# Resolutions of Ideals of Uniform Fat Point Subschemes of $\mathbf{P}^2$

Brian Harbourne

Department of Mathematics and Statistics, University of Nebraska-Lincoln
Lincoln, NE 68588-0323
email: bharbour@math.unl.edu
WEB: http://www.math.unl.edu/~bharbour/

Sandeep Holay

Department of Mathematics, Southeast Community College, Lincoln, NE
email: sholay@math.unl.edu

Stephanie Fitchett

Department of Mathematics, Duke University, Durham, NC
email: sfitchet@math.duke.edu

June 4, 1999

Abstract: Let $I$ be the ideal defining a set of general points $p_1, \ldots, p_n \in \mathbf{P}^2$. There recently has been progress in showing that a naive lower bound for the Hilbert functions of symbolic powers $I^{(m)}$ is in fact attained when $n > 9$. Here, for $m$ sufficiently large, the minimal free graded resolution of $I^{(m)}$ is determined when $n > 9$ is an even square, assuming only that this lower bound on the Hilbert function is attained. Under ostensibly stronger conditions (that are nonetheless expected always to hold), a similar result is shown to hold for odd squares, and for infinitely many $m$ for each nonsquare $n$ bigger than 9. All results hold for an arbitrary algebraically closed field $k$.

## 1. Introduction

The concerns of this paper, in which we fix an algebraically closed ground field $k$, are rooted in work by Dubreil on numbers of generators of homogeneous ideals and in work by Nagata resolving Hilbert's 14th Problem and posing a still open conjecture regarding the minimum degree of a curve with certain assigned multiplicities.

Let $I$ be a homogeneous ideal defined by base point conditions in $\mathbf{P}^2$ (more precisely, in terms defined below, let $I$ be the saturated homogeneous ideal defining the fat point subscheme $m_1p_1 + \cdots + m_np_n$ for points $p_i \in \mathbf{P}^2$ and nonnegative multiplicities $m_i$). If the points are general and if $n > 9$, Nagata [N1, N2] conjectures that the least degree of a nontrivial element of $I$ is always more than $(m_1 + \cdots + m_n)/\sqrt{n}$. He proves this when $n$ is a square, which he then uses in his famous counterexample to Hilbert's 14th Problem.

Nagata's work can be seen as giving a bound on the Hilbert function of $I$ for low degrees. Dubreil's work [Du], with additional developments by [DGM] and [Cam], derives from the Hilbert function of $I$ bounds on numbers of generators of $I$. Recently, by paying attention to geometry (in particular, by keeping in mind the multiplicities $m_i$ attached to the points), [F1] improves these bounds under certain conditions. In various special cases, the minimal free resolution for $I$ has now also been determined: [Cat] and [Ha5] do so for points on plane conics; [F1, F2] does so for $n \leq 6$ general points, extended in unpublished work of the authors to $n = 8$; [Ha6] does so for uniform subschemes $mp_1 + \cdots + mp_n$ with $p_i$ general and $n \leq 9$; and [Id] does so for $n > 9$ for subschemes $2p_1 + \cdots + 2p_n$ when the points $p_i$ are general.

In each of the cases that a resolution was determined, the Hilbert function of $I$ was known previously. One stumbling block to finding minimal free resolutions generally is that the Hilbert function of I is not always known, as evidenced by the fact that Nagata's conjecture is still open. However, for subschemes $m_1p_1 + \cdots + m_np_n$ with $n > 9$ and $p_i$ general, evidence is accumulating [CM1, CM2, R] that the Hilbert function takes a certain minimal form if the multiplicities $m_1, \ldots, m_n$ are fairly uniform. The conjecture that this is indeed what happens is formulated here precisely as Conjecture 3.1. (In fact, Nagata's conjecture is a consequence of Conjecture 3.1.)

This work benefitted from a National Science Foundation grant.

1991 *Mathematics Subject Classification.* Primary 13P10, 14C99. Secondary 13D02, 13H15.

*Key words and phrases.* Ideal generation conjecture, symbolic powers, resolution, fat points, maximal rank.

Under the assumption that this minimal Hilbert function is under certain circumstances achieved, the main result of this paper, Theorem 2.5, shows for each $n > 9$ that there are infinitely many $m$ such that the minimal free resolution for $I$ also takes a certain minimal form, when $I$ corresponds to the subscheme $mp_1 + \cdots + mp_n$ with $p_i$ general. This suggests that this minimal form of the minimal free resolution may be attained for all $m$ for every $n > 9$ when the points $p_i$ are general. This suggestion is formalized by Conjecture 3.2. Additional evidence for Conjecture 3.2 is given in Example 5.2.

Although our results are for $\mathbf{P}^2$, the general problem is of interest for all $\mathbf{P}^N$, $N > 1$. So let $p_1, \ldots, p_n \in \mathbf{P}^N$ be general points of projective space. Let $P_j$ be the homogeneous ideal defining $p_j$ in the homogeneous coordinate ring $R = k[x_0, \ldots, x_N]$ of $\mathbf{P}^N$. An ideal of the form $I(\mathbf{m}; n, N) = P_1^{m_1} \cap \cdots \cap P_n^{m_n}$, where $\mathbf{m}$ denotes the $n$-tuple $(m_1, \ldots, m_n)$ of nonnegative integers $m_j$, defines a *fat point* subscheme $m_1p_1 + \cdots + m_np_n$ of $\mathbf{P}^N$. (Thus, for example, $I((m, \ldots, m); n, N)$ is the $m$-th symbolic power $I((1, \ldots, 1); n, N)^{(m)}$ of the ideal $I((1, \ldots, 1); n, N)$ generated by all forms that vanish at the $n$ points.) Being a homogeneous ideal, $I(\mathbf{m}; n, N)$ is a direct sum of its components $I(\mathbf{m}; n, N)_t$ of each degree $t$. We can regard $I(\mathbf{m}; n, N)_t$ as the linear system of forms of degree $t$ with the imposed base point conditions of vanishing at each point $p_j$ to order at least $m_j$.

A natural problem is that of determining the ($k$ vector space) dimension $h_{I(\mathbf{m};n,N)}(t)$ of $I(\mathbf{m}; n, N)_t$ as a function of $t$; $h_{I(\mathbf{m};n,N)}$ is known as the Hilbert function of $I(\mathbf{m}; n, N)$. A more difficult problem is to determine the minimal number of homogeneous generators of $I(\mathbf{m}; n, N)$ needed in each degree $t$; this is equivalent to determining the Hilbert function of $I(\mathbf{m}; n, N) \otimes_R k$, since the dimension of the homogeneous component $(I(\mathbf{m}; n, N) \otimes_R k)_t$ is just the minimal number of homogeneous generators of $I(\mathbf{m}; n, N)$ needed in degree $t$. The gold standard in connection to problems like these is the determination of a minimal free resolution $0 \to F_{N-1} \to \cdots \to F_0 \to I(\mathbf{m}; n, N) \to 0$, or at least a determination of the modules $F_j$, up to isomorphism as graded $R$-modules, since knowing the modules $F_j$ up to graded isomorphism is sufficient to determine both the numbers of homogeneous generators of $I(\mathbf{m}; n, N)$ in each degree and the Hilbert function of $I(\mathbf{m}; n, N)$.

The situation for $N = 1$ is trivial, since then saturated homogeneous ideals are principal. For $N > 1$, it is only for $N = 2$ that a general conjecture has been posed, and then only for the Hilbert function (see Conjecture 3.3 below, which uses the language of linear systems of sections on blow ups of $\mathbf{P}^2$ at the $n$ points). It is perhaps worth mentioning that complete answers for $N = 2$ have been attained for all of the questions raised above for $n \leq 8$. Under different terminology, the Hilbert function of the ideal $I$ corresponding to any subscheme $m_1p_1 + \cdots + m_np_n \in \mathbf{P}^2$ with $p_i$ general and $n \leq 8$ was determined by Nagata [N2]. (More generally, the Hilbert function of $I$ is known as long as a cubic curve passes through the points $p_i$; see [Ha4].) For $n \leq 8$, recent unpublished work [FHH] of the authors has now determined the minimal free resolution of $I(\mathbf{m}; n, 2)$, but the general situation remains unclear for $n > 8$.

## 2. Background

Since hereafter we restrict our attention to $N = 2$, $R$ will denote the homogeneous coordinate ring $k[x, y, z]$ of $\mathbf{P}^2$ and we will write $I(\mathbf{m}; n)$ for $I(\mathbf{m}; n, 2)$. Moreover, unless otherwise explicitly mentioned, we will hereafter always assume that $I(\mathbf{m}; n)$ involves $n$ *general* points of $\mathbf{P}^2$.

So suppose one knows the Hilbert function $h_{I(\mathbf{m};n)}$ of $I(\mathbf{m}; n)$. The problem of then determining the number of generators in each degree of a minimal set of homogeneous generators for $I(\mathbf{m}; n)$ is equivalent to the problem of determining the modules in the resolution up to graded isomorphism. This is because, since the subscheme $Z_{\mathbf{m},n} \subset \mathbf{P}^2$ defined by $I(\mathbf{m}; n)$ is arithmetically Cohen-Macaulay, the resolution takes the form $0 \to F_1 \to F_0 \to I(\mathbf{m}; n) \to 0$. Knowing the number of generators in each degree gives $F_0$, and, given the Hilbert function of $I(\mathbf{m}; n)$, exactness of the sequence $0 \to F_1 \to F_0 \to I(\mathbf{m}; n) \to 0$ allows one to then determine the Hilbert function of $F_1$ and thus $F_1$ itself.

For $n \geq 9$, however, the solution to these equivalent problems, of determining resolutions or of determining numbers of generators for $I(\mathbf{m}; n)$ given its Hilbert function, is not yet clear even conjecturally. Things improve somewhat if we restrict to uniform $\mathbf{m}$:

**Definition 2.1**: We will say that $\mathbf{m}$ or more generally $I(\mathbf{m}; n)$ is *uniform* if $\mathbf{m} = (m, \ldots, m)$ for some $m$ and *quasiuniform* if $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ with $r \geq 9$ and $m_1 = \cdots = m_9 \geq m_{10} \geq \cdots \geq m_r$.

**Definition 2.2**: It will be helpful to denote $I(\mathbf{m}, n)$ by $I(m; n)$ when the $n$-tuple $\mathbf{m}$ is $(m, \ldots, m)$, or more generally by $I(m_1, m_2, \ldots, m_r; n)$ when $\mathbf{m} = (m_1, m_2, \ldots, m_r, \ldots, m_r)$. Given $\mathbf{m} = (m_1, \ldots, m_n)$, we will denote $(m_1 + 1, m_2 \ldots, m_n)$ by $\mathbf{m}'$ and $(m_1 - 1, m_2 \ldots, m_n)$ by $\mathbf{m}''$; likewise, we will write $I(\mathbf{m}, n)'$ for

$I(\mathbf{m}', n)$ and $I(\mathbf{m}, n)''$ for $I(\mathbf{m}'', n)$.

All is known for $I(m; n)$ for $n \le 9$ ([Ha6]), while for $n \ge 10$ we have Conjecture 3.1 (which is a special case of Conjecture 3.3) for the Hilbert function, and we have Conjecture 3.2 for the resolution.

In preparation for stating these conjectures in the next section, consider the space $R_d$ of all forms of a given degree $d$. The subspace of those vanishing at a given point with multiplicity $m$ or more has dimension exactly $\max\{0, ((d+1)(d+2) - m(m+1))/2\}$. Thus a natural first conjecture is that the subspace of $R_d$ of those vanishing with multiplicity $m_i$ or more at each of $n$ general points $p_i$ has dimension $\max\{0, ((d+1)(d+2) - \sum_{i=1}^n m_i(m_i+1))/2\}$. Accordingly, we make the following definition.

**Definition 2.3**: Given $\mathbf{m} = (m_1, \ldots, m_n)$ with each $m_i \ge 0$, we will say that $I(\mathbf{m}; n)$ is *unhindered* if $h_{I(\mathbf{m};n)}(t) = \max\{0, ((t+1)(t+2) - \sum_{i=1}^n m_i(m_i+1))/2\}$ for all $t \ge 0$.

Sheafifying $I(\mathbf{m}; n)$ gives an ideal sheaf $\mathcal{I}$, and we note that $I(\mathbf{m}; n)$ being unhindered is the same thing as $h^1(\mathbf{P}^2, \mathcal{I}(t))$ vanishing for every $t$ for which $h^0(\mathbf{P}^2, \mathcal{I}(t)) > 0$, where $\mathcal{I}(t)$ denotes the twist $\mathcal{I} \otimes \mathcal{O}_{\mathbf{P}^2}(t)$ by $t$ times the class of a line.

Now, although $I(\mathbf{m}; n)$ is not in general unhindered, the known hindrances (which Conjecture 3.3 is formulated to account for) are fairly special, and in fact Conjecture 3.3 implies that all quasiuniform $I(\mathbf{m}; n)$ are unhindered if $n \ge 10$ (see Remark 3.5). Moreover, [CM1, CM2] shows that $I(m; n)$ is unhindered for all $n \ge 10$ with uniform multiplicity $m \le 12$. Thus it is reasonable to study resolutions of uniform ideals $I(m; n)$ under assumptions of unhinderedness.

Note that if $I(\mathbf{m}; n)$ is unhindered, it follows that the Castelnuovo-Mumford regularity of $I(\mathbf{m}; n)$ is at most $\alpha(\mathbf{m}; n) + 1$, where $\alpha(\mathbf{m}; n)$ is the least degree $t$ such that $I(\mathbf{m}; n)_t \ne 0$ (and where $\alpha(m; n)$ denotes the least degree $t$ such that $I(m; n)_t \ne 0$). This implies (see [DGM] or Lemma 2.9 of [Ha5]) that the first syzygy module $F_0$ in a minimal free resolution of $I(\mathbf{m}; n)$ has generators in at most two degrees, $\alpha(\mathbf{m}; n)$ and $\alpha(\mathbf{m}; n) + 1$. The number of generators in degree $\alpha(\mathbf{m}; n)$ is clearly $h(\mathbf{m}; n) = h_{I(\mathbf{m};n)}(\alpha(\mathbf{m}; n))$, while if $I(\mathbf{m}; n)$ is unhindered the number of generators in degree $\alpha(\mathbf{m}; n) + 1$ is at least $\max\{0, \alpha(\mathbf{m}; n) + 2 - 2h(\mathbf{m}; n)\}$, since $\alpha(\mathbf{m}; n) + 2 - 2h(\mathbf{m}; n)$ is the difference in dimensions of $I(\mathbf{m}; n)_{\alpha(\mathbf{m};n)+1}$ and $I(\mathbf{m}; n)_{\alpha(\mathbf{m};n)} \otimes R_1$. Thus, if $I(\mathbf{m}; n)$ is unhindered, the minimal possible rank for $F_0$ is $\max\{h(\mathbf{m}; n), \alpha(\mathbf{m}; n) + 2 - h(\mathbf{m}; n)\}$. In this paper we will adduce circumstances in which this minimum is achieved. Thus it is convenient to make the following definition:

**Definition 2.4**: We will say that $I(\mathbf{m}; n)$ is *rank minimal* if $I(\mathbf{m}; n)$ is unhindered and if the rank of $F_0$ in the minimal free resolution $0 \to F_1 \to F_0 \to I(\mathbf{m}; n) \to 0$ is $\max\{h(\mathbf{m}; n), \alpha(\mathbf{m}; n) + 2 - h(\mathbf{m}; n)\}$.

An ideal $I(\mathbf{m}; n)$ being rank minimal is equivalent to $I(\mathbf{m}; n)$ being unhindered with the multiplication homomorphism $I(\mathbf{m}; n)_{\alpha(\mathbf{m};n)} \otimes R_1 \to I(\mathbf{m}; n)_{1+\alpha(\mathbf{m};n)}$ having maximal rank (i.e, either being injective or surjective). Moreover, an unhindered ideal $I(\mathbf{m}; n)$ being rank minimal is equivalent to the minimal free resolution taking the following form (where $a = \alpha(\mathbf{m}; n)$, $h = h_{I(\mathbf{m};n)}(a)$, $b = \max\{0, a + 2 - 2h\}$ and $c = \max\{0, 2h - a - 2\}$—hence either $b$ or $c$ is zero):

$$(*) \qquad 0 \to R[-a-2]^{a+1-h} \oplus R[-a-1]^c \to R[-a-1]^b \oplus R[-a]^h \to I(\mathbf{m}; n) \to 0.$$

We recall that $R[a]^b$, for example, denotes the direct sum of $b$ copies of the homogeneous $R$-module $R[a]$, with the grading defined by $R[a]_t = R_{t+a}$.

To justify $(*)$, for an unhindered ideal note that a minimal free resolution of the form $(*)$ immediately implies that $I(\mathbf{m}; n)$ is rank minimal. Conversely, if an unhindered ideal $I(\mathbf{m}; n)$ is rank minimal, then $F_0$ is as given and one obtains $F_1$ as follows. Sheafifying (and suppressing $\mathbf{m}$ and $n$) gives $0 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{I} \to 0$ and $0 \to \mathcal{I} \to \mathcal{O}_{\mathbf{P}^2} \to \mathcal{O}_Z \to 0$, where $\mathcal{F}_0 = \mathcal{O}_{\mathbf{P}^2}(-a)^{\oplus h} \oplus \mathcal{O}_{\mathbf{P}^2}(-a-1)^{\oplus b}$ and $Z$ is the subscheme defined by $\mathcal{I}$. Clearly, $h^2(\mathbf{P}^2, \mathcal{F}_0(a))$ vanishes, while $h^1(\mathbf{P}^2, \mathcal{I}(a)) = 0$ since $I(\mathbf{m}; n)$ is unhindered. From $0 \to \mathcal{F}_1(a) \to \mathcal{F}_0(a) \to \mathcal{I}(a) \to 0$ one sees that $h^0(\mathbf{P}^2, \mathcal{F}_1(a)) = 0$ and that $h^2(\mathbf{P}^2, \mathcal{F}_1(a)) = 0$ and hence that $\mathcal{F}_1 = \mathcal{O}_{\mathbf{P}^2}(-a-1)^{\oplus c} \oplus \mathcal{O}_{\mathbf{P}^2}(-a-2)^{\oplus d}$ for some $c$ and $d$. To determine $c$ and $d$, take $h^0$ of $0 \to \mathcal{F}_1(t) \to \mathcal{F}_0(t) \to \mathcal{I}(t) \to 0$ for $t = a+1$ and $t = a+2$.

Our main result can now be stated.

**Theorem 2.5**: *Consider $n > 9$ general points $p_i \in \mathbf{P}^2$.*

(a) *Say $n$ is not a square. Then there are infinitely many $m > 0$ such that $I(m; n)$ is rank minimal whenever $I(m; n)$ and $I(m; n)'$ are unhindered.*

(b) *Say $n$ is an odd square. Then for all $m \geq (n-9)/8$, $I(m;n)$ is rank minimal whenever $I(m;n)$ and $I(m;n)'$ are unhindered.*

(c) *Say $n = r^2$ is an even square. Then for all $m \geq (r-2)/4$, $I(m;n)$ is rank minimal whenever $I(m;n)$ is unhindered.*

**Proof**: Part (a) is Proposition 4.1, part (b) is handled by Theorem 5.1(b), and part (c) follows directly from Theorem 5.1(a) and Lemma 5.3(i). $\diamondsuit$

The method used here when $n$ is not an even square was employed in [Ha6] for certain special values of $n$; thus Theorem 2.5 extends this to all $n$ which are not even squares. The method of [Ha6] fails when $n$ is an even square, so we develop a new approach special to that case.

The following result will be useful. Define $q(\mathbf{m};n)$ to be $h_{I(\mathbf{m}';n)}(\alpha(\mathbf{m};n))$, and define $l(\mathbf{m};n)$ to be $h_{I(\mathbf{m}'';n)}(-1+\alpha(\mathbf{m};n))$. We will also use $q(m;n)$ for $q(\mathbf{m};n)$ and $l(m;n)$ for $l(\mathbf{m};n)$ in case $\mathbf{m} = (m_1,\ldots,m_n)$ when $m = m_1 = \cdots = m_n$.

**Lemma 2.6**: *Assume $I(\mathbf{m};n)$ is unhindered.*

(a) *If $q(\mathbf{m};n) = 0$ and $l(\mathbf{m};n) = 0$, then $I(\mathbf{m};n)$ is rank minimal.*

(b) *If $\mathbf{m} = (m_1,\cdots,m_n)$ has $m_1 = m_2$, and $q(\mathbf{m};n) = 0$, then $I(\mathbf{m};n)$ is rank minimal.*

(c) *If $\mathbf{m} = (m_1,\cdots,m_n)$ has $m_1 = m_2$, $l(\mathbf{m};n) > 0$, and $I(\mathbf{m};n)'$ and $I(\mathbf{m};n)''$ are unhindered, then $I(\mathbf{m};n)$ is rank minimal.*

**Proof**: (a) Keeping in mind our comment about regularity (preceding Definition 2.4), this follows from Lemma 4.1 of [Ha6].

(b) Let $f$ be the linear form vanishing on the line through the points $p_1$ and $p_2$. Then multiplication by $f$ gives an injection $I(-1+m_1,m_2,m_3,\cdots,m_n;n)_{-1+\alpha(\mathbf{m};n)} \to I(m_1,1+m_2,m_3,\cdots,m_n;n)_{\alpha(\mathbf{m};n)}$, and since the points $p_1,\cdots,p_n$ are general, we know $I(m_1,1+m_2,m_3,\cdots,m_n;n)_{\alpha(\mathbf{m};n)}$ and $I(1+m_1,m_2,\cdots,m_n;n)_{\alpha(\mathbf{m};n)}$ have the same dimension, hence $l(\mathbf{m};n) \leq q(\mathbf{m};n)$. Therefore, if $q(\mathbf{m};n) = 0$, then $I(\mathbf{m};n)$ is rank minimal by (a).

(c) As in (b), we have $l(\mathbf{m};n) \leq q(\mathbf{m};n)$, but now $0 < l(\mathbf{m};n)$, so, applying the unhindered hypotheses, the result follows from Lemma 4.2 of [Ha6]. $\diamondsuit$

## 3. Conjectures

Although Theorem 2.5 employs certain assumptions about unhinderedness, this is not a serious restriction since the assumptions are either known to be met or, as we discuss below, are conjectured to be.

To begin, the Hilbert function of $I(\mathbf{m};n)$ is known (even if the points $p_i$ are not general) for $n \leq 9$ but it is somewhat complicated. It remains unknown (but conjectured) for $n > 9$, but, as indicated by the following Quasiuniform Hilbert function Conjecture (QHC), it is expected that the answer is fairly simple for quasiuniform ideals:

**Conjecture 3.1 (QHC)**: *For $n > 9$ general points of $\mathbf{P}^2$, $I(\mathbf{m};n)$, $I(\mathbf{m};n)'$ and $I(\mathbf{m};n)''$ are unhindered if $I(\mathbf{m};n)$ is quasiuniform.*

Similarly, whereas the problem of determining resolutions is known but complicated for ideals $I(\mathbf{m};n)$ involving $n \leq 8$ general points (and for uniform $\mathbf{m}$ when $n = 9$) and unknown and in general unconjectured for $n > 9$, things are expected to be particularly simple for uniform ideals for $n > 9$ general points. In particular, we have the following Uniform Resolution Conjecture (URC) (which is equivalent to Conjecture 6.3 (Maximal Rank Conjecture) of [Ha6]):

**Conjecture 3.2 (URC)**: *For $n > 9$ general points of $\mathbf{P}^2$, uniform unhindered ideals $I(m;n)$ are rank minimal.*

Example 5.2 gives various examples in which URC is now known to hold.

Whereas it is unclear how URC might be extended to the nonuniform case, QHC is, as we show in Remark 3.5, a special case of a more general conjecture, Conjecture 3.3 given below, in which no assumptions are made on the uniformity of the multiplicities $m_i$. For other equivalent variants of Conjecture 3.3, see [Ha1, Hi, Gi, Ha3]; for a nice survey, see [Mi].

**Conjecture 3.3**: *For each positive integer $d$ there is a nonempty open set $U_d$ of points $(p_1,\ldots,p_n) \in (\mathbf{P}^2)^n$ such that on the surface $X$ obtained by blowing up $p_1,\ldots,p_n$ we have the following (where $L$ is the total* *transform of a line):*

(i) *either $h^0(X, \mathcal{O}_X(F)) = 0$ or $h^1(X, O_X(F)) = 0$ for any numerically effective divisor $F$ with $F \cdot L \le d$;*

(ii) *if $C$ is a prime divisor on $X$ of negative self-intersection with $C \cdot L \le d$, then $C^2 = C \cdot K_X = -1$.*

**Remark 3.4**: As an aside, we mention how by assuming Conjecture 3.3 it is a simple matter to explicitly compute the Hilbert function of an ideal $I(\mathbf{m}; n)$. First, $I(\mathbf{m}; n)_t$ corresponds for each $t$ to the complete linear system $|D_t|$ on $X$ of the divisor $D_t = tL - m_1E_1 - \cdots - m_nE_n$, where $X \to \mathbf{P}^2$ is the blow up of the points $p_1, \ldots, p_n$, $L$ is the total transform to $X$ of a line in $\mathbf{P}^2$, and each $E_i$ is the exceptional locus of the blow up of $p_i$. Next, one subtracts off from $D_t$ irreducible effective divisors $C$ of the form $C^2 = C \cdot K_X = -1$ (all such $C$ are known by [N2]) for which $C \cdot D_t < 0$. Eventually a divisor $D$ is obtained from $D_t$ which either meets all such $C$ nonnegatively or which has $D \cdot L < 0$. In the latter case, $|D|$ is empty and $h^0(X, \mathcal{O}_X(D)) = 0$. In the former case, Conjecture 3.3(ii) implies either that $|D|$ is empty or $D$ is numerically effective, but by duality $h^2(X, \mathcal{O}_X(D)) = 0$ for any numerically effective divisor, so Conjecture 3.3(i) and Riemann-Roch allow one to compute $h^0(X, \mathcal{O}_X(D))$ and hence the Hilbert function of $I(\mathbf{m}; n)$.

**Remark 3.5**: Using the method of proof of Corollary 5.6 of [Ha6], we now show that Conjecture 3.3 implies Conjecture 3.1 (QHC). Using the notation of Remark 3.4, let $D_t = tL - m_1E_1 - \cdots - m_nE_n$, where by quasiuniformity $m_1 = \cdots = m_9$ (we will denote this common multiplicitiy by $m$), and $m \ge m_{10} \ge \cdots$. But $C = 3L - E_1 - \cdots - E_9$ is numerically effective (there is always an irreducible cubic through 9 general points), so, if $t \ge \alpha((m_1, \ldots, m_n); n)$, then $C \cdot D_t \ge 0$, hence $t \ge 3m$ so $D_t = -mK_X + (t-3m)L + (m-m_{10})E_{10} + \cdots + (m - m_n)E_n$. If $D_t$ were not numerically effective, then there would be a reduced, irreducible curve $A$ such that $A \cdot D_t < 0$, but then we would have $A^2 < 0$ and $A \cdot L \le t$, hence, by Conjecture 3.3, $A^2 = A \cdot K_X = -1$. Since $E_i \cdot D_t \ge 0$ for all $i$, we see $A$ is not $E_i$, but then $A \cdot D_t \ge A \cdot (-mK_X) = m \ge 0$, which shows that $D_t$ is numerically effective if $h^0(X, \mathcal{O}_X(D_t)) > 0$; thus $I((m_1, \ldots, m_n); n)$ is unhindered by Conjecture 3.3. To see that $I((m_1, \ldots, m_n); n)'$ is unhindered, take $D_t$ to be $tL - (m_1 + 1)E_1 - m_2E_2 - \cdots - m_nE_n$, where $t \ge \alpha((m_1 + 1, m_2, \ldots, m_n); n)$; then $C \cdot D_t \ge 0$ implies $3t \ge 9m + 1$ so $t \ge 3m + 1$ so $D_t = -mK_X + (t - 3m)L + (L - E_1) + (m - m_{10})E_{10} + \cdots + (m - m_n)E_n$ and the argument now proceeds as before. Finally, for $I((m_1, \ldots, m_n); n)''$, take $D_t = tL - (m_1 - 1)E_1 - m_2E_2 - \cdots - m_nE_n$, where $t \ge \alpha((m_1 - 1, m_2, \ldots, m_n); n)$; then $3t \ge 9m - 1$ so $t \ge 3m$ so $D_t = -mK_X + (t - 3m)L + E_1 + (m - m_{10})E_{11} + \cdots + (m - m_n)E_n$ and again we obtain the result.

## 4. When $n$ is not a square

**Proposition 4.1**: *Let $n > 9$ be a nonsquare. Then for any $n$ general points of $\mathbf{P}^2$, there are infinitely many $m$ such that if $I(m; n)$ and $I(m; n)'$ are unhindered, then $I(m; n)$ is rank minimal.*

**Proof**: We will use a criterion developed in the proof of Corollary 5.9 of [Ha6], which we briefly recall. By Lemma 2.6, if $I(m; n)$ is unhindered and $q(m; n) = 0$, then $I(m; n)$ is rank minimal. If $I(m; n)$ is unhindered, $\alpha(m; n)$ is the solution $x$ to the pair of inequalities $(x+1)(x+2) - n(m+1)m > 0$ and $(x+1)x - n(m+1)m \le 0$. If in addition $I(m; n)'$ is unhindered, then $q(m; n) = 0$ if and only if also $(x+1)(x+2) - n(m+1)m - 2(m+1) \le 0$. But $(x+1)(x+2) - n(m+1)m > 0$ implies that $x \ge m$, and since adding $2(x - m)$ to $(x+1)x - n(m+1)m$ gives $(x + 1)(x + 2) - n(m + 1)m - 2(m + 1)$, we see, assuming $I(m; n)$ and $I(m; n)'$ are unhindered, that $q(m; n) = 0$ if $(x + 1)(x + 2) - n(m + 1)m > 0$ and $(x + 1)(x + 2) - n(m + 1)m - 2(m + 1) \le 0$ have a simultaneous positive integer solution $x$. Let $0 < \epsilon < 1$; by easy but tedious arithmetic, one can check each real $x$ in the interval $[\sqrt{n}m + (\sqrt{n} - 3)/2, \sqrt{n}m + (\sqrt{n} - 3)/2 + \epsilon/\sqrt{n}]$ is, for $m$ sufficiently large (how large depending on $n$ and $\epsilon$), a solution to our pair of inequalities. Thus this interval containing an integer is a criterion for $q(m; n)$ to vanish. But by simplifying, there being an integer $x$ in this interval is equivalent to there being an integer $\eta = x + 1$ satisfying $0 < (2\eta + 1)/(2m + 1) - \sqrt{n} \le 2\epsilon/((2m + 1)\sqrt{n})$.

So now we consider $0 < (2\eta + 1)/(2m + 1) - \sqrt{n} \le 2\epsilon/((2m + 1)\sqrt{n})$. Since $n$ is not a square, we can write $n = a^2 + b$ with $0 < b \le 2a$, by taking $a = [\sqrt{n}]$ to be the integer part of $\sqrt{n}$. We now show that there are infinitely many pairs of odd integers $p, q$ such that $0 < p/q - \sqrt{a^2 + b} \le 2\epsilon/(q\sqrt{a^2 + b})$, thus completing the proof.

Let $f$ and $g$ be positive odd integers such that $f^2 - (a^2+b)g^2$ is positive. As is well known, Pell's equation, $z^2 - (a^2+b)y^2 = 1$, has a solution $z = c$, $y = d$ in positive integers, and we obtain additional solutions $z = u'$, $y = v'$ from $u' + v'\sqrt{a^2 + b} = (c + d\sqrt{a^2 + b})^t$. Moreover, whenever $t$ is even it is easy to check that $u'$ is odd and $v'$ is even. Now, taking $u + v\sqrt{a^2 + b} = (u' + v'\sqrt{a^2 + b})(f + g\sqrt{a^2 + b})$ we obtain a solution $z = u$, $y = v$ to $z^2 - (a^2 + b)y^2 = f^2 - (a^2 + b)g^2$ with $u$ and $v$ both odd. It follows that there are infinitely many such solutions. Moreover, $u/v - \sqrt{a^2 + b} = (f^2 - (a^2 + b)g^2)/(v^2(u/v + \sqrt{a^2 + b})) < (f^2 - (a^2 + b)g^2)/(v^2\sqrt{a^2 + b})$

which is clearly less than or equal to $2\epsilon/(v\sqrt{a^2+b})$ for $v$ sufficiently large. $\diamondsuit$

## 5. When $n$ is a square

Corollary 5.8 of [Ha6] shows that Proposition 4.1 also holds for odd squares $n > 9$. In this section, we will strengthen this result for odd squares. However, our main result in this section is Theorem 5.1(a), which by Lemma 5.3(i) is a precise formulation of the fact that for each sufficiently large $m$, if $I(m;n)$ is unhindered then it is rank minimal, when $n$ is an even square bigger than 9.

**Theorem 5.1**: *Let $n = r^2$ for $r \geq 3$.*

*(a) Assume $r$ is even. Then $I(m;n)$ is rank minimal for $m \geq (r-2)/4$ if $\alpha(m;n) = rm + r/2 - 1$.*

*(b) Assume $r$ is odd. Then $I(m;n)$ is rank minimal for $m \geq (r^2-9)/8$ if $I(m;n)$ and $I(m;n)'$ are unhindered.*

The proof is at the end of this section.

**Example 5.2**: The hypothesis of Theorem 5.1(a), that $\alpha(m;n) = rm + r/2 - 1$, is in several cases known to hold. It follows from [N1] and Lemma 5.3(i) that $\alpha(m;n) = rm + r/2 - 1$ for all $m > 0$ when $r = 4$. Thus URC holds by Theorem 5.1 for all $m \geq 0$ if $n = 16$. Additionally, [CM1, CM2] show that $I(m;n)$ is unhindered for any $n > 9$ general points when $m \leq 12$. Consequently, by Theorem 5.1(a) and Lemma 5.3(i), URC holds for $I(m;r^2)$ with $12 \geq m \geq (r-2)/4$ when $r > 3$ is even. In particular: for any even square $16 < n \leq 50^2 = 2500$, URC holds for $n$ general points taken with multiplicity 12; for any even square $16 < n \leq 46^2 = 2116$, URC holds for $n$ general points taken with multiplicity 11; $\ldots$; and for any even square $16 < n \leq 14^2 = 196$, URC holds for $n$ general points taken with multiplicity 3. (For $m = 2$ [Id] shows that URC holds for all $n > 9$.)

Our proof of Theorem 5.1 uses geometrical arguments. Thus we now will work on the blow-up $X$ of $\mathbf{P}^2$ at the $n$ points $p_1, \ldots, p_n$. Results in this geometrical setting directly translate back to the algebraic setting to which we have up to now mostly confined ourselves, and we refer the reader to, for example, section 3 of [Ha6], for the dictionary to do so. The reader will recall from Remark 3.5 and Remark 3.4 that $E_i$ is the exceptional locus of the blow up of $p_i$, and $L$ is the total transform of a line in $\mathbf{P}^2$. Then the divisor classes $[L], [E_1], \ldots, [E_n]$ give a basis of the divisor class group of $X$. We will denote the divisor $tL - m(E_1 + \cdots + E_n)$ by $F_{t,m}$, and the corresponding line bundle $\mathcal{O}_X(F_{t,m})$ by $\mathcal{F}_{t,m}$. Note that $[L] = [F_{1,0}]$ and $[F_{t,l}] + [F_{s,m}] = [F_{t+s,l+m}]$.

**Lemma 5.3**: *Let $X$ be the blow up of $\mathbf{P}^2$ at $n$ general points where $n = r^2$, and let $t$ and $m$ be nonnegative integers.*

*(i) Let $r$ be even.*

*(a) Then $h^1(X, \mathcal{F}_{t,m}) = 0$ and $h^0(X, \mathcal{F}_{t,m}) > 0$ for all $t \geq rm + (r-2)/2$.*

*(b) If $m \geq (r-2)/4$ and $0 \leq t < rm + (r-2)/2$, then $h^0(X, \mathcal{F}_{t,m}) - h^1(X, \mathcal{F}_{t,m}) \leq 0$.*

*(ii) Let $r$ be odd.*

*(a) Then $h^1(X, \mathcal{F}_{t,m}) = 0$ and $h^0(X, \mathcal{F}_{t,m}) > 0$ for all $t \geq rm + (r-3)/2$.*

*(b) If $m \geq (r-1)(r-3)/(8r)$ and $0 \leq t < rm + (r-3)/2$, then $h^0(X, \mathcal{F}_{t,m}) - h^1(X, \mathcal{F}_{t,m}) \leq 0$.*

**Proof**: Note for $t \geq 0$, that $h^2(X, \mathcal{F}_{t,m}) = h^0(X, \mathcal{F}_{-3-t,1-m})$ by duality, and $h^0(X, \mathcal{F}_{-3-t,1-m})$ vanishes, since $-3-t < 0$. Thus for both (i)(b) and (ii)(b) we have $h^0(X, \mathcal{F}_{t,m}) - h^1(X, \mathcal{F}_{t,m}) = (F_{t,m}^2 - K_X \cdot F_{t,m} + 2)/2$ by Riemann-Roch. Now $F_{t,m}^2 - K_X \cdot F_{t,m} + 2 = (t+2)(t+1) - r^2 m(m+1)$ is an increasing function of $t$ for $t \geq 0$, so to prove (i)(b) and (ii)(b) it is enough to take $t = rm + (r-2)/2 - 1$ when $r$ is even and $t = rm + (r-3)/2 - 1$ when $r$ is odd. For $t = rm + (r-2)/2 - 1$, $(t+2)(t+1) - r^2 m(m+1)$ becomes $(r/4)(r - 4m - 2)$, which is nonpositive for $m \geq (r-2)/4$. This proves (i)(b). For $t = rm + (r-3)/2 - 1$, $(t+2)(t+1) - r^2 m(m+1)$ becomes $-2rm + (r-3)(r-1)/4$, which is nonpositive for $m \geq (r-3)(r-1)/(8r)$. This proves (ii)(b).

To prove both parts (a), consider a specialization of the $r^2$ points to the case of general points $p_i$ on a smooth plane curve $C'$ of degree $r$, and let $C$ be the proper transform to $X$ of $C'$. In the case that $r$ is even (so $t \geq rm + (r-2)/2$), the restriction $\mathcal{F}_{t,m} \otimes \mathcal{O}_C$ to $C$ has degree at least $(r-2)/2 + g$, where $g$ is the genus of $C$. But $\mathcal{F}_{t,m} \otimes \mathcal{O}_C$ is a general bundle of its degree (the points $p_i$ being general points on $C'$), so $\mathcal{F}_{t,m}^{-1} \otimes \mathcal{K}_C$ is a general bundle of degree at most $g - r/2 - 1$, and thus has no nontrivial global sections (since $\mathrm{Pic}^0(C)$ has dimension $g$, there are more line bundles than there are effective divisors for any given degree less than $g$). Thus $h^1(C, \mathcal{F}_{t,m} \otimes \mathcal{O}_C) = 0$ by duality. Since (i)(a) is true for $m = 0$, our result follows for all $m \geq 0$ by induction by taking global sections of the sequence

$$0 \to \mathcal{F}_{t-r,m-1} \to \mathcal{F}_{t,m} \to \mathcal{F}_{t,m} \otimes \mathcal{O}_C \to 0$$

obtained by restriction. The result for general points of $\mathbf{P}^2$ (rather than general points of $C'$) now follows by semicontinuity. Case (ii)(a), that $r$ is odd, is similar, except now $t \geq rm + (r-3)/2$ and $\mathcal{F}_{t,m} \otimes \mathcal{O}_C$ has degree at least $g - 1$. $\diamondsuit$

**Remark 5.4**: To justify our appeal to semicontinuity in the preceding proof, we can appeal to flat families, using results of [Ha2]. Alternatively, consider any $n$ nonnegative multiplicities $m_1, \ldots, m_n$ and any $n$ distinct points $p_1, \ldots, p_n \in \mathbf{P}^2$, with their corresponding ideals $P_i$. We may assume that none of the ideals $P_i \subset k[x, y, z]$ contain $z$. Let $I = \cap_{i=1}^n P_i^{m_i}$ and let $V_i$ be the vector space span in $k[x, y]$ of the monomials of degree less than $m_i$. For each degree $t$ and each point $p_i$ we have a linear map $\lambda_{ti} : R_t \to V_i$, in which a homogeneous polynomial $f(x, y, z)$ of degree $t$ is first sent to $f(x + x_1, y + y_1, 1)$, where $(x_1, y_1)$ are affine coordinates for $p_i$ (taking $z = 0$ to be the line at infinity), and then $f(x + x_1, y + y_1, 1)$ is truncated to drop all terms of degree $m_i$ or more. With $V$ taken to be $V_1 \times \cdots \times V_n$, define $\Lambda_t : R_t \to V$ to be the map $\lambda_{t1} \times \cdots \times \lambda_{tn}$. Thus the kernel of $\Lambda_t$ is $I_t$, and the entries for the matrix defining $\Lambda_t$ in terms of bases of monomials are polynomials in the coordinates of the points $p_i$. If $X$ is the blow up of $\mathbf{P}^2$ at the points $p_i$, and if we set $\mathcal{F} = \mathcal{O}_X(F)$ where $F = tL - m_1E_1 - \cdots - m_nE_n$, then $h^0(X, \mathcal{F}) = \dim I_t$ and by Riemann-Roch $h^0(X, \mathcal{F}) - h^1(X, \mathcal{F}) = \dim R_t - \dim V$. Since $\dim I_t = \dim R_t - l$ where $l$ is the rank of $\Lambda_t$, we see that $h^1(X, \mathcal{F}) = \dim V - l$. But for any given $l$, $\Lambda_t$ having rank at least $l$ is an open condition on the set of all $n$-tuples $(p_1, \ldots, p_n)$ of distinct points with none at infinity, and hence $h^1(X, \mathcal{F})$ is also semicontinuous.

Our proof of Theorem 5.1(a) involves examining certain specializations of the $r^2$ points. The first specialization we will consider is easiest to specify as a subset of the affine plane $\mathbf{A}^2$, rather than of $\mathbf{P}^2$. Regard $\mathbf{A}^2$ as $k^2$, where $k$ is the ground field. Consider $r$ distinct vertical lines $V_1, \ldots, V_r$ in $k^2$ and $s = 3r/2 - 1$ distinct horizontal lines $H_1, \ldots, H_s$. Let $p_{ij}$, for each $1 \leq i \leq r$ and $1 \leq j \leq s$, be the point of intersection of $V_i$ with $H_j$. We choose our $r^2$ points in blocks from among these $rs$ points $p_{ij}$. The first block is $B_1 = \{p_{ij} : 1 \leq i \leq 2, 1 \leq j \leq s\}$, the second is $B_2 = \{p_{ij} : 3 \leq i \leq 4, 1 \leq j \leq s - 2\}$, etc., and the last block is $B_{r/2} = \{p_{ij} : r - 1 \leq i \leq r, 1 \leq j \leq s - (r - 2)\}$. Note that all together, the union of these $r/2$ blocks contains $2s + \cdots + 2(s - (r - 2)) = 2(sr/2 - (2 + 4 + \cdots + (r - 2)) = sr - 4(1 + \cdots + (r/2 - 1)) = (3r/2 - 1)r - 4(r/2 - 1)(r/2)/2 = r^2$ points. It will be convenient to denote $rm + (r - 2)/2$ by $t_m$ and by $\mu_{t_m}$ the map $H^0(X, \mathcal{F}_{t_m,m}) \otimes H^0(X, \mathcal{F}_{1,0}) \to H^0(X, \mathcal{F}_{t_m+1,m})$ given by multiplication.

**Proposition 5.5**: *With respect to the configuration $B_1 \cup \cdots \cup B_{r/2}$ of $r^2$ points of $\mathbf{P}^2$ specified above, $\mu_{t_1}$ is surjective and $h^1(X, \mathcal{F}_{t_1,1}) = 0$.*

**Proof**: Apply the method and results of [GGR]. Let $I \subset R$ be the homogeneous ideal of the $r^2$ points in the homogeneous coordinate ring $R$ of $\mathbf{P}^2$. Let $f \in R$ be a homogeneous element of degree 1 not vanishing on any of the $r^2$ points; thus the image of $f$ in the quotient $R/I$ is not a zero divisor. Hence, as discussed in [GGR], $J = I + (f)/(f)$ has the same minimal number of homogeneous generators in every degree as does $I$, but $J$ is a monomial ideal which, by the discussion in [GGR], is easy to handle explicitly. The result in our case is that there are no generators in degrees greater than $3r/2 - 1$, hence the same is true for $I$, which shows that $\mu_{t_1}$ is surjective, as required. To see that $h^1(X, \mathcal{F}_{t_1,1}) = 0$, it is enough to check that the points impose independent conditions on forms of degree $t_1$; i.e., that $h^0(X, \mathcal{F}_{t_1,1}) = (t_1 + 2)(t_1 + 1)/2 - r^2$. But the fact that $J = I + (f)/(f)$ has the same number of generators in every degree as does $I$ means that $I$ has one generator in degree $r$ and $r/2$ in degree $t_1 = 3r/2 - 1$. Thus the dimension of $I$ in degree $t_1$ is $(r/2) + ((r/2)(r/2 + 1)/2)$, which is indeed $(t_1 + 2)(t_1 + 1)/2 - r^2$.$\diamondsuit$

**Proposition 5.6**: *Let $r \geq 0$ be even with $m \geq 0$. Then for general points $p_1, \ldots, p_{r^2}$ of a general smooth plane curve $C'$ of degree $r$, taking $X$ to be the blow up of $\mathbf{P}^2$ at $p_1, \ldots, p_{r^2}$, the map $\mu_{t_m}$ is surjective and $h^1(X, \mathcal{F}_{t_m,m}) = 0$ for all $m \geq 0$.*

**Proof**: First let $C'$ be any smooth plane curve of degree $r$ and let $C$ be the proper transform of $C'$ to $X$. As in the proof of Lemma 5.3, we have the exact sequence $0 \to \mathcal{F}_{t_m-r,m-1} \to \mathcal{F}_{t_m,m} \to \mathcal{F}_{t_m,m} \otimes \mathcal{O}_C \to 0$, and $h^1(X, \mathcal{F}_{t_m,m}) = 0$ for all $m \geq 0$. The exact sequence leads by the snake lemma (see [Mu] or [Ha5]) to an exact sequence

$$(*) \qquad \text{cok }(\mu_{t_m-r}) \to \text{cok }(\mu_{t_m}) \to \text{cok }(\mu_{C,t_m}) \to 0,$$

where $\mu_{C,t_m}$ is the map $H^0(C, \mathcal{F}_{t_m,m} \otimes \mathcal{O}_C) \otimes H^0(X, \mathcal{F}_{1,0}) \to H^0(X, \mathcal{F}_{t_m+1,m} \otimes \mathcal{O}_C)$. But $\mathcal{F}_{t_m,m} \otimes \mathcal{O}_C$ is a general bundle of degree $r(r-2)/2$, regardless of $m$, so if cok $(\mu_{C,t_1}) = 0$, then cok $(\mu_{C,t_m}) = 0$ for all $m \geq 1$, and cok $(\mu_{C,t_1}) = 0$ follows from $(*)$ if we show that cok $(\mu_{t_1}) = 0$. But cok $(\mu_{t_1}) = 0$ and $h^1(X, \mathcal{F}_{t_1,1}) = 0$ for the configuration of points given in Proposition 5.5, and these points are points of a plane curve $C'$ of degree $r$ (the union of $r$ lines), so by semicontinuity (see Remark 5.7) cok $(\mu_{t_1}) = 0$ holds for general points of a general curve $C'$ of degree $r$.

Finally, induction using $(*)$ gives cok $(\mu_{t_m}) = 0$ for all $m \geq 0$. (Note that $\mu_{t_m-r} = \mu_{t_{m-1}}$ so $\mu_{t_m-rm} = \mu_{t_0}$, and cok $(\mu_{t_m-rm}) = 0$ since $\mathcal{F}_{t_m-rm,0} = \mathcal{F}_{t_0,0}$ can be regarded as $\mathcal{O}_{\mathbf{P}^2}(((r-2)/2)L)$ on $\mathbf{P}^2$, where the result is obvious.) $\diamondsuit$

**Remark 5.7**: We now show that the requirements that cok $(\mu_{t_1}) = 0$ and $h^1(X, \mathcal{F}_{t_1,1}) = 0$ together impose an open condition on $r^2$-tuples of points $(p_1, \ldots, p_{r^2})$. More generally, let $I$ and $\mathcal{F}$ be as in Remark 5.4. As there, we have the map $\Lambda_t : R_t \to V$. Then $I_t \otimes R_1$ is the kernel of $\Lambda_t \otimes \mathrm{id}_{R_1} : R_t \otimes R_1 \to V \otimes R_1$ and the kernel of $\mu_t : I_t \otimes R_1 \to I_{t+1}$ is also the kernel of $\gamma : R_t \otimes R_1 \to (V \otimes R_1) \oplus R_{t+1}$, where $\gamma$ is $(\Lambda_t \otimes \mathrm{id}_{R_1}) \oplus \mu'_t$, and where $\mu_t : I_t \otimes R_1 \to I_{t+1}$ and $\mu'_t : R_t \otimes R_1 \to R_{t+1}$ are given by multiplication. (This is consistent with our usage above, since sections of line bundles on blow ups of $\mathbf{P}^2$ can be identified with subspaces of $R_t$ for appropriate $t$.) Since, as in Remark 5.4, $\gamma$ can be given by a matrix whose entries are polynomials in the coordinates of the points $p_i$, the rank of $\gamma$ and hence the dimension of ker $(\mu_t)$ is semicontinuous. But dim cok $(\mu_t) = h^0(X, \mathcal{F}(L)) - 3h^0(X, \mathcal{F}) +$dim ker $(\mu_t)$, which is dim cok $(\mu_t) = t + F \cdot K_X - F^2 +$dim ker $(\mu_t)$ if $h^1(X, \mathcal{F}) = 0$, and since $h^1(X, \mathcal{F})$ is also semicontinuous, it follows that it is an open condition to require that both cok $(\mu_t)$ and $h^1(X, \mathcal{F})$ vanish.

**Proof** (of Theorem 5.1): (a) Assume that $\alpha(m;n) = rm + r/2 - 1$. Then by definition of $\alpha(m;n)$, we have $I(m;n)_t = 0$ for $t < rm + r/2 - 1$ and by Lemma 5.3(i)(a) we have $h^1(X, \mathcal{F}_{t,m}) = 0$ for $t \geq rm + r/2 - 1$. Thus $I(m;n)$ is unhindered. Moreover, since $I(m;n)_t = 0$ for $t < rm + r/2 - 1$, there are no homogeneous generators for $I(m;n)$ in degrees less than $rm + r/2 - 1$ and there are $h_{I(m;n)}(\alpha(m;n))$ generators in degree $rm + r/2 - 1$. Clearly, then, $I(m;n)$ is rank minimal if $F_0$ has rank $h_{I(m;n)}(\alpha(m;n))$, so it is enough to prove that these $h_{I(m;n)}(\alpha(m;n))$ elements of degree $rm + r/2 - 1$ generate $I(m;n)$.

By Lemma 5.3(i), the Castelnuovo-Mumford regularity of $I(m;n)$ is at most $\alpha(m;n)+1$, so no generators need be taken in degrees greater than $rm + r/2$ (see [DGM] or Lemma 2.9 of [Ha5]). Hence we now need only show that no generators need be taken in degree $rm + r/2$; i.e., that $\mu_{t_m} : H^0(X, \mathcal{F}_{t_m,m}) \otimes H^0(X, \mathcal{F}_{1,0}) \to H^0(X, \mathcal{F}_{t_m+1,m})$ is surjective, where, as in Proposition 5.6, $t_m = rm + r/2 - 1$. But this follows by Proposition 5.6 and semicontinuity (cf. Remark 5.7).

(b) (We note that when $n = r^2 > 9$ is an odd square, [Ha6] noted but did not explicitly show that $I(m;r^2)$ is rank minimal for all but finitely many $m$ for which $I(m;r^2)$ and $I(m;r^2)'$ are unhindered.) By Lemma 2.6(b), if $I(m;r^2)$ is unhindered and $q(m;r^2) = 0$, then $I(m;r^2)$ is rank minimal. But Lemma 5.3(ii) implies that $\alpha(m;r^2) = rm + (r-3)/2$ for $m \geq (r-1)(r-3)/(8r)$. Now, $q(m;r^2) = h_{I(m;n)'}(\alpha(m;r^2))$, and using $t = \alpha(m;r^2)$ and assuming that $I(m;r^2)'$ is unhindered, we have $h_{I(m;n)'}(\alpha(m;r^2)) = \max\{0, (t+2)(t+1)/2 - (r^2-1)m(m+1)/2 - (m+1)(m+2)/2\}$. But $(t+2)(t+1)/2 - (r^2-1)m(m+1)/2 - (m+1)(m+2)/2 \leq 0$ for $r \geq 3$ with $m \geq (r^2-9)/8$, and since $(r^2-9)/8 \geq (r-1)(r-3)/(8r)$ when $r \geq 3$, we see that $I(m;r^2)$ is rank minimal if $m \geq (r^2-9)/8$ for $r \geq 3$, whenever $I(m;r^2)$ and $I(m;r^2)'$ are unhindered. $\diamondsuit$

## 6. Additional Results

Our results on resolutions of uniform ideals in certain cases extend to quasiuniform ideals. This raises the question of whether quasiuniform ideals involving general points of $\mathbf{P}^2$ are always rank minimal; we know of no counterexamples.

**Proposition 6.1**: *Assume $q(m;r) = 0$ for some $m$ and some $r \geq 2$. Let $n \geq r$ and let $\mathbf{m} = (m_1, \ldots, m_n)$ where $m = m_1 = \cdots = m_r$ and $\sum_{i>r}(m_i^2 + m_i)/2 < h_{I(m;r)}(\alpha(m;r))$. If $I(\mathbf{m};n)$ is unhindered, then $I(\mathbf{m};n)$ is rank minimal.*

**Proof**: By the hypothesis $\sum_{i>r}(m_i^2 + m_i)/2 < h_{I(m;r)}(\alpha(m;r))$, we see $\alpha(m;r) = \alpha(\mathbf{m};n)$, so $q(m;r) = 0$ implies $q(\mathbf{m};n) = 0$, and the result follows by Lemma 2.6(b). $\diamondsuit$

Note that if $r \geq 9$ in Proposition 6.1, then, as conjectured by QHC, we expect that $I(\mathbf{m};n)$ is indeed unhindered if $\mathbf{m}$ is quasiuniform. Similarly, we expect that the hypotheses of unhinderedness in the following result always hold (since after reordering $m_i$, if need be, for $i > r^2$, $\mathbf{m}$ is quasiuniform).

**Corollary 6.2**: *Let $n \geq r^2$ where $r \geq 3$ is odd and let $m = m_1 = \cdots = m_{r^2} \geq (r^2-9)/8 \geq \sum_{r^2<i\leq n}(m_i^2 + m_i)/2$. If, for $\mathbf{m} = (m_1,\ldots,m_n)$, $I(\mathbf{m};n)$, $I(m;r^2)$ and $I(m;r^2)'$ are unhindered, then $I(\mathbf{m};n)$ is rank minimal.*

**Proof**: As in the proof of Theorem 5.1(b), $I(m;r^2)$ and $I(m;r^2)'$ being unhindered implies $q(m;r^2)=0$. From Lemma 5.3(ii) we find $h_{I(m;r^2)}(\alpha(m;r^2)) = (r^2-1)/8$, so Proposition 6.1 gives the result. $\diamondsuit$

We close with a result involving no conditional hypotheses of unhinderedness.

**Proposition 6.3**: *Let $m > 0$, let $t \geq 0$, and choose $n \geq 9$ such that $-m-1 \leq n-[9+3tm+(t+1)(t+2)/2] \leq m$. Then $I(\mathbf{m};n)$ is rank minimal for $\mathbf{m} = (m_1,\ldots,m_n)$ with $m = m_1 = \cdots = m_9$ and $m_i = 1$ for $9 < i \leq n$.*

**Proof**: It is known (see [Ha4]) that Conjecture 3.3 holds for 9 general points, and thus (by Remark 3.5) so does QHC, but general simple points impose independent conditions, so QHC also holds when $\mathbf{m}$ satisfies the hypotheses of Proposition 6.3. Thus $I(\mathbf{m};n)$, $I(\mathbf{m};n)'$ and $I(\mathbf{m};n)''$ are unhindered.

Given this, if $8+(3t-1)m+(t+1)(t+2)/2 \leq n \leq 8+3tm+(t+1)(t+2)/2$, we can easily check that $\alpha(\mathbf{m};n) = 3m+t$ and then that $q(\mathbf{m};n)=0$, whereas if $9+3tm+(t+1)(t+2)/2 \leq n \leq 9+(3t+1)m+(t+1)(t+2)/2$, then $\alpha(\mathbf{m};n) = 3m+t+1$ and $l(\mathbf{m};n) > 0$. Now the result follows by Lemma 2.6. $\diamondsuit$

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