote the constant $h^1(X,Z(0)) - h^0(X,Z(0)) + 1$ by $p(Z)$, and refer to it as the <u>hilbert polynomial</u> of Z. By (I.2.1) we see $p(Z) = \frac{1}{2}[m_1^2 + \dots + m_n^2 + m_1 + \dots + m_n]$.

PROPOSITION (II.1.1). Let Z be a 0-cycle of $\mathbb{P}^2$. Then

$$H(d,Z) + h^1(X,Z(d)) = p(Z) .$$

PROOF. The dimension of $\overline{R}_d$ is the difference of the dimensions of R_d and I_d, using the notation above. The former is $\frac{1}{2}(d^2 + 3d) + 1$, which is well known and follows from (I.2.1) since R is the homogeneous ideal of the zero 0-cycle of $\mathbb{P}^2$. Also, the dimension of I_d is $h^0(X,Z(d))$ and by (I.2.1) again we see that this is $h^1(X,Z(d)) + \frac{1}{2}(d^2 + 3d) - p(Z) + 1$. The difference gives $H(d,Z) = -h^1(X,Z(d)) + p(Z)$.

It is well-known that $H(d,Z)$ is an increasing function of d and that for $d \gg 0$, $H(d,Z)$ stabilizes with a constant value of $p(Z)$ [F]. We set τ to be the least such nonnegative d. (This agrees with our definition of τ in Chapter I.) One problem, then, short of determining $H(d,Z)$, is to determine τ.

A different sort of problem is to determine which 0-cycles Z have a generic hilbert function, meaning that (for $d \geqslant 0$) $H(d,Z)$ is of the form $\min(1 + \frac{1}{2}(d^2 + 3d), p(Z))$, or, said differently, $H(d,Z)$, up to the point at which it stabilizes, equals the hilbert function of $\mathbb{P}^2$ itself. Thus a generic hilbert function grows as quickly as possible, or what is the same, $h^0(X,Z(d))$ grows as slowly as possible. We have:

PROPOSITION (II.1.2). The hilbert function $H(d,Z)$ of a 0-cycle Z is generic if and only if $h^0(X,Z(d)) \cdot h^1(X,Z(d))$ is identically zero.

PROOF. Clearly, $H(d,Z)$ is generic if and only if $H(d,Z)$ is always either $p(Z)$ or $1 + \frac{1}{2}(d^2 + 3d)$. But $H(d,Z)$ equals $p(Z)$ precisely when $h^1(X,Z(d)) = 0$ by (II.1.1). On the other hand, $H(d,Z) = 1 + \frac{1}{2}(d^2 + 3d)$ precisely when the degree d component $\overline{R}_d$ of the homogeneous coordinate ring of Z equals the degree d component R_d of the ring of $\mathbb{P}^2$, and thus precisely when the degree d component I_d of the ideal of Z is (0), which is equivalent to $h^0(X,Z(d)) = 0$.

II.2 Points on a curve Q of low degree. In order to apply the results of Chapter I, we will hereafter assume, as in (I.4), that the points $p_1,\ldots,p_n$ lie on the smooth part of an irreducible curve Q of degree three or less. By (II.1.1) and our algorithm in (I.6) (see [H1] for the case $\deg Q = 3$) for computing cohomology we see that we can compute the hilbert function of any nonnegative 0-cycle $Z = m_1p_1 + \ldots + m_np_n$. As noted in (I.6.1), when the degree of Q is no more than two, $H(d,Z)$ depends only on the coefficients m_i. We now want to give simple formulas for the degree τ at which the hilbert function first agrees with the hilbert polynomial, and also to give a complete classification of the 0-cycles for which the hilbert function is generic.

We do this only in case $\deg Q \leqslant 2$. When $\deg Q = 3$ we do not have good answers to these questions. The occurrence of infinitely many exceptional classes on the blowing-up X of $\mathbb{P}^2$ at $p_1,\ldots,p_n$ makes characterizing δ (vid. (I.6.3)) difficult and guarantees a plethora of generic 0-cycles. For example, if Z is a 0-cycle for which $Z(d)$ for some d is an irreducible exceptional class, then Z is a generic 0-cycle. And if $\deg Q = 3$, there may be infinitely many such exceptional classes.

II.3 Q is a line. Hilbert functions of points on a line are completely analyzed by Geramita and Davis in [DG]. However it seems appropriate to deduce the following result from our techniques. We will follow our previously established conventions. In particular, $p_1,\ldots,p_n$ are distinct points of Q, Z is a nonnegative formal sum of these points, $Z(d)$ is the divisor class on X of degree d which corresponds to Z (vid. (II.1)), where X is the blowing-up of $\mathbb{P}^2$ at $p_1,\ldots,p_n$, and δ is the least d for which $Z(d)$ is numerically effective. We also denote by Q the proper transform of Q on X. Also, since $H(d,Z)$ depends only on the coefficients of the points in the sum Z, we will sometimes represent Z by its coefficients $(m_1,\ldots,m_n)$. Moreover, to compute cohomology of $Z(d)$, it is enough by (I.3.1)(3) to take X to be the blowing-up of $\mathbb{P}^2$ only at the points whose coefficients are not zero, and this is what we do.

THEOREM (II.3.1). Suppose Q is a line.

(1) For any nonzero nonnegative 0-cycle, τ equals $\delta - 1$, while $\tau = \delta = 0$ for the zero 0-cycle.

(2) Up to permuting the coefficients, the 0-cycles having generic hilbert functions are precisely $(m, 0, \ldots, 0)$, and $(m, 1, 0, \ldots, 0)$, where m is a nonnegative integer.

PROOF. (1) If Z is the trivial 0-cycle then $\tau = \delta = 0$ as we saw in the proof of (I.6.3). Otherwise, $Z(\delta) - Q$ is numerically effective by (I.5.2)(3). Likewise $Z(\delta) - Q - E_0$ is also numerically effective, as long as $Q \cdot Q < 0$ on X. Thus, in case $Q \cdot Q < 0$, we see $h^i(X, Z(\delta) - Q - E_0) = 0$, $i = 1,2$, by (I.4.1) and (I.3.1), while $h^1(Q, (Z(\delta) - E_0) \otimes \mathcal{O}_Q) = 0$ since $(Z(\delta) - E_0) \cdot Q \geqslant -1$. So taking cohomology of

$$0 \longrightarrow Z(\delta) - E_0 - Q \longrightarrow Z(\delta) - E_0 \longrightarrow (Z(\delta) - E_0) \otimes \mathcal{O}_Q \longrightarrow 0$$

we find that $h^1(X, Z(\delta) - E_0) = 0$. Since $Z(\delta) - E_0 = Z(\delta - 1)$, we see that $\tau \leqslant \delta - 1$, and $\tau = \delta - 1$ now follows from (I.6.3).

If, however, $Q \cdot Q = 0$, then $Z = (m, 0, \ldots, 0)$, so $Z(\delta - 1)$ is $(m-1)E_0 - mE_1$. Since $F = E_0 - E_1$ is numerically effective and $Z(\delta - 1) \cdot F < 0$, we see $h^0(X, Z(\delta - 1)) = 0$. By (I.3.1), $h^2(X, Z(\delta - 1)) = 0$, and now, plugging into Riemann-Roch (I.2.1), we find $h^1(X, Z(\delta - 1)) = 0$. As before, τ equals $\delta - 1$.

(2) Since $h^1(X, Z(\delta - 1)) = 0$ for any nonnegative 0-cycle Z, it follows by (II.1.2) that Z has a generic hilbert function if $h^0(X, Z(\delta - 2)) = 0$. For $Z = (m, 0, \ldots, 0)$ it is easy to see that $\delta = m$ and hence $Z(\delta - 2) \cdot (E_0 - E_1) < 0$. Since $(E_0 - E_1)$ is numerically effective, $Z(\delta - 2)$ cannot be effective and so Z is generic. For $Z = (m, 1, 0, \ldots, 0)$ we find $\delta = m + 1$, and again $Z(\delta - 2) \cdot (E_0 - E_1) < 0$ so Z is generic. However, if Z has at least three nonzero entries, say $m_1, m_2, m_3 > 0$, then $[Z(\delta - 2) - (E_0 - E_1 - E_2 - E_3)]$ is numerically effective, hence effective (I.5.2). But $E_0 - E_1 - E_2 - E_3$ is effective, so $Z(\delta - 2)$ is also. In particular, $Z(\delta - 2)$ is not generic. If Z has two entries, say m_1, m_2, both at least two, then $Z(\delta - 2) - 2(E_0 - E_1 - E_2)$ is numerically effective, and as before Z is not generic.

Therefore, Z is generic if and only if Z has at most two positive entries and of these at most one is larger than one.

II.4 <u>Q is a conic</u>. We employ the same conventions as stated in the first paragraph of (II.3):

THEOREM (II.4.1). Suppose Q is an irreducible conic.

(1) For any nonnegative 0-cycle $Z = (m_1, \ldots, m_n)$, τ equals δ if and

only if $Z(\delta)\cdot Q = 0$; otherwise $\tau = \delta - 1$.

(2) A nonnegative 0-cycle Z has a generic hilbert function if and only if $Z(\delta - 1)$ is not effective or has the form $N + F' + Q'$ where: N is numerically effective but $N - E_0$ is not effective; Q' is either 0 or Q; F' is, up to reindexing, either 0, $3E_0 - 2E_1 - 2E_2 - 2E_3$, or $dE_0 - dE_1 - E_2 - \dots - E_{d+1}$, $d \geq 1$; $N\cdot F' = 0$; and $(N + Q)\cdot Q = -1$ if $Q' = Q$.

The proof of (II.4.1) follows the next result, which classifies the numerically effective classes N such that $N - E_0$ is not effective. We note from (II.4.1)(2) and (II.4.2) one can generate a list of all possible generic 0-cycles. We give this list in (II.4.3).

PROPOSITION (II.4.2). Let Z be a nonnegative 0-cycle of points on Q. Then $Z(\delta - 1)$ is not effective if and only if Z is, up to reindexing, one of the following cycles $(m \geq 0)$:

(m) , $(1, 1, 1)$, (m, m, m, m) ,
$(m + 2, m + 2, m + 2, m + 1)$, $(m + 2, m + 2, m + 2, m + 1, 1)$,
$(m + 2, m + 1, m + 1, m + 1)$, $(m + 2, m + 1, m + 1, m + 1, 1)$,
$(4, 2, 2, 2, 2)$, or
$(d - 1 , 1,\dots,1)$ where 1 occurs c times and $4 \leq d \leq c \leq d + 1$.

PROOF. We first check that $Z(\delta - 1)$ is not effective for each of the given 0-cycles Z . If $Z = 0$ this is clear, so assume $Z \neq 0$. If $Z = (m)$, then $Z(\delta) = m(E_0 - E_1)$ by (I.5.3)(2). But then $Z(\delta - 1)\cdot Z(\delta) < 0$ and since $Z(\delta)$ is numerically effective, $Z(\delta - 1)$ is not effective. Indeed, in each case $Z(\delta)\cdot Z(\delta - 1) < 0$, and the same reasoning applies.

Now we show that this list is complete by showing if $Z(\delta)$ is a numerically effective divisor class such that $Z(\delta - 1)$ is not effective then the 0-cycle Z to which it corresponds is on the list. Suppose $Z(\delta)$ is one of the generators of the numerically effective cone (vid. (I.5.3)(3)), and that $Z(\delta - 1)$ is not effective. Then $Z(\delta)$ must be (up to reindexing) either 0, $E_0 - E_1$, $2E_0 - E_1 - E_2 - E_3$, or $dE_0 - (d - 1)E_1 - E_2 - \dots - E_{d+2}$, $d \geq 2$; otherwise $Z(\delta - 1)$ is effective. We note that the corresponding 0-cycle Z for each of these classes is on the list.

Now if F is a numerically effective divisor such that $F - E_0$ is not effective and if F is the sum $G + H$ of numerically effective divisors, then also $G - E_0$ and $H - E_0$ are not effective. Thus it is enough now to show for each class $Z(\delta)$ corresponding to a 0-cycle Z on the list, and for each generator F of the numerically effective cone, either $Z(\delta) + F$ is on the list or $Z(\delta - 1) + F$ is effective. This is straightforward, albeit tedious, and the details are left to the reader.

We now prove (II.4.1):

PROOF of (II.4.1). (1) If $Z(\delta)\cdot Q = 0$ then by (I.5.3) we see that $\delta \geqslant 2$, unless $Z = 0$. In the latter case $\delta = 0$ while $\tau = 0$ by definition. If Z is not zero, then $h^2(X, Z(\delta - 1) - Q) = 0$ by (I.3.1) and $h^1(Q, Z(\delta - 1) \otimes \mathcal{O}_Q) > 0$ since $Z(\delta - 1)\cdot Q = -2$. Now by taking cohomology of:

$$0 \longrightarrow Z(\delta - 1) - Q \longrightarrow Z(\delta - 1) \longrightarrow Z(\delta - 1) \otimes \mathcal{O}_Q \longrightarrow 0$$

we see $h^1(X, Z(\delta - 1)) > 0$ and now $\delta = \tau$ follows from (I.6.3).

If $Z(\delta)\cdot Q > 0$, then it follows from (I.5.3)(3) that $Z(\delta) = G + H$, where H is numerically effective and G is, up to indexing, either E_0 $2E_0 - E_1 - E_2 - E_3$ or $E_0 - E_1$. By (I.2.1) and (I.3.1) we see that $h^1(X, G - E_0) = 0$. Now we show by induction that $h^1(X, N + G - E_0) = 0$ for any numerically effective divisor N. Indeed, we only need to show that if $h^1(X, N + G - E_0) = 0$ then $h^1(X, F + N + G - E_0) = 0$, where F is a generator of the numerically effective cone (vid. (I.5.3)(3)). By (I.5.4), F has an irreducible section C. Consider the exact sequence:

$$0 \longrightarrow N + G - E_0 \longrightarrow F + N + G - E_0 \longrightarrow (F + N + G - E_0) \otimes \mathcal{O}_C \longrightarrow 0$$

It follows from (I.5.3)(3) by examining $(F + N + G - E_0)\cdot C$ that $h^1(C, (F + N + G - E_0) \otimes \mathcal{O}_C) = 0$, so taking cohomology of the sequence we see $h^1(X, F + N + G - E_0) = 0$ if $h^1(X, N + G - E_0) = 0$. By induction we now see $h^1(X, Z(\delta - 1)) = 0$ and therefore $\tau = \delta - 1$.

(2) By (II.1.2) and (1) above, Z is a generic 0-cycle if and only if either $Z(\delta)\cdot Q = 0$ and $Z(\delta - 1)$ is not effective or $Z(\delta)\cdot Q > 0$ and $Z(\delta - 2)$ is not effective. In particular, the 0-cycles of (II.4.2) are generic, since $Z(\delta - 2)$ is not effective if $Z(\delta - 1)$ is not. Therefore, we now determine when Z is generic, assuming that $Z(\delta - 1)$ is effective and $Z(\delta)\cdot Q > 0$.

Since $Z(\delta - 1)$ is effective, we can write $Z(\delta - 1) = N + F$ where F is the sum of fixed components of the linear system $|Z(\delta - 1)|$ and N is therefore numerically effective. Since Z generic implies $Z(\delta - 2)$ is not effective we also have $N - E_0$ is not effective. Such numerically effective divisors N are classified in (II.4.2).

Now consider F. It is a nonnegative sum of irreducible exceptional classes $E_0 - E_i - E_j = E_{ij}$, $0 \neq i \neq j \neq 0$, E_i, $i \geqslant 1$, and Q. But if E and E' are distinct exceptional classes that actually occur as summands of F, then $E\cdot E' = 0$. For if $E\cdot E' > 0$ then $E + E'$ is numerically effective and hence not fixed. Likewise, if $E\cdot N > 0$ then $E + N$ is numerically effective and so E is not a fixed component. Therefore, we also have $N\cdot E = 0$, but since $Z(\delta) = N + F + E_0$ is numerically effective, we see, moreover, for any fixed component E_i of $Z(\delta - 1)$ that $0 \leq E_i \cdot Z(\delta) = E_i \cdot F = E_i \cdot Z(\delta - 1)$, while for any fixed component E_{ij} of $Z(\delta - 1)$ we have $E_{ij} \cdot Z(\delta) = E_{ij} \cdot Z(\delta - 1) + 1 \geq 0$. Thus any fixed component E_{ij} of $Z(\delta - 1)$ occurs with multiplicity one and, moreover, $Z(\delta - 1) \cdot E_{ij} = -1$.

Therefore we may write $Z(\delta - 1)$ as $N + F' + Q'$ where F' is a sum of distinct and mutually orthogonal classes E_{ij}, where Q' is the sum of fixed components Q and E_i, $i \geq 1$, and $F' \cdot N = F' \cdot Q = 0$.

A sum F' of mutually orthogonal and distinct classes E_{ij} must be, up to reindexing, either E_{ij} alone, or $3E_0 - 2E_1 - 2E_2 - 2E_3$ or $dE_0 - dE_1 - E_2 - \dots - E_{d+1}$. For if $E_{ij} \cdot E_{rs} = 0$ then E_{ij} and E_{rs} have precisely one index in common. If all the summands of F' have the same index in common the result is $dE_0 - dE_1 - E_2 \dots - E_{d+1}$ while if no index is simultaneously common to all of the summands there can be at most three distinct summands and $F' = 3E_0 - 2(E_1 + E_2 + E_3)$.

We now examine Q'. If E_i is a fixed component of $Z(\delta - 1) = N + F' + Q'$ then we already noted that $E_i \cdot N = E_i \cdot F' = 0$ and $E_i \cdot Z(\delta - 1) \geq 0$. Therefore, $E_i \cdot Q' \geq 0$, and so after reindexing we may write Q' as a nonnegative sum of classes Q_j where $j \geq 5$ and for each Q_j actually occurring in this sum we have $j \geq \ell$ whenever $E_\ell \cdot (N + F') > 0$, where $Q_j = 2E_0 - E_1 - \dots - E_j$.

Since we only blow-up points of Z having positive coefficients, if $Z(\delta - 1)$ has any fixed component Q_j, one of these must be Q itself, and if this is the case then $Q \cdot Z(\delta - 1) = -1$. For $Q \cdot Z(\delta) > 0$ by assumption so $Q \cdot Z(\delta - 1) \geq -1$. But $Q \cdot F' = 0$, so $Q \cdot Z(\delta - 1) = Q \cdot (N + Q')$. But as we saw above, $E \cdot (N + Q') \geq 0$ for any exceptional class E_{ij} or E_i. If $Q \cdot (N + Q') \geq 0$ then $N + Q'$ is numerically effective and hence has no fixed components. In particular, this would mean $Q' = 0$, and so Q would not be a fixed component. Thus $Q \cdot (N + Q') = -1$ so $(N + Q' - Q) \cdot Q \geq 0$ and therefore, $N + Q' - Q$ is numerically effective and in particular has no fixed components. Thus we see that either $Q' = Q$ or $Q' = 0$.

Therefore, we conclude that if Z is generic then $Z(\delta - 1) = N + F' + Q'$ where N is a numerically effective class such that $N - E_0$ is not effective (these are classified in (II.4.2)); F' is up to reindexing one of the classes 0, $3E_0 - 2E_1 - 2E_2 - 3E_3$, or $dE_0 - dE_1 - E_2 - \dots - E_{d+1}$, $d \geq 1$; and Q' is either 0 or Q. Moreover, we also have $N \cdot F' = 0$ and, in case $Q' = Q$, $(N + Q) \cdot Q = -1$. Finally, since for any such class, $N + F' + Q' - E_0$ is not effective, while a direct calculation shows $h^1(X, N + F' + Q') = 0$, we see that $N + F' + Q'$ does arise from a generic 0-cycle Z.

II.4.3. For the reader's convenience we list here each 0-cycle having a generic hilbert function. Our notation is explained by example: $(3_2, 2, 1)$ means the 0-cycle having coefficient vector $(3, 3, 2, 1)$. The list follows:

$0, (1_5), (2_3), (3_3, 1_2)$;
$(m, 1_n), (m + 1, 2_n, 1_i), 0 \leq n \leq m, 0 \leq i = 4 + m - 2n$;
$(1_3), (2_2, 1), (3, 2_2), (3_3)$,
$(2_3, 1_3), (3_2, 2, 1_3), (4, 3_2, 1_3), (4_3, 1_3)$;
$(m_4), ((m + 1)_2, m_2), ((m + 2), (m + 1)_2, m)$,
$((m + 3), (m + 1)_3), ((m + 2)_3, m)$,
$((m + 1)_4, 1), ((m + 2)_2, (m + 1)_2, 1), ((m + 3), (m + 2)_2, m + 1, 1)$,
$((m + 4), (m + 2)_3, 1), ((m + 3)_3, m + 1, 1), m \geq 1$;
$((m + 2)_3, m + 1), ((m + 3)_2, m + 2, m + 1)$,
$(m + 4, (m + 3)_2, m + 1), ((m + 4)_3, m + 1)$,
$((m + 3)_3, m + 2, 1_2), ((m + 4)_2, m + 3, m + 2, 1_2)$,
$(m + 5, (m + 4)_2, m + 2, 1_2), ((m + 5)_3, m + 2, 1_2), m \geq 0$;
$((m + 2)_3, m + 1, 1), ((m + 3)_2, m + 2, m + 1, 1)$,
$(m + 4, (m + 3)_2, m + 1, 1), ((m + 4)_3, m +1, 1)$,
$((m + 3)_3, m + 2, 2), ((m + 4)_2, m + 3, m + 2, 2)$,
$(m + 5, (m + 4)_2, m + 2, 2), ((m + 5)_3, m + 2, 2), m \geq 0$;
$(m + 2, (m + 1)_3), (m + 3, m + 2, (m + 1)_2), (m + 4, (m + 2)_2, m + 1)$,
$(m + 5, (m + 2)_3), (m + 3, (m + 2)_3, 1_2), (m + 4, m + 3, (m+2)_2, 1_2)$,
$(m + 5, (m + 3)_2, m + 2, 1_2), (m + 6, (m + 3)_3, 1_2), m \geq 0$;
$(m+2, (m+1)_3, 1), (m+3, m+2, (m+1)_2, 1), (m+4, (m+2)_2, m+1, 1)$,
$(m+5, (m+2)_3, 1), (m+3, (m+2)_3\ 2), (m+4, m+3, (m+2)_2, 2)$,
$(m + 5, (m + 3)_2, m + 2, 2), (m + 6, (m + 3)_3, 2), m \geq 0$;
$(6, 2_4), (7, 3_4)$;
$(4, 2_4), (5, 3, 2_3), (6, 3_2, 2_2), (7, 3_3, 2), (8, 3_4)$,
$(5, 3_4), (6, 4, 3_3), (7, 4_2, 3_2), (8, 4_3, 3)\ (9, 4_4)$;
$(4, 2_4, 1), (5, 3, 2_3, 1), (6, 3_2, 2_2, 1), (7, 3_3, 2, 1)$,
$(8, 3_4, 1), (5, 2_5), (6, 3, 2_4), (7, 3_2, 2_3)$,
$(8, 3_3, 2_2), (9, 3_4, 2), (10, 3_5)$;
$(d + m - 1, 2_m, 1_{d-m}), d \geq 4 , d \geq m \geq 0$;
$(d + m - 1, 2_m, 1_{d+1-m}), d \geq 4 , d + 1 \geq m \geq 0$.

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Brian Harbourne
Department of Mathematics
University of Texas at Austin
Austin, Texas 78712 U.S.A.