Linear systems with multiple base points in $\mathbf{P}^2$

Brian Harbourne

Department of Mathematics and Statistics
University of Nebraska-Lincoln
Lincoln, NE 68588-0323
email: bharbour@math.unl.edu
WEB: http://www.math.unl.edu/∼bharbour/

Joaquim Roé

Departament d'Àlgebra i Geometria
Universitat de Barcelona
Barcelona 08007, Spain
email: jroevell@mat.ub.es

January 30, 2003

Abstract: Conjectures for the Hilbert function $h(n;m)$ and minimal free resolution of the mth symbolic power $I(n;m)$ of the ideal of n general points of $\mathbf{P}^2$ are verified for a broad range of values of m and n where both m and n can be large, including (in the case of the Hilbert function) for infinitely many m for each square $n > 9$ and (in the case of resolutions) for infinitely many m for each even square $n > 9$. All previous results require either that n be small or be a square of a special form, or that m be small compared to n. Our results are based on a new approach for bounding the least degree among curves passing through n general points of $\mathbf{P}^2$ with given minimum multiplicities at each point and for bounding the regularity of the linear system of all such curves. For simplicity, we work over the complex numbers.

I. Introduction

Consider the ideal $I(n;m) \subset R = \mathbf{C}[\mathbf{P}^2]$ generated by all forms having multiplicity at least m at n given general points of $\mathbf{P}^2$. This is a graded ideal, and thus we can consider the Hilbert function $h(n;m)$ whose value at each nonnegative integer t is the dimension $h(n;m)(t) = \dim I(n;m)_t$ of the homogeneous component $I(n;m)_t$ of $I(n;m)$ of degree t. It is well known that $h(n;m)(t) \geq \max(0, \binom{t+2}{2} - n\binom{m+1}{2})$, with equality for t sufficiently large. Denote by $\alpha(n;m)$ the least degree t such that $h(n;m)(t) > 0$ and by $\tau(n;m)$ the least degree t such that $h(n;m)(t) = \binom{t+2}{2} - n\binom{m+1}{2}$; we refer to $\tau(n;m)$ as the *regularity* of $I(n;m)$.

For $n \leq 9$, the Hilbert function [N2] and minimal free resolution [H2] of $I(n;m)$ are known. For $n > 9$, there are in general only conjectures:

Conjecture I.1: *Let $n \geq 10$ and $m \geq 0$; then:*

(a) $\alpha(n;m) \geq m\sqrt{n}$;

(b) $h(n;m)(t) = \max(0, \binom{t+2}{2} - n\binom{m+1}{2})$ *for each integer $t \geq 0$; and*

(c) *the minimal free resolution of* $I(n;m)$ *is an exact sequence*

$$0 \to R[-\alpha-2]^d \oplus R[-\alpha-1]^c \to R[-\alpha-1]^b \oplus R[-\alpha]^a \to I(n;m) \to 0,$$

where $\alpha = \alpha(n;m)$, $a = h(n;m)(\alpha)$, $b = \max(h(n;m)(\alpha+1) - 3h(n;m)(\alpha), 0)$, $c = \max(-h(n;m)(\alpha+1) + 3h(n;m)(\alpha), 0)$, $d = a+b-c-1$, *and* $R[i]^j$ *is the direct sum of* j *copies of the ring* $R = \mathbf{C}[\mathbf{P}^2]$, *regarded as an* R*-module with the grading* $R[i]_k = R_{k+i}$.

Note that Conjecture I.1(c) implies Conjecture I.1(b) which implies Conjecture I.1(a). Conjecture I.1(a) was posed in [N1] in the form "$\alpha(n;m) > m\sqrt{n}$ for $n \geq 10$ and $m > 0$", together with a proof in case $n > 9$ is a square. Conjecture I.1(c) was posed in [H2], together with a determination of the resolution for $n \leq 9$. Conjecture I.1(b) is a special case of more general conjectures posed in different but equivalent forms by a number of people. In particular, given general points $p_1, \ldots, p_n \in \mathbf{P}^2$, let $\mathbf{m} = (m_1, \ldots, m_n)$ be any sequence of nonnegative integers, and define $I(\mathbf{m})$ to be the ideal generated by all forms having multiplicity at least m_i at p_i. We can in the obvious and analogous way define $\alpha(\mathbf{m})$, $\tau(\mathbf{m})$ and $h(\mathbf{m})$. Equivalent conjectures for $h(\mathbf{m})$ have been posed in [H1], [Gi] and [Hi1]. Ciliberto and Miranda have recently pointed out that these conjectures are also equivalent to what seemed to be a weaker conjecture posed in [S]; see [CM3], or [H4]. However, no general conjecture for the minimal free resolution of $I(\mathbf{m})$ has yet been posed when $\mathbf{m}$ is arbitrary.

These conjectures for $h(\mathbf{m})$ have been verified in certain special cases: for $n \leq 9$ by [N2]; for any n as long as $m_i \leq 4$ for all i by [Mg]; and by [AH] for any m_i as long as the maximum of the m_i is sufficiently small compared to the number of points for which $m_i > 0$. In addition, Conjecture I.1(b) was shown to be true by [Hi2] for $m \leq 3$, and by [CM2] for $m \leq 12$. (After our paper was submitted for publication, this was extended to $m \leq 20$ by [CCMO].) The only result before ours with $n \geq 10$ and arbitrarily large multiplicities was that of [E2], which verifies Conjecture I.1(b) for all m as long as n is a power of 4. (This has now been extended to include all n which are products of powers of 4 and 9; see [BZ].) Similarly, [E1] verifies Conjecture I.1(a) as long as m is no more than about $\sqrt{n}/2$.

Results for resolutions are more limited. A complete solution for the resolution of $I(\mathbf{m})$ for arbitrary $\mathbf{m}$ was given in [C2] as long as $n \leq 5$. This has been extended to $n \leq 8$ by [FHH]. The resolution of $I(n;m)$ was found for $n \leq 9$ for any m by [H2]. Also, [GGR] shows that Conjecture I.1(c) holds for $m = 1$. This was extended to $m = 2$ by [I] and to $m = 3$ by [GI]. In addition, [HHF] applies the result of [E2] to verify Conjecture I.1(c) when n is a power of 4, as long as m is not too small.

In this paper we obtain substantial improvements on these prior results for all three parts of Conjecture I.1. For example, we have:

Corollary I.2: *Let* $n \geq 10$. *Then:*

(a) *Conjecture I.1(a) holds as long as* $m \leq (n - 5\sqrt{n})/2$;

(b) *Conjecture I.1(b) holds for infinitely many* m *for each square* $n \geq 10$; *and*

(c) *Conjecture I.1(c) holds for infinitely many* m *for each even square* $n \geq 10$.

We also verify Conjecture I.1(a,b,c) for many other values of m and n (see Corollary IV.1, Corollary V.1 and Corollary VI.1). Although the nature of our approach makes it difficult to give a simple description of all m and n which we can handle, see Figures 1–4 for graphical representations of some of our results.

Our approach combines and extends the methods of [H3], [HHF], [R1], and [R2], to obtain improved bounds on $\alpha(\mathbf{m})$ and $\tau(\mathbf{m})$. Sufficiently good bounds determine these quantities exactly, which in many situations is sufficient to also determine $h(\mathbf{m})$ and the minimal free resolution of $I(\mathbf{m})$. Our bounds are algorithmic (for an example of the algorithm applied in a specific case, see Example II.4). The input to the algorithm is the sequence $\mathbf{m} = (m_1, \ldots, m_n)$ of multiplicities of the n general points. One also must pick a positive integer d and a positive integer $r \leq n$, corresponding to a specialization of the n points in which the first point is a general smooth point of an irreducible plane curve of degree d, each additional point is infinitely near the previous one, and exactly r of them lie on the curve. Using properties of linear series on curves, we then obtain bounds for such sets of specialized points; by semicontinuity these bounds also apply in the case of general points. For fixed $\mathbf{m}$, note that different choices of d and r can lead to different bounds, as is shown particularly clearly by Figures 3 and 4 at the end of this paper.

An analysis of our algorithm leads to explicit formulas for these bounds in certain cases. To write down these formulas, given $d > 0$, $0 < r \leq n$ and $\mathbf{m} = (m_1, \ldots, m_n)$, define integers u and ρ via $M_n = ur + \rho$, where $u \geq 0$, $0 < \rho \leq r$ and $M_i = m_1 + \cdots + m_i$ for each $1 \leq i \leq n$. Note that we can equivalently define $u = \lceil M_n/r \rceil - 1$ and $\rho = M_n - ru$. Also, given an integer $0 < r \leq n$, we say that $(m_1, \ldots, m_n)$ is *r-semiuniform* if $m_r + 1 \geq m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$. Note that a nonincreasing sequence $(m_1, \ldots, m_n)$ of nonnegative integers is r-semiuniform if and only if m_i is either m_r or $m_r + 1$ for every $1 \leq i \leq r$; thus, for example, $(6, 6, 6, 5, 5, 5, 5, 5, 4, 4, 4)$ is r-semiuniform for each r up to 7, but not for $r = 8, 9$ or 10. We now have:

Theorem I.3: *Given integers $0 < d$ and $0 < r \leq n$, let $\mathbf{m} = (m_1, \ldots, m_n)$ be r-semiuniform, define M_i, u and ρ as above, denote the genus $(d-1)(d-2)/2$ of a plane curve of degree d by g and let s be the largest integer such that we have both $(s+1)(s+2) \leq 2\rho$ and $0 \leq s < d$.*

- *(a) If $r \leq d^2$, then*
  - *(i) $\alpha(\mathbf{m}) \geq 1 + min(\lfloor (M_r + g - 1)/d \rfloor, s + ud)$ whenever $d(d+1)/2 \leq r$, while*
  - *(ii) $\tau(\mathbf{m}) \leq max(\lceil (\rho + g - 1)/d \rceil + ud, ud + d - 2)$.*
- *(b) If $2r \geq n + d^2$, then*
  - *(i) $\alpha(\mathbf{m}) \geq s + ud + 1$, and*
  - *(ii) $\tau(\mathbf{m}) \leq max(\lceil (M_r + g - 1)/d \rceil, ud + d - 2)$.*
- *(c) Say for some m we have $m_i = m$ for all i, and that $rd(d+1)/2 \leq r^2 \leq d^2 n$; then $\alpha(n; m) \geq 1 + \min(\lfloor (mr + g - 1)/d \rfloor, s + ud)$.*

In Section II, we develop the results needed to state and analyze our algorithm. In Section III we prove Theorem I.3. In Section IV we apply Theorem I.3 to prove results less easily stated but stronger than Corollary I.2(a), from which Corollary I.2(a) is an easy consequence. Similarly, Corollary I.2(b) is an immediate consequence of more comprehensive results that we deduce from Theorem I.3 in Section V, and Corollary I.2(c) is an immediate consequence of more comprehensive results that we deduce in Section VI from Theorem I.3 using [HHF].

Since it is hard to easily describe all of the cases that our method handles, we include for this purpose some graphs in Section VII, together with some explicit comparisons of our bounds on α and τ with previously known bounds.

II. Algorithms

In this section we derive algorithms giving bounds on $\alpha(\mathbf{m})$ and $\tau(\mathbf{m})$. It is of most interest to give lower bounds for α and upper bounds for τ, since upper bounds for α and lower bounds for τ are known which are conjectured to be sharp. (See [H4] for a discussion.)

Our method involves a specialization of the n points as in [H3] (which in turn was originally inspired by that of [R1]), together with properties of linear series on curves. Recall that a flex for a linear series V of dimension a on a curve C is a point $p \in C$ such that $V - (a+1)p$ is not empty. In Lemma II.1 we use the known result (see p. 235, [Mr]) that the set of flexes of a linear series is finite; it is the only place that we need the characteristic to be zero (although, of course, everything else refers to this Lemma). In positive characteristics, complete linear series can indeed have infinitely many flexes [Ho].

Before deriving our algorithms, we need two lemmas.

Lemma II.1: *Let C be an irreducible plane curve of degree d, so $g = (d-1)(d-2)/2$ is the genus of C, and let q be a general point of C. Take $D = tL_C - vq$, where $t \geq 0$ and $v \geq 0$ are integers and L_C is the restriction to C of a line L in $\mathbf{P}^2$.*

(1) If $t \geq d-2$ then $h^0(C,D) = 0$ for $t \leq (v+g-1)/d$ and $h^1(C,D) = 0$ for $t \geq (v+g-1)/d$.

(2) If $t < d$ then $h^0(C,D) = 0$ for $(t+1)(t+2) \leq 2v$.

Proof: We have $L, C \subset \mathbf{P}^2$. The linear system $|tL_C|$ is the image of $|tL|$ under restriction to C. Note that $h^1(C,tL_C) = 0$ if (and only if) $t \geq d-2$, in which case $h^0(C,tL_C) = td-g+1$, whereas, for $t < d$, $h^0(C,tL_C) = h^0(\mathbf{P}^2,tL) = (t+1)(t+2)/2$.

Since $h^1(C,tL_C) = 0$ for $t \geq d-2$, as long as vq imposes independent conditions on $|tL_C|$ (i.e., $h^0(C,tL_C - vq) = h^0(C,tL_C) - v$), then $h^1(C,tL_C - vq) = 0$ too. Also, if we show that $h^1(C,tL_C - vq) = 0$, then it is easy to see that $h^1(C,tL_C - v'q) = 0$ for all $v' \leq v$, and if we show that $h^0(C,tL_C - vq) = 0$, then it is easy to see that $h^0(C,tL_C - v'q) = 0$ for all $v' \geq v$. So it is enough to show $h^0(C,tL_C - vq) = 0$ for $v = h^0(C,tL_C)$. Since q is general in C, we can assume it is not a flex of $|tL_C|$, and therefore the claim follows. $\diamondsuit$

Consider n distinct points $p_1, \ldots, p_n$ of $\mathbf{P}^2$ and let X be the blow up of the points. More generally, we can allow the possibility that some of the points are infinitely near by taking $p_1 \in \mathbf{P}^2 = X_0$, $p_2 \in X_1$, ..., $p_n \in X_{n-1}$, where X_i, for $0 < i \leq n$, is the blow up of X_{i-1} at p_i, and we take $X = X_n$. Given integers t and $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$, we denote by F_t the divisor $F_t = tL - m_1E_1 - \cdots - m_nE_n$ on X, where E_i is the divisorial inverse image of p_i under the blow up morphisms $X = X_n \to X_{n-1} \to \cdots \to X_{i-1}$, and L is the pullback to X of a general line in $\mathbf{P}^2$. Note that the divisor classes $[L], [E_1], \ldots, [E_n]$ give a basis for the divisor class group $\mathrm{Cl}(X)$ of X.

Now, given positive integers $r \le n$ and d, we choose our points $p_1, p_2, \dots, p_n$ such that p_1 is a general smooth point of an irreducible plane curve C' of degree d, and then choose points $p_2, \dots, p_n$ so that p_i is infinitely near p_{i-1} for $i \le n$ and so that p_i is a point of the proper transform of C' on X_{i-1} for $i \le r$ (more precisely, so that $[E_i - E_{i+1}]$ is the class of an effective, reduced and irreducible divisor for $0 < i < n$ and so that the class of the proper transform of C' to X is $[dL - E_1 - \cdots - E_r]$). Let C denote the proper transform of C' in X.

Define divisors D_j and D'_j such that $D_0 = F_t$, $D'_j = D_j - (dL - E_1 - \cdots - E_r)$, and such that D_{j+1} is obtained from D'_j by *unloading* multiplicities (i.e., if $D'_j = a_0 L - a_1 E_1 - \cdots - a_n E_n$, then we permute the $a_1, a_2, \dots, a_n$ so that $a_1 \ge a_2 \ge \cdots \ge a_n$ and set to 0 each which is negative).

Part (1) of the following lemma is used in our algorithms. Parts (2) and (3) are used in the proof of Theorem I.3(a,b).

Lemma II.2: *Let $r \le n$ and d be positive integers. Given $F_t = tL - m_1 E_1 - \cdots - m_n E_n$ with $m_1 \ge m_2 \ge \cdots \ge m_n \ge 0$, define C, D_j and D'_j for $j \ge 0$ as above.*

(1) For every $j > 0$, $h^i(X, D'_{j-1}) = h^i(X, D_j)$, $i = 0, 1$.

(2) If $r \le d^2$, then $D_j \cdot C \le D_{j-1} \cdot C$ for every $j > 0$.

(3) If $2r \ge n + d^2$, then $D_j \cdot C \ge D_{j-1} \cdot C$ as long as $D_{j-1} \cdot E_r > 0$.

Proof: Write $D'_{j-1} = a_0 L - a_1 E_1 - \cdots - a_n E_n$. Because of the definition of F_t and D_i, we have either $a_1 \ge a_2 \ge \cdots \ge a_n \ge -1$, or $a_1 \ge a_2 \ge \cdots \ge a_i = a_{i+1} - 1$ and $a_{i+1} \ge \cdots \ge a_n \ge -1$ for some $i \le r$. Therefore, the unloading procedure leading from D'_{j-1} to D_j consists in a number of unloading steps, each of which either transposes a_k and a_{k+1} (whenever $a_k = a_{k+1} - 1$) or sets a_n to 0 (whenever in the course of the transpositions we find $a_n = -1$). If we denote the proper transform of the exceptional divisor of blowing up p_k by $\tilde{E}_k$ (hence $[\tilde{E}_k] = [E_k - E_{k+1}]$ for $k < n$ and $\tilde{E}_n = E_n$), this is the same as iteratively subtracting $\tilde{E}_k$ from D'_{j-1} whenever $D'_{j-1} \cdot \tilde{E}_k = -1$, and it is enough to show that doing so does not affect cohomology in order to prove (1).

Consider the exact sequence $0 \to \mathcal{O}_X(D'_{j-1} - \tilde{E}_k) \to \mathcal{O}_X(D'_{j-1}) \to \mathcal{O}_X(D'_{j-1}) \otimes \mathcal{O}_{\tilde{E}_k} \to 0$. Since $\tilde{E}_k \cong \mathbf{P}^1$ and $D'_{j-1} \cdot \tilde{E}_k = -1$, we have $h^i(\mathcal{O}_X(D'_{j-1}) \otimes \mathcal{O}_{\tilde{E}_k}) = 0$. Thus, taking cohomology of the sequence, we see that the cohomology of $\mathcal{O}_X(D'_{j-1} - \tilde{E}_k)$ and $\mathcal{O}_X(D'_{j-1})$ coincide, as claimed.

To prove (2), let $D'_{j-1} = a_0 L - a_1 E_1 - \cdots - a_n E_n$ as above and observe that $D'_{j-1} \cdot C = D_{j-1} \cdot C - d^2 + r$. Passing from D'_{j-1} to D_j by unloading might increase some of the coefficients a_i with $0 < i \le r$ but cannot decrease any of these coefficients and hence cannot increase the intersection with C; i.e., $D_j \cdot C \le D'_{j-1} \cdot C = D_{j-1} \cdot C - d^2 + r \le D_{j-1} \cdot C$.

To prove (3), let $D'_{j-1} = a_0 L - a_1 E_1 - \cdots - a_n E_n$ as above. If $D_{j-1} \cdot E_r > 0$, then $a_i \ge 0$ for all $i > 0$ and $a_i + 1 \ge a_j$ for all $0 < i \le r < j \le n$. Thus passing from D'_{j-1} to D_j may involve swapping some of the coefficients a_i, $0 < i \le r$, with some of the coefficients a_j, $j > r$, but there are only $n - r$ coefficients a_j with $j > r$, each of which is at most 1 bigger than the least coefficient a_i with $0 < i \le r$, so passing from D'_{j-1} to D_j can decrease the intersection with C by at most $n - r$; i.e., $D_j \cdot C \ge D'_{j-1} \cdot C - (n - r) = D_{j-1} \cdot C - d^2 + r - (n - r) \ge D_{j-1} \cdot C$. $\diamondsuit$

For Theorem I.3(c) we need a slightly more general version of the preceding lemma, restricted however to the case of uniform multiplicities:

Lemma II.3: *Let $r \leq n$ and d be positive integers. Let $F_t = tL - mE_1 - \cdots - mE_n$, with respect to which we take C, D_i and D_i' for all $i \geq 0$ as in Lemma II.2, and let ω' be the least i such that $D_i \cdot E_i = 0$ for all $i > 0$. Then for all $0 \leq i \leq \omega' - 1$ we have*

$$i\left(\frac{r^2}{n} - d^2\right) - \left(r - \frac{r^2}{n}\right) \leq D_i \cdot C - D_0 \cdot C \leq i\left(\frac{r^2}{n} - d^2\right).$$

Proof: Let $A_0 = 0$, and for $0 < k \leq n$ let $A_k = -E_1 - \cdots - E_k$ and define t_i to be $D_i \cdot L$. For $0 \leq i < \omega'$, it is not hard to check that $D_i = (t_0 - id)L - (m - i + q)E_1 - \cdots - (m - i + q)E_n + A_\rho$, where $i(n-r) = qn + \rho$ with $0 \leq \rho < n$, and therefore

$$D_i \cdot C - D_0 \cdot C = i(r - d^2) - rq + A_\rho \cdot C.$$

On the other hand, $A_\rho \cdot C = -\min(\rho, r)$, and it is easy to see that

$$i(n-r)\frac{r}{n} \leq rq + \min(\rho, r) \leq i(n-r)\frac{r}{n} + r - \frac{r^2}{n},$$

from which the claim follows. $\diamondsuit$

We now derive our algorithms. The reader may find the algorithm easier to follow by looking at the example we give below. Let X be obtained by blowing up n general points of $\mathbf{P}^2$, let $F_t = tL - m_1E_1 - \cdots - m_nE_n$, where we assume that $m_1 \geq \cdots \geq m_n \geq 0$, and choose any integers d and r such that $d > 0$ and $0 < r \leq n$. Next, specialize the points as in Lemma II.2. We then have the specialized surface X'. It is convenient to denote the basis of the divisor class group of X' corresponding to the specialized points also by L, $E_1, \ldots, E_n$, since it will always be clear whether we are working on X' or on X. By semicontinuity, we know $h^i(X', F_t) \geq h^i(X, F_t)$, so for any t with $h^i(X', F_t) = 0$ we also have $h^i(X, F_t) = 0$.

As in Lemma II.2, we have on X' the curve C with class $[dL - E_1 - \cdots - E_r]$ and genus $g = (d-1)(d-2)/2$ and we have the sequence of divisors D_j, where $D_0 = F_t$, $D_j' = D_j - (dL - E_1 - \cdots - E_r)$, and D_{j+1} is obtained from D_j' as above.

Whenever $h^0(X', F_t) = 0$, we get the bound $t + 1 \leq \alpha(m_1, \ldots, m_n)$, and whenever $h^1(X', F_t) = 0$, we get the bound $t \geq \tau(m_1, \ldots, m_n)$. But, using the exact sequences

$$0 \to \mathcal{O}_{X'}(D_j') \to \mathcal{O}_{X'}(D_j) \to \mathcal{O}_{X'}(D_j) \otimes \mathcal{O}_C \to 0 \qquad (*)$$

and Lemma II.2(1), we see that $h^0(X', F_t) = 0$ if for some $i = I$ we have $h^0(X', D_I) = 0$ and for all $0 \leq i < I$ we have $h^0(C, \mathcal{O}_{X'}(D_i) \otimes \mathcal{O}_C) = 0$. Similarly, $h^1(X', F_t) = 0$ if for some $j = J$ we have $h^1(X', D_J) = 0$ and for all $0 \leq j < J$ we have $h^1(C, \mathcal{O}_{X'}(D_j) \otimes \mathcal{O}_C) = 0$. But, $\mathcal{O}_{X'}(D_j) \otimes \mathcal{O}_C = \mathcal{O}_{X'}((t - jd)L) \otimes \mathcal{O}_C(-v_jq)$, where $v_j = (t - jd)d - D_j \cdot C$ and q is the point of C infinitely near to p_1. Thus we can apply the criteria of Lemma II.1 to control $h^0(C, \mathcal{O}_{X'}(D_i) \otimes \mathcal{O}_C)$ for $0 \leq i < I$ and $h^1(C, \mathcal{O}_{X'}(D_j) \otimes \mathcal{O}_C)$ for $0 \leq j < J$.

In particular, for a given value of t we have $h^0(X', D_I) = 0$ if we take I to be the least i such that $D_i \cdot (L-E_1) < 0$. Then, if t is not too large, for all $0 \le i < I$ we also have either $D_i \cdot C \le g-1$ and $t-id \ge d-2$, or $t-id < 0$, or $t-id < d$ and $(t-id+1)(t-id+2) \le 2v_i$, which guarantees by Lemma II.1 that $h^0(C, D_i) = 0$ for all $0 \le i < I$. The largest such t then gives a lower bound for $-1+\alpha(m_1, \ldots, m_n)$; i.e., $t+1 \le \alpha(m_1, \ldots, m_n)$.

Similarly, let J be the least j such that D_j is a multiple of L (i.e., such that if $D_j = a_0L - a_1E_1 - \cdots - a_nE_n$, then $a_1 = \cdots = a_n = 0$, in which case we have $h^1(X', D_j) = 0$). If we have chosen t sufficiently large, then $D_j \cdot L \ge d-2$ and $D_j \cdot C \ge g-1$, for all $0 \le j < J$; the least such t then gives an upper bound for $\tau(m_1, \ldots, m_n)$.

Example II.4: For this example, suppose we wish to get information on $n = 18$ general points of multiplicity $m = 2$. We first must pick d and r. Any integers $d > 0$ and $0 < r \le n$ will do, but values such that r/d is close to $\sqrt{n}$ tend to be best; here we will choose $d = 4$ and $r = 17$. Now we pick specialized points $p_1, \ldots, p_{18}$ such that p_1 is a general point of a reduced irreducible plane curve C' of degree $d = 4$, and p_2 is the point on C' infinitely near to p_1, and so on for the first $r = 17$ points. Thus p_{17} is the point on C' infinitely near to p_{16}, but each point p_i for $i > 17$ is taken to be a general point infinitely near to p_{i-1} and so in particular p_i is not on C'. For this example, $n = 18$ so there is only one such point, p_{18}.

Now let C be the proper transform of C' to the surface X' obtained by blowing up each point in turn. Thus the class of C is $dL - E_1 - \cdots - E_{17}$, and $F_t = tL - 2E_1 - \cdots - 2E_{18}$. We want to find the largest t such that we can show that $h^0(X', F_t) = 0$. If we can show that $h^0(X', F_t) = 0$, then we try $t+1$, and we continue increasing t until we are unable to show that $h^0(X', F_t) = 0$. For example, say $t = 8$. Then, using the notation above, $D_0 = F_8$ and $D_0' = F_8 - C$, and we use Lemma II.1 to determine whether or not $h^0(C, \mathcal{O}_{X'}(D_0) \otimes \mathcal{O}_C) = 0$. In this case $h^0(C, \mathcal{O}_{X'}(D_0) \otimes \mathcal{O}_C) = 0$ is obvious since $D_0 \cdot C = 8 \cdot 4 - 17 \cdot 2 = -2$. By the sequence $(*)$ of sheaves above, it follows that $h^0(X', D_0) = h^0(X', D_0')$. Next, we *unload* $D_0' = 4L - E_1 - \cdots - E_{17} - 2E_{18}$ to obtain D_1. This just amounts to reordering the coefficients of the E_i to be nondecreasing; thus $D_1 = 4L - 2E_1 - E_2 - \cdots - E_{17} - E_{18}$. (Since p_{18} is infinitely near p_{17}, $N_{17} = E_{17} - E_{18}$ is the class of a reduced irreducible curve. But $D_0' \cdot N_{17} < 0$, so $h^0(X', D_0') = h^0(X', D_0' - N_{17})$, and $(D_0' - N_{17}) \cdot N_{16} < 0$, so $h^0(X', D_0' - N_{17}) = h^0(X', D_0' - N_{17} - N_{16})$. Note that $D_0' - N_{17} = 4L - E_1 - \cdots - E_{16} - 2E_{17} - E_{18}$, and $D_0' - N_{17} - N_{16} = 4L - E_1 - \cdots - E_{15} - 2E_{16} - E_{17} - E_{18}$; unloading consists of continuing to subtract classes $N_i = E_i - E_{i+1}$ until the coefficients of the E_i are nondecreasing. Fully unloading D_0' results in the class D_1, for which we see we have $h^0(X', D_0') = h^0(X', D_1)$.) Now we repeat the process above, applied to D_1 in place of D_0. Again we have $h^0(C, \mathcal{O}_{X'}(D_1) \otimes \mathcal{O}_C) = 0$, so $h^0(X', D_1) = h^0(X', D_1') = h^0(X', D_2)$, but now $D_2 = -E_1 - E_2$, so $h^0(X', D_2) = 0$, hence $h^0(X', F_8) = 0$, so (by semicontinuity) we know $\alpha(18; 2) > 8$. At this point we repeat the whole process with $t = 9$, in hopes of improving our bound. Thus D_0 is now F_9. However, if for some i it ever happens that either $h^0(C, \mathcal{O}_{X'}(D_i) \otimes \mathcal{O}_C) \ne 0$ or $h^0(X', D_i) \ne 0$, then we will be unable to conclude that $h^0(X', F_9) = 0$. In fact, D_2 in this case is $L - E_1 - E_2$, so $h^0(X', D_2) \ne 0$, so we are indeed unable to conclude that $h^0(X', F_9) = 0$. Thus our algorithm cannot improve on the bound $\alpha(18; 2) > 8$ so our bound in the end is $9 \le \alpha(18; 2)$. Our bound on τ works similarly, except we start with t large enough so that h^1 vanishes both for D_j for some j and for the restrictions of D_i to C for all $0 \leq i < j$. The least such t gives us the bound $t \geq \tau(n;m)$. In the case of our example, we find $9 \geq \tau(18;2)$. Thus the 54 conditions imposed by 18 general double points on the 55 dimensional space of forms of degree 9 are independent, so in fact $h^0(X,F_9) = 1$ and $\alpha(18;2) = 9$. Moreover, since $\alpha(18;2) = 9 = \tau(18;2)$ and $h^0(X,F_9) = 1$, we know a minimal set of homogeneous generators of $I(18;2)$ contains a single generator in degree 9 and $h^0(X,F_{10}) - 3h^0(X,F_9) = 12 - 3 = 9$ generators in degree 10. Generators are never needed in degrees greater than $\tau + 1$, so it follows that the resolution of $I(18;2)$ is $0 \to R[-11]^9 \to R[-10]^9 \oplus R[-9] \to I(18;2) \to 0$.

III. Proof of Theorem I.3

In this section we prove the explicit bounds on α and τ claimed in Theorem I.3. The analysis of the algorithm of the previous section, which leads to the proof, goes differently depending on whether r is relatively small (a) or big (b) compared to d; we also include a separate analysis for the case of uniform multiplicities (c).

We begin with (a)(i), so assume $d(d+1)/2 \leq r \leq d^2$ and $\mathbf{m} = (m_1,\ldots,m_n)$. To show $1+t \leq \alpha(\mathbf{m})$ for $t = \min(\lfloor (M_r+g-1)/d \rfloor, s+ud)$, it is by our algorithm in Section II enough (taking $I = u+1$) to show that $D_I \cdot (L - E_1) < 0$, and for all $0 \leq i < I$ that either $D_i \cdot C \leq g-1$ and $t - id \geq d-2$, or $t - id < 0$, or $0 \leq t - id < d$ and $(t-id+1)(t-id+2) \leq 2v_i$, where $v_j = (t-jd)d - D_j \cdot C$, as before.

First, by Lemma II.2(2), $D_0 \cdot C \geq D_1 \cdot C \geq \cdots$, and by hypothesis $t \leq \lfloor (M_r+g-1)/d \rfloor$, so $g-1 \geq td - M_r = D_0 \cdot C \geq D_1 \cdot C \geq \cdots$, as required. Also by hypothesis, we have $t \leq s+ud$. It follows that $D_I \cdot L = t - Id$ and hence that $D_I \cdot (L-E_1) < 0$, as required. Thus it is now enough to check that $(t-id+1)(t-id+2) \leq 2v_i$ for the largest i (call it i'') such that $t - id \geq 0$. If $i'' = I-1$ we have $t - i''d = t - ud \leq s$ by hypothesis and hence $(t-i''d+1)(t-i''d+2) \leq (s+1)(s+2) \leq 2\rho = 2v_{I-1}$ by definition of s. If $i'' < I-1$, we at least have $t - i''d \leq d-1$, so $(t-i''d+1)(t-i''d+2) \leq d(d+1)$. But $v_{I-2} \geq r$ by r-semiuniformity, so $M_r = v_0 \geq v_1 \geq \cdots \geq v_{I-2} \geq r \geq v_{I-1} = \rho > v_I = 0$, hence $2v_{i''} \geq 2r \geq d(d+1) \geq (t-i''d+1)(t-i''d+2)$, as we wanted.

We now prove (a)(ii). As always we have $r \leq n$; in addition we assume $r \leq d^2$. It follows from semiuniformity that D_J is a multiple of L for $J = u+1$, so it suffices to show for $t = \max(\lceil (\rho+g-1)/d \rceil + ud, ud+d-2)$ that $D_j \cdot L \geq d-2$ and $D_j \cdot C \geq g-1$, for all $0 \leq j < J$; it then follows by our algorithm that $t \geq \tau(\mathbf{m})$. But $t \geq ud+d-2$ ensures that $D_j \cdot L \geq d-2$ for all $0 \leq j < J$, and, since $D_0 \cdot C \geq D_1 \cdot C \geq \cdots \geq D_u \cdot C = (t-ud)d - \rho$ by Lemma II.2, $t \geq \lceil (\rho+g-1)/d \rceil + ud$ ensures that $D_j \cdot C \geq g-1$, for all $0 \leq j < J$.

Next, consider (b). Now we assume that $2r \geq n + d^2$ and that $\mathbf{m}$ is r-semiuniform. By semiuniformity we have $D_i \cdot E_r > 0$ for $i < u$. Now by Lemma II.2(3) we have $D_0 \cdot C \leq D_1 \cdot C \leq \cdots \leq D_u \cdot C$.

Starting with (b)(i), let $I = u+1$ and $t = s+ud$. It suffices to check that $D_I \cdot (L-E_1) < 0$, and for all $0 \leq i < I$ that either $D_i \cdot C \leq g-1$ and $t - id \geq d-2$, or $t - id < d$ and $(t-id+1)(t-id+2) \leq 2v_i$. But $D_I \cdot (L-E_1) = t - (u+1)d = s - d < 0$, as required, so now consider D_u. Here we have $D_u = sL - (E_1 + \cdots + E_\rho)$. But by hypothesis $s < d$ and $(s+1)(s+2) \leq 2\rho = 2v_u$, as required. Finally, consider D_i for $0 \leq i < u$. Then $D_i \cdot L = t - id = s + (u-i)d \geq d$, and $sd - \rho = D_u \cdot C \geq D_i \cdot C$, so if we prove that $s \leq (\rho+g-1)/d$ we will have $D_i \cdot C \leq g-1$ as we want. But it is easy to see that

$sd-g+1=(s+1)(s+2)/2-(s-d+1)(s-d+2)/2\leq(s+1)(s+2)/2$, and therefore the hypothesis $2\rho\geq(s+1)(s+2)$ implies $\rho\geq sd-g+1$, so we are done.

Next, we prove (b)(ii). Let $J=u+1$; then $D_J=(t-Jd)L$ is a multiple of L, as by our algorithm we would want. Now assume in addition that $t\geq\max(\lceil(M_r+g-1)/d\rceil, ud+d-2)$. We want to verify that $D_j\cdot L\geq d-2$ and $D_j\cdot C\geq g-1$, for all $0\leq j<J$. First consider $j=0$; since $t\geq(M_r+g-1)/d$ and $t\geq ud+d-2\geq d-2$, we have $dt-M_r=D_0\cdot C\geq g-1$ and $t=D_0\cdot L\geq d-2$. As for $0<j<J$, we have $t-jd\geq(ud+d-2)-(ud)=d-2$ and $D_j\cdot C\geq D_0\cdot C\geq g-1$, which ends the proof of Theorem I.3(b).

Finally, we prove (c). In the notation of Lemma II.3 and its proof, it is easy to check that $\omega'=\lceil mn/r\rceil=u+1$, so if $t\leq s+ud$, it follows that $t_{\omega'}\leq s-d<0$, and thus $\omega'\geq\omega$, where ω is the least i such that $t_i<0$. Since $r^2/n-d^2\leq0$, it follows from Lemma II.3 that $D_i\cdot C\leq D_0\cdot C$ for all $0\leq i\leq\omega-2$. If $t\leq\lfloor(mr+g-1)/d\rfloor$, then $D_i\cdot C\leq D_0\cdot C=td-mr\leq g-1$. To conclude that $\alpha(n;m)\geq t+1$, it is now enough to check that $(t-i'd+1)(t-i'd+2)\leq 2v_{i'}$ for $i'=\omega-1$. If $i'=u$ (i.e., $\omega'=\omega$) we have $t-i'd=t-ud\leq s$ by hypothesis and hence $(t-i'd+1)(t-i'd+2)\leq(s+1)(s+2)\leq2\rho=2v_{i'}$ by definition of s. If $i'<u$ (so $\omega'>\omega$), by definition of i' we at least have $t-i'd\leq d-1$, so $(t-i'd+1)(t-i'd+2)\leq d(d+1)$. But $\omega'>\omega$ implies $v_{i'}\geq r$, and by hypothesis $rd(d+1)/2\leq r^2$ (so $d(d+1)\leq 2r$); therefore $2v_{i'}\geq 2r\geq d(d+1)\geq(t-i'd+1)(t-i'd+2)$ as we wanted. $\diamondsuit$

Remark III.1: Even for uniform multiplicities, sometimes the best bound determined by Theorem I.3 comes from parts (a) or (c), and sometimes it comes from part (b), depending on n and m. Sometimes, of course, one can do better applying our algorithm for values of r and d for which Theorem I.3 does not apply.

For example, let $\alpha_c(n;m)$ denote the conjectural value of $\alpha(n;m)$ and let $\tau_c(n;m)$ denote the conjectural value of $\tau(n;m)$ (i.e., the values of each assuming Conjecture I.1(b) holds). Then $\alpha_c(33;29)=168$; the best bound given by Theorem I.3 is $\alpha(33;29)\geq165$, obtained in part (b) using $r=29$ and $d=5$, or in part (c) using $r=17$ and $d=3$. Applying our algorithm with $r=23$ and $d=4$, however, gives $\alpha(33;29)\geq168$ (and hence $\alpha(33;29)=\alpha_c(33;29)$). On the other hand, $\alpha_c(38;16)=101$ and indeed we obtain $\alpha(38;16)\geq101$ via Theorem I.3(b) using $r=37$ and $d=6$, while the best bound obtainable via Theorem I.3(c) is $\alpha(38;16)\geq98$, gotten using $r=36$ and $d=6$. In contrast, we obtain $\alpha(119;13)\geq146=\alpha_c(119;13)$ via Theorem I.3(c) using $r=109$ and $d=10$, while the best bound obtainable via Theorem I.3(b) is $\alpha(119;13)\geq144$, gotten using $r=100$ and $d=9$.

Similarly, $\tau_c(33;29)=168$; applying our algorithm with $r=23$ and $d=4$, gives $\tau(33;29)\leq169$. The best bound given by Theorem I.3(b) is $\tau(33;29)\leq170$, obtained using $r=29$ and $d=5$, while the best bound given by Theorem I.3(a) is $\tau(33;29)\leq175$, obtained using $r=33$ and $d=6$. On the other hand, $\tau_c(38;16)=101$ and indeed we obtain $\tau(38;16)\leq101$ via Theorem I.3(b) using $r=37$ and $d=6$, while the best bound obtainable via Theorem I.3(c) is $\tau(38;16)\leq103$, gotten using $r=36$ and $d=6$. In contrast, $\tau_c(119;13)=146$, and the best bound obtainable using our algorithm is $\tau(119;13)\leq147$, obtained using $r=119$ and $d=11$ (and hence Theorem I.3(a) applies), while the best bound obtainable via Theorem I.3(b) is $\tau(119;13)\leq148$, gotten using

$r = 111$ and $d = 10$.

IV. Nagata's Conjecture

In this section we prove Corollary I.2(a) as an immediate easy-to-state consequence of our following more involved result. Because Conjecture I.1(a) is known when n is a square, we need not consider that case.

Corollary IV.1: *Given an integer $n > 9$, let $d = \lfloor\sqrt{n}\rfloor$ (hence $d \geq 3$) and $\Delta = n - d^2$. Then $\alpha(n; m) \geq m\sqrt{n}$ holds whenever:*

(a) Δ is odd and $m \leq$ max$(d(d-3), d(d-2)/\Delta)$, or

(b) $\Delta > 0$ is even, and $m \leq$ max$(d(d-3)/2, 2d^2/\Delta)$.

Proof: Both claims follow from Theorem I.3, with $d = \lfloor\sqrt{n}\rfloor$ and appropriate choices of r. Consider part (a). We first prove $\alpha(n; m) \geq m\sqrt{n}$ if Δ is odd and $m \leq d(d-2)/\Delta$. Apply Theorem I.3(a) with $r = d^2$, $u = m$ and $\rho = m\Delta$; it has to be checked that $md + s + 1 \geq m\sqrt{n}$ and $\lfloor(mr + g - 1)/d\rfloor + 1 \geq m\sqrt{n}$. The first inequality is equivalent to $(s+1)^2 + 2(s+1)md \geq m^2\Delta$. If $s = 0$ then $m\Delta < 3$ and the inequality follows from $d \geq 3$, whereas if $s = d-1$ then $d^2 > d(d-2) \geq m\Delta$ says $d^2 + 2md^2 > m^2\Delta$. In all intermediate cases one has $2d(s+1) \geq (s+2)(s+3)$ and $(s+2)(s+3) > 2\rho = 2m\Delta$ which also imply $(s+1)^2 + 2(s+1)md \geq m^2\Delta$ easily. To prove the second inequality it is enough to see that $md + d/2 - 1 \geq m\sqrt{n}$, which is equivalent to $(d/2-1)^2 + md(d-2) \geq m^2\Delta$, and this follows from $m \leq d(d-2)/\Delta$.

We now prove $\alpha(n; m) \geq m\sqrt{n}$ if Δ is odd and $m \leq d(d-3)$. We can write $\Delta = 2t+1$ for some nonnegative integer t, hence $n = d^2+2t+1$. Apply Theorem I.3(c) with $r = d^2+t$ (i.e., $r = \lfloor d\sqrt{n}\rfloor$, hence $rd(d+1)/2 \leq r^2 \leq d^2n$). Note $\lfloor(mr + g - 1)/d\rfloor + 1 > (mr + g - 1)/d = m(d^2+t)/d + (d-3)/2$, but $m(d^2+t)/d + (d-3)/2 > m\sqrt{d^2 + 2t + 1} = m\sqrt{n}$ for $m \leq d(d-3)$. On the other hand, since $r^2 \leq d^2n$, we see that $m\sqrt{n} \leq mnd/r$, so it suffices to show that $mnd/r \leq s+ud+1$. If $s = d-1$, then $s+ud+1 = (u+1)d = \lceil mn/r\rceil d \geq mnd/r$ as required, so assume $(s+1)(s+2) \leq 2\rho < (s+2)(s+3)$ and $s+2 \leq d$. Then $r(s+ud+1) = r(s+1)+mnd-d\rho$, so we need only check that $r(s+1)+mnd-d\rho \geq mnd$, or even that $r(s+1) \geq d(s+2)(s+3)/2$ (which is clear if $s = 0$ since $d \geq 3$) or that $r \geq d^2(s+3)/(2(s+1))$ (which is also clear since now we may assume $s \geq 1$).

Now consider (b). First assume $m \leq 2d^2/\Delta$. Let $\Delta = 2t$ and apply Theorem I.3(b) with $r = d^2 + t$ (so again $r = \lfloor d\sqrt{n}\rfloor$), $u = m$ and $\rho = mt$; it has to be checked that $md + s + 1 \geq m\sqrt{n}$ or, equivalently, that $(s+1)^2 + 2(s+1)md \geq 2m^2t$. If $s = 0$ then $mt < 3$ and the inequality follows from $d \geq 3$, whereas if $s = d-1$ then $m \leq 2d^2/\Delta$ implies $d^2 + 2md^2 > 2m^2t$. In all intermediate cases one has $2d(s+1) \geq (s+2)(s+3)$ and $(s+2)(s+3) > 2\rho = 2mt$ which also imply $(s+1)^2 + 2(s+1)md \geq 2m^2t$ easily.

Finally, assume $m \leq d(d-3)/2$. Again $\Delta = 2t$ so $n = d^2 + 2t$; take $r = d^2 + t - 1$ and apply Theorem I.3(c) in the same manner as previously. $\diamondsuit$

Proof of Corollary I.2(a): It follows from Corollary IV.1 that $\alpha(n; m) \geq m\sqrt{n}$ holds for all $n \geq 10$ if $m \leq d(d-3)/2$, where $d = \lfloor\sqrt{n}\rfloor$ (given that $\alpha(n; m) \geq m\sqrt{n}$ is known and indeed easy to prove when n is a square). But $\sqrt{n} \geq d$ and $d^2 + 2\sqrt{n} \geq n$, so obviously $d(d-3)/2 \geq (n - 5\sqrt{n})/2$. $\diamondsuit$

V. Hilbert Functions

We now consider the problem of determining the Hilbert function of an ideal of the form $I(n;m)$. Typically Theorem I.3(b) gives a lower bound $\lambda_\alpha(n;m)$ on $\alpha(n;m)$ which is smaller than the upper bound $\Lambda_\tau(n;m)$ it gives for $\tau(n;m)$, but there are in fact many cases for which $\lambda_\alpha(n;m) \geq \Lambda_\tau(n;m)$. In any such case, it follows that $\alpha(n;m) \geq \tau(n;m)$, which clearly implies Conjecture I.1(b) for the given n and m. This is precisely the method of proof of the next result.

Corollary V.1: *Let $d \geq 3$, $\varepsilon > 0$ and $i > 0$ be integers, and consider $n = d^2 + 2\varepsilon$. Then Conjecture I.1(b) holds for the given n and m if m falls into one of the following ranges:*

(a) $(d-1)(d-2)/(2\varepsilon) \leq m < (d+2)(d+1)/(2\varepsilon)$;
(b) $(i(d^2+\varepsilon) + (d-1)(d-2)/2)/\varepsilon \leq m \leq (id^2 + d(d-1)/2)/\varepsilon$;
(c) $(i(d^2+\varepsilon) + d(d-1)/2)/\varepsilon \leq m \leq (id^2 + (d+1)d/2)/\varepsilon$; *and*
(d) $(i(d^2+\varepsilon) + (d+1)d/2)/\varepsilon \leq m \leq (id^2 + d(d+3)/2)/\varepsilon$.

Proof: Case (a) is most easily treated by considering three subcases: (a1) $(d-1)(d-2)/(2\varepsilon) \leq m < d(d-1)/(2\varepsilon)$; (a2) $d(d-1)/(2\varepsilon) \leq m < (d+1)d/(2\varepsilon)$; and (a3) $(d+1)d/(2\varepsilon) \leq m < (d+2)(d+1)/(2\varepsilon)$.

For the proof, apply Theorem I.3 with $r = d^2 + \varepsilon$, $u = m + i$ and $\rho = m\varepsilon - ir$ (with $i = 0$ for part (a)). The reader will find in cases (a1) and (b) that $s = d-3$, while $s = d-2$ in cases (a2) and (c), and $s = d-1$ in cases (a3) and (d). It follows from Theorem I.3 that $\lambda_\alpha(n;m) \geq \Lambda_\tau(n;m)$, and hence, as discussed above, Conjecture I.1(b) holds for the given n and m. $\diamondsuit$

For each n, there is a finite set of values of d, ε and i to which Corollary V.1 can be profitably applied. For example, in parts (b), (c) and (d) of Corollary V.1 we may assume $i \leq (d-1)/\varepsilon$, $i \leq d/\varepsilon$ and $i \leq d/\varepsilon$ respectively, as otherwise the corresponding range of multiplicities is empty. Thus, Corollary V.1 determines a finite set V_n of values of m for which Conjecture I.1(b) must hold. Between the least and largest m in V_n there can also be many integers m which are not in V_n. For example, of the 4200 pairs (n,m) with $10 \leq n = d^2 + 2\varepsilon \leq 100$ and $1 \leq m \leq 100$, there are 723 with $m \in V_n$. Of these, 308 have $m \leq 12$ (and thus for these Conjecture I.1(b) was verified by [CM2]); the other 415 were not known before our results.

It is also noteworthy that in many cases we verify Conjecture I.1(b) for quite large values of m. In particular, if $n = d^2 + 2$, it follows from Corollary V.1 that $m \in V_n$ for $m = d(d^2+1) + d(d+1)/2$. Thus we have $243 \in V_{38}$, for example, and $783 \in V_{83}$. Apart from special cases when n is a square (in particular, when n is a power of 4; see [E2]), no other method we know can handle such large multiplicities. On the other hand, as indicated by Corollary I.2(b), if n is any square larger than 10, our method also handles arbitrarily large values of m, as we now prove. For the purpose of stating the result, given any positive integer i, let l_i be the largest integer j such that $j(j+1) \leq i$.

Corollary V.2: *Consider $10 \leq n = \sigma^2$ general points of $\mathbf{P}^2$. Let k be any nonnegative integer, and let $m = x + k(\sigma - 1)$, where x is an integer satisfying $\sigma/2 - l_\sigma \leq x \leq \sigma/2$ if σ is even, or $(\sigma+1)/2 - l_{2\sigma} \leq x \leq (\sigma+1)/2$ if σ is odd. Then Conjecture I.1(b) holds for $I(n;m)$.*

Proof: We apply Theorem I.3(c) with $d = \sigma - 1$, $r = d\sigma$, $u = \lceil mn/r \rceil - 1 = m + k$ and $\rho = mn - ur = x\sigma$. We claim that $t_0 \leq \min(\lfloor (mr + g - 1)/d \rfloor, s + ud)$, where $t_0 = m\sigma + \sigma/2 - 2$ if σ is even and $t_0 = m\sigma + (\sigma - 1)/2 - 2$ if σ is odd. But $t_0 \leq (mr + g - 1)/d$ because $(mr + g - 1)/d = (md\sigma + d(d-3)/2)/d = m\sigma + (\sigma - 4)/2$. To see $t_0 \leq s + ud$, note that $t_0 \leq s + ud$ simplifies to $x + \sigma/2 - 2 \leq s$ if σ is even and to $x + (\sigma - 1)/2 - 2 \leq s$ if σ is odd. Therefore (by definition of s) we have to check that $x + \sigma/2 - 1 \leq d$ and $(x + \sigma/2 - 1)(x + \sigma/2) \leq 2\sigma x$ if σ is even, and that $x + (\sigma - 1)/2 - 1 \leq d$ and $(x + (\sigma - 1)/2 - 1)(x + (\sigma - 1)/2) \leq 2\sigma x$ if σ is odd. The first inequality follows from $x \leq \sigma/2$ and $x \leq (\sigma + 1)/2$ respectively. For the second, substituting $\sigma/2 - j$ for x if σ is even and $(\sigma + 1)/2 - j$ for x if σ is odd, $(x + \sigma/2 - 1)(x + \sigma/2) \leq 2\sigma x$ and $(x + (\sigma - 1)/2 - 1)(x + (\sigma - 1)/2) \leq 2\sigma x$ resp. become $j(j + 1) \leq \sigma$ if σ is even and $j(j + 1) \leq 2\sigma$ if σ is odd. Thus $(x + \sigma/2 - 1)(x + \sigma/2) \leq 2\sigma x$ and $(x + (\sigma - 1)/2 - 1)(x + (\sigma - 1)/2) \leq 2\sigma x$ resp. hold if x is an integer satisfying $\sigma/2 - l_\sigma \leq x \leq \sigma/2$ if σ is even, and $(\sigma + 1)/2 - l_{2\sigma} \leq x \leq (\sigma + 1)/2$ if σ is odd.

This shows by Theorem I.3(c) that $\alpha(n; m) \geq m\sigma + \sigma/2 - 1$ if σ is even and $\alpha(n; m) \geq m\sigma + (\sigma - 1)/2 - 1$ if σ is odd. But since n points of multiplicity m impose at most $n\binom{m+1}{2}$ conditions on forms of degree t, it follows that $h(n; m)(t) \geq \binom{t+2}{2} - n\binom{m+1}{2}$, and it is easy to check that $\binom{t+2}{2} - n\binom{m+1}{2} > 0$ whenever $t \geq m\sigma + \sigma/2 - 1$ if σ is even and $t \geq m\sigma + (\sigma - 1)/2 - 1$ if σ is odd. Thus in fact we have $\alpha(n; m) = m\sigma + \sigma/2 - 1$ if σ is even and $\alpha(n; m) = m\sigma + (\sigma - 1)/2 - 1$ if σ is odd, whenever m is of the form $m = x + k(\sigma - 1)$, with x as given in the statement of Corollary V.2.

Of course, $h(n; m)(t) = 0$ for all $t < \alpha(n; m)$, and by [HHF], we know that $h(n; m)(t) = \binom{t+2}{2} - n\binom{m+1}{2}$ for all $t \geq \alpha(n; m)$ (apply Lemma 5.3 of [HHF], keeping in mind our explicit expression for $\alpha(n; m)$). $\diamondsuit$

Note that Corollary I.2(b) is an immediate consequence of the preceding result.

VI. Resolutions

We now show how our results verify many cases of Conjecture I.1(c) also, including cases with m arbitrarily large. Indeed, in addition to the case that n is an even square treated in Corollary VI.2 below, we have by the following proposition the resolution for 121 of the 723 pairs (n, m) with $m \in V_n$ mentioned above, and of these 121, 91 have $m > 2$ and hence were not known before.

Corollary VI.1: *Let $d \geq 3$ and $\varepsilon \geq 1$ be integers. Then Conjecture I.1(c) holds for each of the following values of n and m, whenever m is an integer:*

(a) *$n = d^2 + 2\varepsilon$ and $m = d(d \pm 1)/(2\varepsilon)$, in which case $\alpha = \alpha(n; m) = md + d - 1/2 \pm 1/2$ and the minimal free resolution of $I(n; m)$ is*

$$0 \to R[-\alpha - 1]^{\alpha} \to R[-\alpha]^{\alpha+1} \to I(n; m) \to 0;$$

(b) *$n = d^2 + 2\varepsilon$ and $m = (d(d \pm 1)/2 - 1)/\varepsilon$, in which case $\alpha = \alpha(n; m) = md + d - 3/2 \pm 1/2$ and the minimal free resolution of $I(n; m)$ is*

$$0 \to R[-\alpha - 2]^{b+m} \to R[-\alpha - 1]^{b} \oplus R[-\alpha]^{m+1} \to I(n; m) \to 0,$$

where $b = (m + 1)(d - 2) + 1/2 \pm 1/2$;

(c) $n = d^2 + 2$ *and* $m = d^2 + d(d \pm 1)/2$, *in which case* $\alpha = \alpha(n; m) = (m+1)d + d - 3/2 \pm 1/2$ *and the minimal free resolution of* $I(n; m)$ *is*

$$0 \to R[-\alpha - 2]^{a+b-1} \to R[-\alpha - 1]^b \oplus R[-\alpha]^a \to I(n; m) \to 0,$$

where $a = d(d \pm 1)/2$ *and* $b = \alpha + 2 - d(d \pm 1)$.

Proof: For case (a), apply Corollary V.1(a2) for $m = d(d-1)/(2\varepsilon)$ and Corollary V.1(a3) for $m = d(d+1)/(2\varepsilon)$. It turns out that $\alpha(n; m) > \tau(n; m)$ in these cases, but it is well known that $I(n; m)$ is generated in degrees $\tau(n; m) + 1$ and less, hence in degree $\alpha(n; m)$, from which it follows (see the displayed formula following Definition 2.4 of [HHF]) that the minimal free resolution is as claimed.

For cases (b) and (c), it turns out that $\alpha(n; m) = \tau(n; m)$: for case (b), apply Corollary V.1(a1-2), resp., while for case (c), apply Corollary V.1(b,c), resp., using $i = \varepsilon = 1$. To obtain the resolution, consider $\mathbf{m} = (m_1, \ldots, m_n)$, where $m_1 = m+1$ and $m_2 = \cdots = m_n = m$, and apply Theorem I.3 to $\alpha(\mathbf{m})$ using $r = d^2 + \varepsilon$. It turns out that $\alpha(\mathbf{m}) > \alpha(n; m)$. Now by Lemma 2.6(b) of [HHF] it follows that Conjecture I.1(c) holds and that the minimal free resolutions are as claimed (again, see the displayed formula following Definition 2.4 of [HHF]). $\diamondsuit$

When n is an even square, Corollary V.2, together with Theorem 5.1(a) of [HHF], directly implies:

Corollary VI.2: *Consider* $n = \sigma^2$ *general points of* $\mathbf{P}^2$, *where* $\sigma > 3$ *is even. Let* k *be any nonnegative integer, and let* $m = x + k(\sigma - 1)$, *where* x *is an integer satisfying* $\sigma/2 - l_\sigma \leq x \leq \sigma/2$. *Then Conjecture I.1(c) holds for* $I(n; m)$.

Note that Corollary I.2(c) is an immediate consequence of the preceding result.

VII. Comparisons

It is interesting to carry out some comparisons with previously known bounds. Let, as before, $\alpha_c(n; m)$ denote the conjectural value of $\alpha(n; m)$ and let $\tau_c(n; m)$ denote the conjectural value of $\tau(n; m)$ (i.e., the values of each assuming Conjecture I.1(b) holds).

Suppose that $rd(d+1)/2 \leq r^2 < nd^2$; then for m large enough the bound from Theorem I.3(c) is $\alpha(n; m) \geq 1 + \lfloor (mr + g - 1)/d \rfloor$. This is better than the bound of Corollary IV.1.1.2 of [H4] (which generalizes the main theorem of [H3]; see [H5] for further generalizations and related results), which is just $\alpha(n; m) \geq mr/d$. On the other hand, suppose that $2r \geq n + d^2$. Then $r^2 \geq nd^2$ (because the arithmetic mean is never less than the geometric mean), so the main theorem of [H3] applies and gives $\alpha(n; m) \geq mnd/r$. Typically Theorem I.3(b) gives a better bound than this, but if in addition r divides mn, the bound from Theorem I.3(b) simplifies, also giving $\alpha(n; m) \geq mnd/r$.

In fact, for given r and d, the bound in [H3], and the generalization given in Theorem IV.1.1.1 of [H4], can be shown (see [H4]) never to give a better bound on α than that given by the algorithms of Section II. When m is large enough compared with n, [H3] shows its bound on $\alpha(n; m)$ is better than those of [R1], and thus so are the bounds here.

When m is not too large compared with n, the bounds on α given by Theorem I.3, like the bound given by the unloading algorithm of [R1], are among the few that sometimes give bounds better than the bound $\alpha(n;m) \geq \lfloor m\sqrt{n}\rfloor + 1$ conjectured in [N1]. Consider, for example, $n = 1000$ and $m = 13$: [R1] gives $\alpha(n;m) \geq 421$ and Theorem I.3, using $r = 981$ and $d = 31$, gives $\alpha(n;m) \geq 424$, whereas $\lfloor m\sqrt{n}\rfloor + 1 = 412$; $\alpha_c(n;m)$ is 426 in this case. See Corollary IV.1 for more examples. Moreover, Theorem I.3 is the only result that we know which sometimes determines $\alpha(n;m)$ exactly even for m reasonably large compared to n, when n is not a square.

Here are some comparisons for τ. Bounds on $\tau(n;m)$ given by Hirschowitz [Hi1], Gimigliano [Gi] and Catalisano [C1] are on the order of $m\sqrt{2n}$. Thus, for sufficiently large m, the bound $\tau(n;m) \leq m\lceil\sqrt{n}\rceil + \lceil(\lceil\sqrt{n}\rceil - 3)/2\rceil$ given in [HHF] for $n > 9$ is better. In fact, [HHF] shows that $\tau(n;m) \leq m\lceil\sqrt{n}\rceil + \lceil(\lceil\sqrt{n}\rceil - 3)/2\rceil$ is an equality when $n > 9$ is a square and m is sufficiently large. However, when n is a square, Theorem I.3(a), using $d^2 = r = n$, also gives this bound (this is to be expected, since the method we use is based on the method used in [HHF]), and when n is not a square, Theorem I.3(a), using $d = \lceil\sqrt{n}\rceil$ and $r = n$, gives a bound that is less than or equal to that of [HHF] (although never more than 2 smaller). But one can also apply Theorem I.3(b) using other values of r and d, and often do much better. In addition, as was pointed out for α above, Theorem I.3 is the only result that we know that sometimes determines $\tau(n;m)$ exactly for values of m and n that can be large, even when n is not a square.

Other bounds on τ have also been given. Bounds given by Xu [X] and Ballico [B] are on the order of $m\sqrt{n}$, but nonethless the bound from [HHF] (and hence Theorem I.3) is better than Xu's when n is large enough and better than Ballico's when m is large enough. For large m, the bound given in [R2] is also better than those of [B] and [X], and by an argument similar to the one used in [H3] to compare the bounds on α, the bounds here on $\tau(n;m)$ are better than those of [R2] when m is large enough compared with n.

For example, for $n = 190$ and $m = 100$, then $\tau_c(n;m) = 1384$, while Theorem I.3(b), using $r = 180$ and $d = 13$, gives $\tau(n;m) \leq 1390$, and we have in addition:

- $\tau(Z) \leq 1957$ from [Hi1],
- $\tau(Z) \leq 1900$ from [Gi],
- $\tau(Z) \leq 1899$ from [C1],
- $\tau(Z) \leq 1487$ from [B],
- $\tau(Z) \leq 1465$ from [X],
- $\tau(Z) \leq 1440$ from [R2] and
- $\tau(Z) \leq 1406$ from [HHF].

For a different perspective, we close with some graphs which show our results in certain ranges. Figure 1 is a graphical representation of Corollary V.1 and Corollary V.2, while Figure 2 is a graphical representation of Corollary VI.1 and Corollary VI.2. Figure 3 shows all (n,m) for which Theorem I.3, using $d = \lfloor\sqrt{n}\rfloor$ and $r = \lfloor d\sqrt{n}\rfloor$, implies Conjecture I.1(a). It was this graph that led us to the statement of Corollary I.2(a). Figure 4 shows for comparison all (n,m) for which Theorem I.3 implies Conjecture I.1(a), but this time using $d = \lfloor\sqrt{n}\rfloor$ and $r = \lfloor(n + d^2)/2\rfloor$. This choice of r and d gives a higher density of cases for which we can conclude that Conjecture I.1(a) is true, but the graph has a very complicated structure which does not seem to suggest any simply stated result.

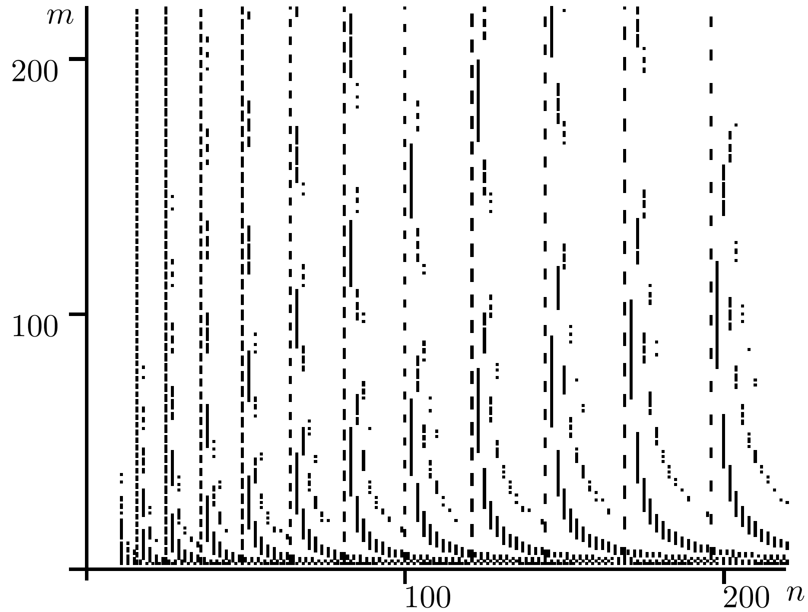

Figure 1: Graph of all (n, m) with $10 \leq n \leq 220$ and $0 \leq m \leq 220$ such that Corollary V.1 and Corollary V.2 determine the Hilbert function of $I(n; m)$ of n points of multiplicity m in $\mathbf{C}P^2$.

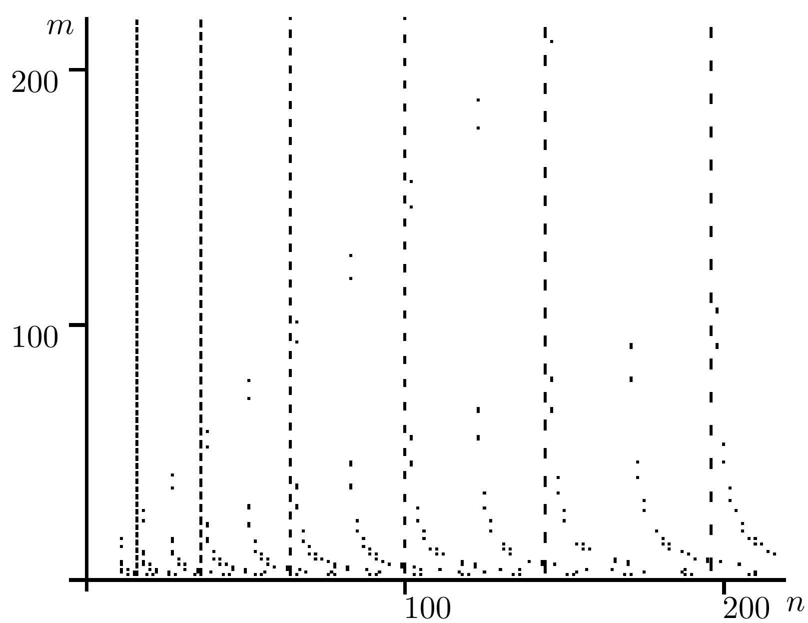

Figure 2: Graph of all (n, m) with $10 \leq n \leq 220$ and $0 \leq m \leq 220$ such that Corollary VI.1 and Corollary VI.2 determine the minimal free resolution of $I(n; m)$ of n points of multiplicity m in $\mathbf{C}P^2$.

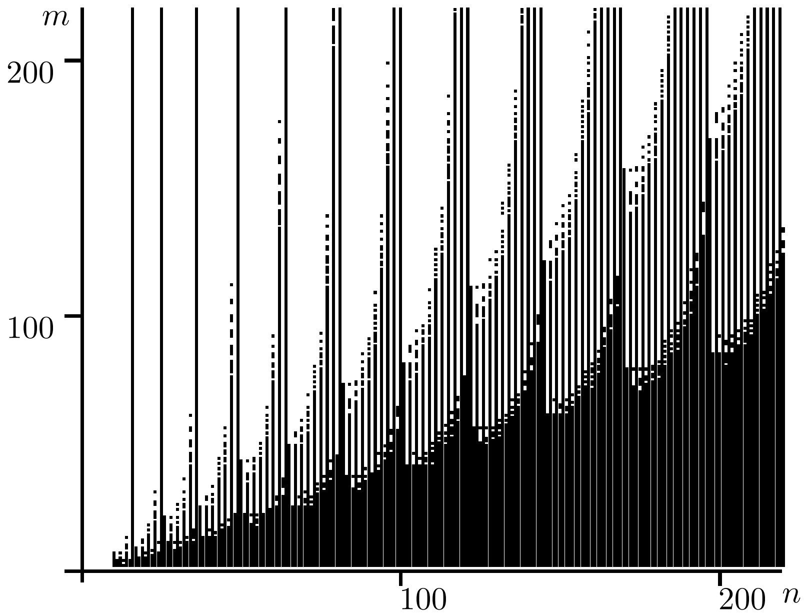

Figure 3: Graph of all (n, m) for n points of multiplicity m in $\mathbf{C}P^2$ with $10 \le n \le 220$ and $0 \le m \le 220$, such that Theorem I.3, using $d = \lfloor\sqrt{n}\rfloor$ and $r = \lfloor d\sqrt{n}\rfloor$, implies $\alpha(n; m) \ge m\sqrt{n}$.

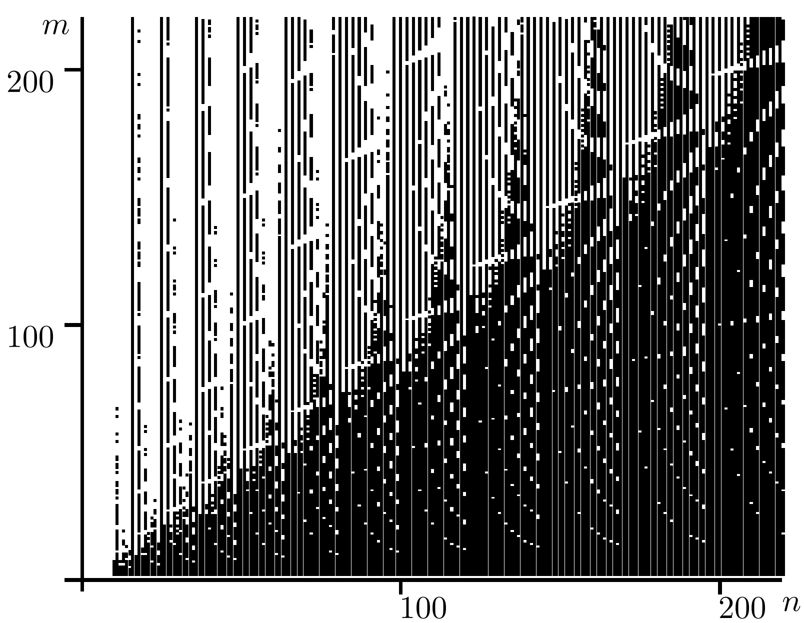

Figure 4: Graph of all (n, m) for n points of multiplicity m in $\mathbf{C}P^2$ with $10 \le n \le 220$ and $0 \le m \le 220$, such that Theorem I.3, using $d = \lfloor\sqrt{n}\rfloor$ and $r = \lfloor(n+d^2)/2\rfloor$, implies $\alpha(n; m) \ge m\sqrt{n}$.

References

[AH] J. Alexander and A. Hirschowitz. *An asymptotic vanishing theorem for generic unions of multiple points*, Invent. Math. 140 (2000), no. 2, 303–325.

[B] E. Ballico. *Curves of minimal degree with prescribed singularities*, Illinois J. Math. 45 (1999), 672–676.

[BZ] A. Buckley and M. Zompatori. *Linear systems of plane curves with a composite number of base points of equal multiplicity*, preprint (2001).

[C1] M. V. Catalisano. *Linear Systems of Plane Curves through Fixed "Fat" Points of* $\mathbf{P}^2$, J. Alg. 142 (1991), 81-100.

[C2] M. V. Catalisano. *"Fat" points on a conic*, Comm. Alg. 19(8) (1991), 2153–2168.

[CCMO] C. Ciliberto, F. Cioffi, R. Miranda and F. Orecchia. *Bivariate Hermite interpolation via computer algebra and algebraic geometry techniques.*, Preprint (2001).

[CM1] C. Ciliberto and R. Miranda. *Degenerations of planar linear systems*, Journ. Reine Angew. Math. 501 (1998), 191–220.

[CM2] C. Ciliberto and R. Miranda. *Linear systems of plane curves with base points of equal multiplicity*, Trans. Amer. Math. Soc. 352 (2000), 4037–4050.

[CM3] C. Ciliberto and R. Miranda. *The Segre and Harbourne-Hirschowitz conjectures*, in: Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001), 37–51, NATO Sci. Ser. II Math. Phys. Chem., 36, Kluwer Acad. Publ., Dordrecht, 2001.

[E1] L. Évain. *Une minoration du degre des courbes planes à singularités imposées*, Bull. Soc. Math. France 126 (1998), no. 4, 525–543.

[E2] L. Évain. *La fonction de Hilbert de la réunion de* 4^h *gros points génériques de* $\mathbf{P}^2$ *de même multiplicité*, J. Algebraic Geometry (1999), 787–796.

[FHH] S. Fitchett, B. Harbourne and S. Holay. *Resolutions of Fat Point Ideals involving Eight General Points of* $\mathbf{P}^2$, J. Algebra 244 (2001), 684–705.

[GGR] A. V. Geramita, D. Gregory, L. Roberts. *Monomial ideals and points in projective space*, J. Pure Appl. Algebra 40 (1986), 33–62.

[Gi] A. Gimigliano. *Regularity of Linear Systems of Plane Curves*, J. Alg. 124 (1989), 447–460.

[GI] A. Gimigliano and M. Idà. *The ideal resolution for generic 3-fat points in* $\mathbf{P}^2$, preprint (2002).

[H1] B. Harbourne. *The geometry of rational surfaces and Hilbert functions of points in the plane*, Can. Math. Soc. Conf. Proc. 6 (1986), 95–111.

[H2] B. Harbourne. *The Ideal Generation Problem for Fat Points*, J. Pure Appl. Algebra 145 (2000), 165–182.

[H3] B. Harbourne. *On Nagata's Conjecture*, J. Alg. 236 (2001), 692–702.

[H4] B. Harbourne. *Problems and Progress: A survey on fat points in* $\mathbf{P}^2$, v. 123, 2002, Queen's papers in pure and applied mathematics, The curves seminar at Queen's.

[H5] B. Harbourne. *Seshadri constants and very ample divisors on algebraic surfaces*, Journ. Reine Angew. Math. (to appear).

[HHF] B. Harbourne, S. Holay and S. Fitchett. *Resolutions of ideals of quasiuniform fat point subschemes of* $\mathbf{P}^2$, Trans. Amer. Math. Soc. 355 (2003), no. 2, 593–608.

[Hi1] A. Hirschowitz. *Une conjecture pour la cohomologie des diviseurs sur les surfaces rationelles génériques*, Journ. Reine Angew. Math. 397 (1989), 208–213.

[Hi2] A. Hirschowitz. *La méthode d'Horace pour l'interpolation à plusieurs variables*, Manus. Math. 50 (1985), 337–388.

[Ho] M. Homma. *A souped up version of Pardini's theorem and its aplication to funny curves*, Comp. Math. 71 (1989), 295–302.

[I] M. Idà. *The minimal free resolution for the first infinitesimal neighborhoods of n general points in the plane*, J. Alg. 216 (1999), 741–753.

[Mg] T. Mignon. *Systèmes de courbes planes à singularités imposées: le cas des multiplicités inférieures ou égales à quatre*, J. Pure Appl. Algebra 151 (2000), no. 2, 173–195.

[Mr] R. Miranda. *Algebraic Curves and Riemann Surfaces*, Graduate Studies in Mathematics 5, Amer. Math. Soc., (1995), xxi + 390 pp.

[N1] M. Nagata. *On the 14-th problem of Hilbert*, Amer. J. Math. 81 (1959), 766–772.

[N2] M. Nagata. *On rational surfaces, II*, Mem. Coll. Sci. Univ. Kyoto, Ser. A Math. 33 (1960), 271–293.

[R1] J. Roé. *On the existence of plane curves with imposed multiple points*, J. Pure Appl. Alg. 156 (2001), 115–126.

[R2] J. Roé. *Linear systems of plane curves with imposed multiple points*, Illinois J. Math. 45 (2001), 895–906.

[S] B. Segre. *Alcune questioni su insiemi finiti di punti in Geometria Algebrica*, Atti del Convegno Internaz. di Geom. Alg., Torino (1961).

[X] G. Xu. *Ample line bundles on smooth surfaces*, Journ. Reine Angew. Math. 469 (1995), 199–209.