

Free Resolutions of Fat Point Ideals on $\mathbf{P}^2$

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Minimal free resolutions for homogeneous ideals corresponding to certain 0-dimensional subschemes of $\mathbf{P}^2$ defined by sheaves of complete ideals are determined implicitly. All work is over an algebraically closed field of arbitrary characteristic.

I. Introduction

Given distinct points $p_1, \dots, p_r$ of a smooth variety V (over an algebraically closed field k) and positive integers m_i , $Z = m_1p_1 + \dots + m_rp_r$ denotes the subscheme defined locally at each point p_i by $\mathcal{I}_i^{m_i}$, where $\mathcal{I}_i$ is the maximal ideal in the local ring $\mathcal{O}_{V,p_i}$ at p_i of the structure sheaf. More briefly, we say Z is a *fat point subscheme* of V . In the case that V is $\mathbf{P}^n$ for some n , it is of interest to study the homogeneous ideal I_Z defining Z as a subscheme of $\mathbf{P}^n$; I_Z is called an *ideal of fat points*.

Given an ideal I_Z of a fat point subscheme $Z \subset \mathbf{P}^n$, one first may want to determine its Hilbert function h_{I_Z} , defined for each d by $h_{I_Z}(d) = \dim_k((I_Z)_d)$, where $(I_Z)_d$ is the homogeneous component of I_Z of degree d (i.e., $(I_Z)_d$ is the k -vector space of all homogeneous forms of degree d in I_Z). One next may wish to study a minimal free resolution $\dots \rightarrow F_1 \rightarrow F_0 \rightarrow I_Z \rightarrow 0$ of I_Z , beginning with determining F_0 . Determining F_0 as a graded module over the homogeneous coordinate ring $k[\mathbf{P}^n]$ of $\mathbf{P}^n$ is equivalent to finding the number $\nu_d(Z)$ of generators in each degree d in a minimal homogeneous set of generators for I_Z , since $F_0 = \bigoplus_d R[-d]^{\nu_d}$ (where R denotes $k[\mathbf{P}^n]$ graded by total degree, and $R[-d]$ signifies that the degree has been shifted such that constants have degree d).

The Hilbert function h_{I_Z} in the case that the points $p_1, \dots, p_r \subset \mathbf{P}^n$ are general has attracted attention (see [16] in general, or [8], [15], [5] and [9] for $n = 2$) but much remains conjectural. Most work done on minimal free resolutions of I_Z has been restricted to the case that Z is a smooth union of general points (cf. [17]). More can be said in the case of subschemes of $\mathbf{P}^2$ involving small numbers of points or points in special position. For example, by [11], h_{I_Z} is completely understood for any fat point subscheme $Z = m_1p_1 + \dots + m_rp_r \subset \mathbf{P}^2$ where $p_1, \dots, p_r$ are points (even possibly infinitely near) of a plane cubic (possibly reducible and nonreduced), by [2] one can determine a minimal homogeneous set of generators for I_Z in case $p_1, \dots, p_r$ lie on a smooth plane conic, and by [13] one can determine a minimal homogeneous set of generators for I_Z in case $Z = m(p_1 + \dots + p_r)$, where $p_1, \dots, p_r$ are $r \leq 9$ general points of $\mathbf{P}^2$ ([13] also conjectures a result for $r > 9$).

Here we will be concerned with determining minimal homogeneous sets of generators for ideals I_Z where $Z = m_1p_1 + \dots + m_rp_r \subset \mathbf{P}^2$ in the case that $p_1, \dots, p_r$ lie on a plane curve of degree at most 3. We obtain results in some special cases in the case of a smooth curve of degree 3, but we obtain complete results for curves of degrees 1 and 2. (Our results for points on a conic are distinguished from those of [2] in that we consider arbitrary conics and we allow infinitely near points.)

Since a fat point ideal I_Z is perfect, one feature of working on $\mathbf{P}^2$ is that a minimal free resolution of I_Z is of the form $0 \rightarrow F_1 \rightarrow F_0 \rightarrow I_Z \rightarrow 0$ for some graded free modules F_0 and F_1 . Moreover, F_0 and F_1 are determined by the Hilbert function h_{I_Z} of I_Z and a minimal homogeneous set of generators for I_Z . In

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particular, we saw above that $F_0 = \bigoplus_d R[-d]^{\nu_d}$ (where here $R = k[\mathbf{P}^2]$) which means the Hilbert function h_{F_0} of F_0 is $h_{F_0}(n) = \sum_d \nu_d \binom{n+2-d}{2}$. But the Hilbert function of F_1 is determined by the Hilbert functions of I_Z and F_0 , and the Hilbert function of F_1 determines F_1 , since any finitely generated free graded module is determined by its Hilbert function.

Thus on $\mathbf{P}^2$ the problem of determining the modules in a resolution for I_Z reduces to finding h_{I_Z} and to computing for each d the number ν_d of generators of degree d in a minimal homogeneous set of generators for I_Z .

Another feature of working on $\mathbf{P}^2$ is the ease with which one can extend the notion of fat point subschemes supported at distinct points to include the possibility of infinitely near points. In fact, it is this extended notion of fat point subschemes that we will use in this paper and which we now introduce.

We first put the notion of fat point subscheme into a context which will make our extended notion natural. Let $p_1, \dots, p_r \subset \mathbf{P}^2$ be distinct points of the plane. Let $\pi : X \rightarrow \mathbf{P}^2$ be the blowing up of each of the points. Then the divisor class group $\text{Cl}(X)$ (elements of which, when convenient, we will identify with the corresponding associated invertible sheaves) of X is a free abelian group of rank $r + 1$, with basis $e_0, \dots, e_r$, where e_0 is the pullback to X of the class of a line, and $e_1, \dots, e_r$ are the classes of the exceptional divisors $E_1, \dots, E_r$ of the blowings up of the points $p_1, \dots, p_r$. Let $\mathcal{I}_Z$ be the sheaf of ideals of some fat point subscheme $Z = m_1 p_1 + \dots + m_r p_r$, and let $\mathcal{F}_n$ denote $ne_0 - m_1 e_1 - \dots - m_r e_r$. Then $\mathcal{I}_Z = \pi_*(\mathcal{F}_0)$, and $\dim((I_Z)_d) = h^0(\mathbf{P}^2, \mathcal{I}_Z \otimes \mathcal{O}_{\mathbf{P}^2}(d)) = h^0(X, \mathcal{F}_d)$ for every d (so I_Z is isomorphic as a graded module to $\bigoplus_d H^0(X, \mathcal{F}_d)$), and in fact $h^i(\mathbf{P}^2, \mathcal{I}_Z \otimes \mathcal{O}_{\mathbf{P}^2}(d)) = h^i(X, \mathcal{F}_d)$ for every i .

Moreover, $(I_Z)_d \otimes R_1 \rightarrow (I_Z)_{d+1}$ corresponds to $H^0(X, \mathcal{F}_d) \otimes H^0(X, e_0) \rightarrow H^0(X, \mathcal{F}_{d+1})$, and so the kernel and cokernel of the former have the same dimension as for the latter. Given divisor classes $\mathcal{G}$ and $\mathcal{H}$, we will denote the kernel and cokernel of $H^0(X, \mathcal{G}) \otimes H^0(X, \mathcal{H}) \rightarrow H^0(X, \mathcal{G} + \mathcal{H})$ by $\mathcal{R}(\mathcal{G}, \mathcal{H})$ and $\mathcal{S}(\mathcal{G}, \mathcal{H})$, respectively, and their dimensions by $\mathcal{R}(\mathcal{G}, \mathcal{H})$ and $\mathcal{S}(\mathcal{G}, \mathcal{H})$. Thus $h_{I_Z}(d) = h^0(X, \mathcal{F}_d)$ and $\nu_{d+1}(Z) = \mathcal{S}(\mathcal{F}_d, e_0)$, so we see the Hilbert function of I_Z and the number of generators of I_Z in each degree can be found by studying invertible sheaves on X . And indeed, this is the approach we take in this paper.

To subsume the case of infinitely near points we now define the notion of *essentially distinct* points. Let $p_1 \in X_0 = \mathbf{P}^2$, and let $p_2 \in X_1, \dots, p_r \in X_{r-1}$, where, for $0 \leq i \leq r-1$, $\pi_i : X_{i+1} \rightarrow X_i$ is the blowing up of p_{i+1} . We will denote X_r by X and the composition $X \rightarrow \mathbf{P}^2$ by π . We call the indexed points $p_1, \dots, p_r$ *essentially distinct* points of $\mathbf{P}^2$; note that p_j for $j > i$ may be infinitely near p_i . Denoting the class of the 1-dimensional scheme-theoretic fiber E_i of $X_r \rightarrow X_i$ by e_i and the pullback to X_r of the class of a line in $\mathbf{P}^2$ by e_0 , we have, as in the case of distinct points, the basis $e_0, \dots, e_r$ of the divisor class group of X corresponding to $p_1, \dots, p_r$, and which we will call an *exceptional configuration*. Then $\pi_*(-m_1 e_1 - \dots - m_r e_r)$ is a coherent sheaf of ideals on $\mathbf{P}^2$ defining a 0-dimensional subscheme Z generalizing the usual notion of fat point subscheme. In analogy with the notation used above, we will denote Z by $m_1 p_1 + \dots + m_r p_r$ and refer to Z as a fat point subscheme. As an aside we also note that the stalks of $\pi_*(-m_1 e_1 - \dots - m_r e_r)$ are complete ideals in the stalks of the local rings of the structure sheaf of $\mathbf{P}^2$, and that conversely if $\mathcal{I}$ is a coherent sheaf of ideals on $\mathbf{P}^2$ whose stalks are complete ideals and if $\mathcal{I}$ defines a 0-dimensional subscheme, then there are essentially distinct points $p_1, \dots, p_r$ of $\mathbf{P}^2$ and integers m_i such that with respect to the corresponding exceptional configuration we have $\mathcal{I} = \pi_*(-m_1 e_1 - \dots - m_r e_r)$. Thus our extended notion of fat points is precisely what is obtained by considering 0-dimensional subschemes defined by coherent sheaves of ideals whose stalks are complete ideals.

Allowing the possibility of infinitely near points necessitates dealing with certain technicalities. In particular, the subscheme Z does not uniquely determine $-m_1 e_1 - \dots - m_r e_r$. For example, if p_1 and p_2 are distinct points of $\mathbf{P}^2$, then $\pi_*(-e_1 + e_2) = \pi_*(-e_1)$ both give the sheaf of ideals defining the subscheme $Z = p_1$. To get uniqueness, we recall that the divisor class group of X supports an intersection form, with respect to which the exceptional configuration $e_0, \dots, e_r$ is orthogonal with $-1 = -e_0^2 = e_1^2 = \dots = e_r^2$. The inequalities $(-m_1 e_1 - \dots - m_r e_r) \cdot C_{ij} \geq 0$, where the index i runs over the divisors E_i and j runs over the components C_{ij} of E_i , corresponds to what older terminology called the *proximity inequalities*. Thus we will say that a divisor class $\mathcal{F}$ on X satisfies the proximity inequalities if $\mathcal{F} \cdot C \geq 0$ for every component C of each divisor E_i . Moreover, given essentially distinct points $p_1, \dots, p_r$ and a subscheme $Z = m_1 p_1 + \dots + m_r p_r$, we will abbreviate saying that the class $-m_1 e_1 - \dots - m_r e_r$ coming from the coefficients $m_1, \dots, m_r$ used to define Z satisfies the proximity inequalities by simply saying that $m_1 p_1 + \dots + m_r p_r$ satisfies the proximity inequalities. Uniqueness is now a consequence of the fact that if $\pi_*(-a_1 e_1 - \dots - a_r e_r) = \pi_*(-b_1 e_1 - \dots - b_r e_r)$,

where $-a_1e_1 - \dots - a_re_r$ and $-b_1e_1 - \dots - b_re_r$ both satisfy the proximity inequalities, then $a_i = b_i$ for each i . In particular, we have a bijection between subschemes of fat points in $\mathbf{P}^2$ and 0-cycles $m_1p_1 + \dots + m_rp_r$, where $p_1, \dots, p_r$ are essentially distinct points of $\mathbf{P}^2$ and $m_1p_1 + \dots + m_rp_r$ satisfies the proximity inequalities.

From another point of view, the significance of the proximity inequalities is given by an old and well known result saying that the linear system of sections of $de_0 - m_1e_1 - \dots - m_re_r$ is fixed component free for d sufficiently large if (and only if) $-m_1e_1 - \dots - m_re_r$ satisfies the proximity inequalities. The proximity inequalities also manifest themselves even in the usual case that $p_1, \dots, p_r$ are distinct points of $\mathbf{P}^2$, since in this case $m_1p_1 + \dots + m_rp_r$ satisfying the proximity inequalities just means that each coefficient m_i is nonnegative, which is generally taken for granted without comment.

We now discuss in more detail the approach we take in this paper. Let $Z = m_1p_1 + \dots + m_rp_r$ be a fat point subscheme of $\mathbf{P}^2$, satisfying the proximity inequalities. Let $e_0, \dots, e_r$ be the exceptional configuration corresponding to the essentially distinct points $p_1, \dots, p_r$. For each d , let $\mathcal{F}_d$ be the class $de_0 - m_1e_1 - \dots - m_re_r$. As we have seen above, determining the graded modules in a minimal free resolution of I_Z amounts to computing $h^0(X, \mathcal{F}_d)$ and $s(\mathcal{F}_d, e_0)$ for each d . If $\mathcal{F}_d$ is not the class of an effective divisor, then clearly $h^0(X, \mathcal{F}_d) = 0$ and $s(\mathcal{F}_d, e_0) = h^0(X, \mathcal{F}_{d+1})$. If, however, $\mathcal{F}_d$ is the class of an effective divisor, then $\mathcal{F}_d$ has a *Zariski decomposition*: $\mathcal{F}_d = \mathcal{H} + \mathcal{N}$, where $h^0(X, \mathcal{N}) = 1$ and $\mathcal{H}$ is *numerically effective* (i.e., meets every effective divisor nonnegatively) with $h^0(X, \mathcal{F}_d) = h^0(X, \mathcal{H})$. By Lemma II.10, $s(\mathcal{F}_d, e_0) = s(\mathcal{H}, e_0) + h^0(X, \mathcal{F}_{d+1}) - h^0(X, \mathcal{H} + e_0)$. Thus, to implicitly determine a resolution of I_Z , it is enough to determine: the monoid EFF of divisor classes of effective divisors; a Zariski decomposition for each class in EFF; and $h^0(X, \mathcal{H})$ and $s(\mathcal{H}, e_0)$ for each class $\mathcal{H}$ in the cone NEFF of numerically effective classes.

If X is any smooth projective rational surface for which $-K_X$ is effective, then EFF and NEFF can be found, as can a Zariski decomposition for any class in EFF and $h^0(X, \mathcal{H})$ for any numerically effective class $\mathcal{H}$, by applying the results of [11]. In particular, since in terms of an exceptional configuration $e_0, \dots, e_r$ on a blowing up X of $\mathbf{P}^2$ at essentially distinct points $p_1, \dots, p_r$ we always have $-K_X = 3e_0 - e_1 - \dots - e_r$, we see $-K_X$ is the class of an effective divisor whenever $p_1, \dots, p_r$ lie on a curve of degree 3 or less. Thus the novel part in determining resolutions for fat point subschemes of $\mathbf{P}^2$ supported at points of a cubic is in determining $s(\mathcal{H}, e_0)$ for numerically effective classes on the blowing up X of the points. What we will see here is that if X is the blowing up of points on a conic, $s(\mathcal{H}, e_0) = 0$ for any numerically effective class $\mathcal{H}$. This is no longer true for points on a cubic, which is partly why our results for the cubic case are less comprehensive.

II. Background on Surfaces

This section recalls results on surfaces that we will need later. For those results which are not standard or well known we give an indication of proof; for the reader's convenience at the least we usually provide references for the rest. Given a subvariety $C \in X$ and a class $\mathcal{L}$ on X , it will be convenient, if our meaning is clear, to write $H^i(C, \mathcal{L})$ for the cohomology of the restriction, rather than $H^i(C, \mathcal{L} \otimes \mathcal{O}_C)$.

Lemma II.1: *Let $\pi : Y \rightarrow X$ be a birational morphism of smooth projective rational surfaces, $\pi^* : \text{Cl}(X) \rightarrow \text{Cl}(Y)$ the corresponding homomorphism on divisor class groups, and let $\mathcal{L}$ be a divisor class on X . Then π^* is an injective intersection-form preserving map of free abelian groups of finite rank; there is a natural isomorphism $H^i(X, \mathcal{L}) = H^i(Y, \pi^*\mathcal{L})$ for every i ; and $\mathcal{L}$ is the class of an effective divisor (resp., numerically effective) if and only if $\pi^*\mathcal{L}$ is.*

Proof: See [11] for an indication of proof. ◇

By Lemma II.1, little harm is done by identifying $\text{Cl}(X)$ with its image in $\text{Cl}(Y)$. This is also compatible with exceptional configurations. For suppose that $X = X_r$ is obtained by blowing up essentially distinct points $p_1, \dots, p_r$ of $\mathbf{P}^2 = X_0$, where, for $1 \leq i \leq r$, X_i is the blowing up of $p_1, \dots, p_i$. If $\pi : X_j \rightarrow X_i$ is, for some $i < j \leq r$, the blowing up of $p_{i+1}, \dots, p_j$, with $e_0, \dots, e_i$ the exceptional configuration on X_i corresponding to $p_1, \dots, p_i$ and $e'_0, \dots, e'_j$ the exceptional configuration on X_j corresponding to $p_1, \dots, p_j$, then $\pi^*(e_l) = e'_l$ for $l \leq i$. For simplicity then, we will for each $0 \leq l \leq r$ simply denote the exceptional

configuration on X_l corresponding to $p_1, \dots, p_l$ by $e_0, \dots, e_l$, and leave to context which surface X_j , $j \geq i$ we wish at any given time to regard e_i as a class on.

We now recall some facts, the most important of which is (a), the formula of Riemann-Roch for a rational surface ([14]). For proofs of (b) and (c), we refer to [10].

Lemma II.2: *Let X be a smooth projective rational surface, and let $\mathcal{F}$ be a divisor class on X .*

- (a) *We have: $h^0(X, \mathcal{F}) - h^1(X, \mathcal{F}) + h^2(X, \mathcal{F}) = (\mathcal{F}^2 - K_X \cdot \mathcal{F})/2 + 1$.*
- (b) *If $\mathcal{F}$ is the class of an effective divisor, then $h^2(X, \mathcal{F}) = 0$.*
- (c) *If $\mathcal{F}$ is numerically effective, then $h^2(X, \mathcal{F}) = 0$ and $\mathcal{F}^2 \geq 0$.*

Here we recall some results from [18], where, in line with the notation in Section I and following [18], given coherent sheaves $\mathcal{A}$ and $\mathcal{B}$ on a scheme T , we will denote the cokernel of the natural map $H^0(X, \mathcal{A}) \otimes H^0(X, \mathcal{B}) \rightarrow H^0(X, \mathcal{A} \otimes \mathcal{B})$ by $\mathcal{S}(\mathcal{A}, \mathcal{B})$ and the kernel by $\mathcal{R}(\mathcal{A}, \mathcal{B})$. Also, Γ denotes the global sections functor.

Proposition II.3: *Let T be a closed subscheme of projective space, let $\mathcal{A}$ and $\mathcal{B}$ be coherent sheaves on T and let $\mathcal{C}$ be the class of an effective divisor C on T .*

- (a) *If the restriction homomorphisms $H^0(T, \mathcal{A}) \rightarrow H^0(C, \mathcal{A} \otimes \mathcal{O}_C)$ and $H^0(T, \mathcal{A} \otimes \mathcal{B}) \rightarrow H^0(C, \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{O}_C)$ are surjective (for example, if $h^1(T, \mathcal{A} \otimes \mathcal{C}^{-1}) = 0 = h^1(T, \mathcal{A} \otimes \mathcal{C}^{-1} \otimes \mathcal{B})$), then mapping the terms of the exact sequence $(0 \rightarrow \Gamma(\mathcal{A} \otimes \mathcal{C}^{-1}) \rightarrow \Gamma(\mathcal{A}) \rightarrow \Gamma(\mathcal{A} \otimes \mathcal{O}_C) \rightarrow 0) \otimes \Gamma(\mathcal{B})$ to those of $0 \rightarrow \Gamma(\mathcal{A} \otimes \mathcal{C}^{-1} \otimes \mathcal{B}) \rightarrow \Gamma(\mathcal{A} \otimes \mathcal{B}) \rightarrow \Gamma((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{O}_C) \rightarrow 0$ leads to the exact sequence*

$$\begin{aligned} 0 \rightarrow \mathcal{R}(\mathcal{A} \otimes \mathcal{C}^{-1}, \mathcal{B}) \rightarrow \mathcal{R}(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{R}(\mathcal{A} \otimes \mathcal{O}_C, \mathcal{B}) \rightarrow \\ \mathcal{S}(\mathcal{A} \otimes \mathcal{C}^{-1}, \mathcal{B}) \rightarrow \mathcal{S}(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{S}(\mathcal{A} \otimes \mathcal{O}_C, \mathcal{B}) \rightarrow 0. \end{aligned}$$

- (b) *If $H^0(T, \mathcal{B}) \rightarrow H^0(C, \mathcal{B} \otimes \mathcal{O}_C)$ is surjective (for example, if $h^1(T, \mathcal{B} \otimes \mathcal{C}^{-1}) = 0$), then $\mathcal{S}(\mathcal{A} \otimes \mathcal{O}_C, \mathcal{B}) = \mathcal{S}(\mathcal{A} \otimes \mathcal{O}_C, \mathcal{B} \otimes \mathcal{O}_C)$.*
- (c) *If T is a smooth curve of genus g , and $\mathcal{A}$ and $\mathcal{B}$ are line bundles of degrees at least $2g + 1$ and $2g$, respectively, then $\mathcal{S}(\mathcal{A}, \mathcal{B}) = 0$.*

Proof: See [18] for (a) and (c); we leave (b) as an easy exercise for the reader. $\diamond$

It will be helpful to generalize Proposition II.3(c) to nonsmooth curves with $g = 0$. We do so in Lemma II.5, using the following technical result, proved in [11].

Lemma II.4: *Let X be a smooth projective rational surface and let $\mathcal{N}$ be the class of a nontrivial effective divisor N on X . If $\mathcal{N} + K_X$ is not the class of an effective divisor and $\mathcal{F}$ meets every component of $\mathcal{N}$ nonnegatively, then $h^0(N, \mathcal{F}) > 0$, $h^1(N, \mathcal{F}) = 0$, $N^2 + N \cdot K_X < -1$, and every component M of N is a smooth rational curve (of negative self-intersection, if M does not move).*

Lemma II.5: *Let X be a smooth projective rational surface, and let $\mathcal{N}$ be the class of an effective divisor N on X such that $h^0(X, \mathcal{N} + K_X) = 0$. If $\mathcal{F}$ and $\mathcal{G}$ are the restrictions to N of divisor classes $\mathcal{F}'$ and $\mathcal{G}'$ on X which meet each component of N nonnegatively, then $\mathcal{S}(\mathcal{F}, \mathcal{G}) = 0$.*

Proof: To prove the lemma, induct on the sum n of the multiplicities of the components of N . By Lemma II.4, $h^1(N, \mathcal{O}) = 0$ and every component of N is a smooth rational curve. Thus the case $n = 1$ is trivial (since then $N = \mathbf{P}^1$, and the space of polynomials of degree f in two variables tensor the space of polynomials of degree g in two variables maps onto the space of polynomials of degree $f + g$). So say $n > 1$.

As in the proof of Theorem 1.7 of [1] (or see the proof of Lemma II.6 of [10]), N has a component C such that $(N - C) \cdot C \leq 1$. Let L be the effective divisor $N - C$ and let $\mathcal{L}$ be its class. Thus we have an exact sequence $0 \rightarrow \mathcal{O}_C \otimes (-\mathcal{L}) \rightarrow \mathcal{O}_N \rightarrow \mathcal{O}_L \rightarrow 0$. Now, $-L \cdot C \geq -1$, and both $\mathcal{F}'$ and $\mathcal{G}'$ meet C nonnegatively. We may assume $\mathcal{F}' \cdot C \geq \mathcal{G}' \cdot C$, otherwise reverse the roles of $\mathcal{F}'$ and $\mathcal{G}'$. Since $C = \mathbf{P}^1$, we see that $h^1(C, \mathcal{O}_C \otimes (\mathcal{F}' - \mathcal{L}))$, $h^1(C, \mathcal{O}_C \otimes (\mathcal{G}' - \mathcal{L}))$ and $h^1(C, \mathcal{O}_C \otimes (\mathcal{F}' + \mathcal{G}' - \mathcal{L}))$ all vanish. An argument similar to that used to prove Proposition II.3(a, b) now shows that we have an exact sequence

$\mathcal{S}(\mathcal{O}_C \otimes (\mathcal{F}' - \mathcal{L}), \mathcal{O}_C \otimes \mathcal{G}') \rightarrow \mathcal{S}(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{S}(\mathcal{O}_L \otimes \mathcal{F}, \mathcal{O}_L \otimes \mathcal{G}) \rightarrow 0$. Since $\mathcal{S}(\mathcal{O}_L \otimes \mathcal{F}, \mathcal{O}_L \otimes \mathcal{G}) = 0$ by induction, it suffices to show $\mathcal{S}(\mathcal{O}_C \otimes (\mathcal{F}' - \mathcal{L}), \mathcal{O}_C \otimes \mathcal{G}') = 0$. If $C \cdot (\mathcal{F}' - \mathcal{L}) \geq 0$, then the latter is 0 (as in the previous paragraph). Otherwise, we must have $0 = \mathcal{F}' \cdot C = \mathcal{G}' \cdot C$ and $C \cdot L = 1$, so $\mathcal{O}_C(-1) = \mathcal{O}_C \otimes (\mathcal{F}' - \mathcal{L})$ and $\mathcal{O}_C = \mathcal{O}_C \otimes \mathcal{G}'$, which means $h^0(\mathcal{O}_C, \mathcal{O}_C \otimes (\mathcal{F}' + \mathcal{G}' - \mathcal{L})) = 0$ and hence again $\mathcal{S}(\mathcal{O}_C \otimes (\mathcal{F}' - \mathcal{L}), \mathcal{O}_C \otimes \mathcal{G}') = 0$. $\diamond$

When working with distinct points $p_1, \dots, p_r$ it can be convenient to reindex them; for example, so that an expression $m_1 p_1 + \dots + m_r p_r$ has $m_1 \geq \dots \geq m_r$. If $p_1, \dots, p_r$ are only essentially distinct, reindexing in this way, properly speaking, makes no sense (since p_i lives on the surface obtained by blowing up $p_1, \dots, p_{i-1}$). Nonetheless, a reindexation that preserves the natural partial ordering of “infinite nearness” seems intuitively acceptable. We make a short digression to justify this intuition.

Suppose $p_1, \dots, p_r$ and $q_1, \dots, q_r$ are essentially distinct points of $\mathbf{P}^2$. Let X be the blowing up of $p_1, \dots, p_r$ with $e_0, \dots, e_r$ being the associated exceptional configuration, and let X' be the blowing up of $q_1, \dots, q_r$ with $e'_0, \dots, e'_r$ being the associated exceptional configuration. If there is an isomorphism $f : X \rightarrow X'$ such that $f^*(e'_0) = e_0$, then there is a unique permutation σ_f of $\{1, \dots, r\}$ such that $f^*(e'_i) = e_{\sigma_f(i)}$ for every $i \geq 1$, and it follows that any subscheme $Z = m_1 p_1 + \dots + m_r p_r$ is projectively equivalent to $Z' = m_{\sigma_f(1)} q_1 + \dots + m_{\sigma_f(r)} q_r$. Thus we shall say that a bijection $\sigma : \{q_1, \dots, q_r\} \rightarrow \{p_1, \dots, p_r\}$ is an *equivalence* and that $p_1, \dots, p_r$ and $q_1, \dots, q_r$ are *equivalent* if for some f as above we have $\sigma(q_i) = p_{\sigma_f(i)}$ for every i . Similarly, we shall say that a permutation of $\{1, \dots, r\}$ is an *equivalence* if it is σ_f for some such f .

By the next lemma we see that we may always assume that a subscheme $Z = m_1 p_1 + \dots + m_r p_r$, satisfying the proximity inequalities, also satisfies $m_1 \geq \dots \geq m_r$ up to equivalence.

Lemma II.6: *Let $p_1, \dots, p_r$ be essentially distinct points of $\mathbf{P}^2$.*

- (a) *Any permutation σ of $\{1, \dots, r\}$, such that $\sigma(j) \geq \sigma(i)$ whenever p_j is infinitely near p_i , is an equivalence.*
- (b) *If $Z = m_1 p_1 + \dots + m_r p_r$ satisfies the proximity inequalities, then $m_j \leq m_i$ whenever p_j is infinitely near to p_i ; in particular, up to equivalence we have $m_1 \geq \dots \geq m_r$.*

Proof: Let X be the blowing up of $p_1, \dots, p_r$, with $e_0, \dots, e_r$ being the corresponding exceptional configuration.

(a) One merely needs to check that $e_0, e'_1, \dots, e'_r$ is also an exceptional configuration, where $e'_i = e_{\sigma(i)}$ for each $i > 0$. But this follows from Theorem 1.1 of [7].

(b) Since p_j being infinitely near to p_i implies $e_i - e_j$ is the class of an effective divisor, the proximity inequalities imply $-(m_1 e_1 + \dots + m_r e_r) \cdot (e_i - e_j) \geq 0$ and hence $m_i \geq m_j$. In particular, we can choose l such that $m_l \leq m_i$ for all $1 \leq i \leq r$ and such that only p_l among $p_1, \dots, p_r$ is infinitely near to p_l ; then the permutation σ which is the identity on $\{1, \dots, l-1\}$, and for which $\sigma(l) = r$ and $\sigma(i) = i-1$ for $l < i \leq r$, is an equivalence by (a). I.e., up to equivalence we may assume m_r is least among $m_1, \dots, m_r$, and the result follows by induction. $\diamond$

We now have a lemma well known for distinct points. As a consequence of Lemma II.4, it holds more generally for essentially distinct points too.

Lemma II.7: *Let $p_1, \dots, p_r$ be essentially distinct points of $\mathbf{P}^2$ and let $e_0, e_1, \dots, e_r$ be the corresponding exceptional configuration on the blowing up X of $\mathbf{P}^2$ at the points. Suppose that $m_1, \dots, m_r$ are nonnegative integers such that $-(m_1 e_1 + \dots + m_r e_r)$ satisfies the proximity inequalities. Then $h^0(X, \mathcal{F}_d) > 0$ and $h^1(X, \mathcal{F}_d) = 0$ for $d \geq -1 + (m_1 + \dots + m_r)$, where $\mathcal{F}_d = de_0 - (m_1 e_1 + \dots + m_r e_r)$.*

Proof: Suppose there is at most one index i with $m_i > 0$. By Lemma II.6 we may assume $r = 1$, in which case we just need to check that $h^1(X, de_0 - m_1 e_1) = 0$ for $d \geq -1 + m_1$, which is straightforward, by restricting to a general section B of $e_0 - e_1$.

We now consider $\mathcal{F}_d$ in case $m_i > 0$ for at least two indices i . By Lemma II.1 we may assume that $m_r > 0$. Then $-(m_1 e_1 + \dots + (m_r - 1)e_r)$ satisfies the proximity inequalities, so by induction we may assume that $h^1(X, (d-1)e_0 - (m_1 e_1 + \dots + (m_r - 1)e_r)) = 0$. Also note that $(d-1)e_0 - (m_1 e_1 + \dots + (m_r - 1)e_r) = \mathcal{F}_d - (e_0 - e_r)$. Clearly $e_0 - e_r$ is the class of an effective divisor; p_r is infinitely near a bona fide point of $\mathbf{P}^2$,

and that point is p_i for some $i \leq r$. Then $e_0 - e_r = (e_0 - e_i) + (e_i - e_r)$; $e_0 - e_i$ is numerically effective, its linear system of sections being the pencil of lines through the point p_i , while E_r is a component of E_i , and $e_i - e_r$ is the class of the difference $E_i - E_r$. Thus we may choose a section B of $e_0 - e_r$ whose components, apart from a numerically effective divisor, are all components of E_i . If C is a component of E_i , then $F_d \cdot C = -(m_1 e_1 + \cdots + m_r e_r) \cdot C \geq 0$ by hypothesis. Since any numerically effective divisor meets any effective divisor nonnegatively, it will follow that $\mathcal{F}_d$ meets every component of B nonnegatively if we show $\mathcal{F}_d$ is the class of an effective divisor. But $d+1 \geq m_1 + \cdots + m_r$, the right hand side of which involves at least two terms; thus $(d+1)^2 > m_1^2 + \cdots + m_r^2$, hence $\mathcal{F}_d^2 - K_X \cdot \mathcal{F}_d = -1 + [(d+1)^2 - (m_1^2 + \cdots + m_r^2)] + [d - (m_1 + \cdots + m_r)] \geq -1$, so by Lemma II.2, $h^0(X, \mathcal{F}_d) > 0$ (since $d \geq 0$ implies $h^2(X, \mathcal{F}_d) = 0$ by duality).

Next, $e_0 \cdot (K_X + (e_0 - e_r)) < 0$ (because $e_0 \cdot K_X = -3$), so $h^0(X, K_X + (e_0 - e_r)) = 0$, since e_0 is numerically effective. We now can apply Lemma II.4 to obtain $h^1(B, \mathcal{F}_d \otimes \mathcal{O}_B) = 0$, from which $h^1(X, \mathcal{F}_d) = 0$ follows by taking cohomology of $0 \rightarrow \mathcal{F}_d - (e_0 - e_r) \rightarrow \mathcal{F}_d \rightarrow \mathcal{F}_d \otimes \mathcal{O}_B \rightarrow 0$. $\diamond$

The next result concerns vanishing of $\mathcal{S}(\mathcal{F}, \mathcal{G})$.

Theorem II.8: *Let X be the blowing up of essentially distinct points $p_1, \dots, p_r$ of $\mathbf{P}^2$, and let $e_0, \dots, e_r$ be the corresponding exceptional configuration. Suppose $\mathcal{F} = a_0 e_0 - a_1 e_1 - \cdots - a_r e_r$ and $\mathcal{G} = b_0 e_0 - b_1 e_1 - \cdots - b_r e_r$ satisfy the proximity inequalities and that $a_0 \geq a_1 + \cdots + a_r$ and $b_0 \geq b_1 + \cdots + b_r$. Then $\mathcal{S}(\mathcal{F}, \mathcal{G}) = 0$.*

Proof: The result is true and easy to see if $\mathcal{F}$ and $\mathcal{G}$ are multiples of e_0 , so assume that a_r say is positive. Then $\mathcal{H} = \mathcal{F} - (e_0 - e_r)$ satisfies the proximity inequalities and also has $e_0 \cdot \mathcal{H} \geq \sum_{i>0} e_i \cdot \mathcal{H}$, so by induction we may assume that $\mathcal{S}(\mathcal{H}, \mathcal{G}) = 0$. By Lemma II.7, we have $h^1(X, \mathcal{H}) = 0 = h^1(X, \mathcal{H} + \mathcal{G})$. If $b_r > 0$, then Lemma II.7 also gives $h^1(X, \mathcal{G} - (e_0 - e_r)) = 0$. If $b_r = 0$, then $h^1(X, \mathcal{G} - e_0) = 0$ by Lemma II.7 and e_r is a fixed component of $\mathcal{G} - (e_0 - e_r)$, so $h^0(X, \mathcal{G} - (e_0 - e_r)) = h^0(X, \mathcal{G} - e_0)$. This, together with $(\mathcal{G} - (e_0 - e_r))^2 - K_X \cdot (\mathcal{G} - (e_0 - e_r)) = (\mathcal{G} - e_0)^2 - K_X \cdot (\mathcal{G} - e_0)$ and Lemma II.2 implies $h^1(X, \mathcal{G} - (e_0 - e_r)) = h^1(X, \mathcal{G} - e_0) = 0$. Now from Proposition II.3(a,b) we have an exact sequence $0 = \mathcal{S}(\mathcal{H}, \mathcal{G}) \rightarrow \mathcal{S}(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{S}(\mathcal{F} \otimes \mathcal{O}_L, \mathcal{G} \otimes \mathcal{O}_L) \rightarrow 0$, where L is a general section of $e_0 - e_r$. As in the proof of Lemma II.7, $K_X + (e_0 - e_r)$ is not the class of an effective divisor and $\mathcal{F}$ and $\mathcal{G}$ meet each component of L nonnegatively, so $\mathcal{S}(\mathcal{F} \otimes \mathcal{O}_L, \mathcal{G} \otimes \mathcal{O}_L) = 0$ by Lemma II.5, and hence $\mathcal{S}(\mathcal{F}, \mathcal{G}) = 0$, as required. $\diamond$

The following result, giving another vanishing criterion, is well known (see Proposition 3.7 of [4]) and follows easily by appropriately applying Proposition II.3; it essentially says that no generator of I_Z need be taken in degrees greater than the regularity of I_Z .

Lemma II.9: *Let $e_0, \dots, e_r$ be the exceptional configuration corresponding to a blowing up $X \rightarrow \mathbf{P}^2$ at essentially distinct points $p_1, \dots, p_r$. Let $Z = m_1 p_1 + \cdots + m_r p_r$ satisfy the proximity inequalities, and let $\mathcal{F}_d$ denote $d e_0 - m_1 e_1 - \cdots - m_r e_r$. If $h^1(X, \mathcal{F}_t) = 0$, then $\mathcal{S}(\mathcal{F}_d, e_0) = 0$ for $d > t$.*

In the following lemma, given the class $\mathcal{F}$ of an effective divisor F with N denoting the fixed components of the complete linear system $|F|$, we denote the class of N by $(\mathcal{F})_f$ (this is the *fixed* part of $\mathcal{F}$) and $\mathcal{F} - (\mathcal{F})_f$ by $(\mathcal{F})_m$ (the free or *moving* part). Thus $\mathcal{F} = (\mathcal{F})_m + (\mathcal{F})_f$ is a Zariski decomposition of $\mathcal{F}$.

Lemma II.10: *Let $e_0, \dots, e_r$ be an exceptional configuration on a surface X and let $\mathcal{F}$ be a class on X .*

- (a) *If $\mathcal{F}$ is not the class of an effective divisor, then $s(\mathcal{F}, e_0) = h^0(X, \mathcal{F} + e_0)$.*
- (b) *If $\mathcal{F}$ is the class of an effective divisor with a Zariski decomposition $\mathcal{F} = \mathcal{H} + \mathcal{N}$ (with $\mathcal{H}$ being the numerically effective part), then $s(\mathcal{F}, e_0) = s(\mathcal{H}, e_0) + h^0(X, \mathcal{F} + e_0) - h^0(X, \mathcal{H} + e_0)$.*
- (c) *If $\mathcal{F}$ is the class of an effective divisor and $(\mathcal{F})_f \neq (\mathcal{F} + e_0)_f$, then $s(\mathcal{F}, e_0) \geq h^0(X, \mathcal{F} + e_0) - h^0(X, (\mathcal{F})_m + e_0) > 0$.*

Proof: (a) This is clear so consider (b). Regarding $\mathcal{H}$ and $\mathcal{F}$ as sheaves, we have an inclusion $\mathcal{H} \rightarrow \mathcal{F}$ which induces an isomorphism on global sections. Thus we have a commutative diagram with exact columns

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(X, \mathcal{H}) \otimes H^0(X, e_0) & \rightarrow & H^0(X, \mathcal{H} + e_0) \\
\downarrow & & \downarrow \\
H^0(X, \mathcal{F}) \otimes H^0(X, e_0) & \rightarrow & H^0(X, \mathcal{F} + e_0) \\
\downarrow & & \\
0 & &
\end{array}$$

and the image of $H^0(X, \mathcal{H}) \otimes H^0(X, e_0) \rightarrow H^0(X, \mathcal{F} + e_0)$ equals the image of $H^0(X, \mathcal{F}) \otimes H^0(X, e_0) \rightarrow H^0(X, \mathcal{F} + e_0)$, which means that $s(\mathcal{F}, e_0) = s(\mathcal{H}, e_0) + h^0(X, \mathcal{F} + e_0) - h^0(X, \mathcal{H} + e_0)$.

(c) By (b) we know $s(\mathcal{F}, e_0) \geq h^0(X, \mathcal{F} + e_0) - h^0(X, (\mathcal{F})_m + e_0)$. But $h^0(X, \mathcal{F} + e_0) - h^0(X, (\mathcal{F})_m + e_0)$ just measures the extent to which there are sections of $\mathcal{F} + e_0$ not containing the full fixed part of $\mathcal{F}$. Since $(\mathcal{F})_f \neq (\mathcal{F} + e_0)_f$, this is positive. $\diamond$

III. Resolutions

In this section we will, under certain restrictions, study minimal free resolutions of homogeneous ideals I_Z (over the homogeneous coordinate ring $k[\mathbf{P}^2]$ of $\mathbf{P}^2$, which hereafter we will denote by R), where $p_1, \dots, p_r$ are essentially distinct points of $\mathbf{P}^2$ and $Z = m_1 p_1 + \dots + m_r p_r$ satisfies the proximity inequalities. By Lemma II.6, we may, if it is convenient, assume that $m_1 \geq \dots \geq m_r > 0$.

Given essentially distinct points $p_1, \dots, p_r$ of $\mathbf{P}^2$, we say that the points are points of a curve of degree n if, on the blowing up X of $\mathbf{P}^2$ at the points, $ne_0 - e_1 - \dots - e_r$ is the class of an effective divisor. We say that $p_1, \dots, p_r$ are points of a curve of degree n with some property (say smooth or irreducible, etc.) if $ne_0 - e_1 - \dots - e_r$ is the class of such a curve. Our results involve essentially distinct points $p_1, \dots, p_r$ on curves in $\mathbf{P}^2$ of degree at most 3. Our results allow one recursively to determine the graded modules in a minimal free resolution of I_Z for any fat point subscheme $Z = m_1 p_1 + \dots + m_r p_r$ satisfying the proximity inequalities, as long as $p_1, \dots, p_r$ are points of a conic, or, in certain cases, points of a smooth cubic. Since essentially distinct points on a line certainly are points on a conic, our results apply also to points on a line; the case of points on a line turns out to be simple enough to write down the resolution completely explicitly, which we do in Example III.i.4.

III.i. Points on a conic

We now consider points, possibly infinitely near, on a conic, possibly nonsmooth. If X is the blowing up of $\mathbf{P}^2$ at such points, we show $s(\mathcal{F}, e_0) = 0$ for any $\mathcal{F} \in \text{NEFF}$. As demonstrated in Example III.i.3, this allows us to work out resolutions in any specific case of points on a conic. Also, in the special case of points on a line (which is subsumed by the case of points on a conic), we give a completely general and explicit result in Example III.i.4.

We begin with a lemma: part (a) will be used in the proof of Theorem III.i.2; the other parts will be helpful references in working out examples, such as Example III.i.3.

Lemma III.i.1: *Let X be the blowing up of essentially distinct points $p_1, \dots, p_r$ of a conic in $\mathbf{P}^2$; i.e., $2e_0 - e_1 - \dots - e_r$ is the class of an effective divisor, where $e_0, \dots, e_r$ is the exceptional configuration corresponding to $p_1, \dots, p_r$. Also, for each $i > 0$, recall E_i denotes the unique effective divisor whose class is e_i .*

- (a) *If $\mathcal{C}$ is the class of a reduced and irreducible curve of negative self-intersection, then $\mathcal{C}$ is either: the class of a component of E_i , $i > 0$; the class of a component of an effective divisor Q with $\mathcal{C} \cdot Q < 0$, where $Q = 2e_0 - e_1 - \dots - e_r$ is the class of Q ; or the class $e_0 - e_i - e_j$, for some $0 < i < j$.*
- (b) *If $\mathcal{F} \in \text{NEFF}$, then $\mathcal{F} \in \text{EFF}$, and $\mathcal{F}$ is regular (i.e., $h^1(X, \mathcal{F}) = 0$) and its linear system of sections is base point (and hence fixed component) free.*
- (c) *If $r \geq 2$, then:*

- (i) NEFF consists of the classes $\mathcal{F} \in \text{Cl}(X)$ such that $\mathcal{F} \cdot \mathcal{C} \geq 0$ whenever $\mathcal{C}$ is the class of an irreducible curve of negative self-intersection, and
- (ii) EFF is generated by the classes of curves of negative self-intersection.

Proof: (a) Let $\mathcal{C}$ be the class of a reduced and irreducible curve C of arithmetic genus g and of negative self-intersection. Since C is effective and e_0 is numerically effective, we have $C \cdot e_0 \geq 0$.

If $C \cdot e_0 = 0$, then C is a component of some E_i , $i > 0$.

If $C \cdot e_0 = 1$, then $\mathcal{C}$ must be of the form $e_0 - e_{i_1} - \cdots - e_{i_l}$, with $0 < i_1 < \cdots < i_l$. Since $C^2 < 0$, we have $l \geq 2$; if $l > 2$ then C meets $\mathcal{Q}$ negatively and hence is a component of $\mathcal{Q}$.

Say $C \cdot e_0 = 2$. Then $C \cdot e_i \geq 0$ for all $i > 0$, since C is clearly not a component of any E_i . Also, since any reduced and irreducible plane conic is smooth, we see $C \cdot e_i \leq 1$, $i > 0$. Thus $\mathcal{C}$ is the sum of $\mathcal{Q}$ with those e_i with $C \cdot e_i = 0$, so we see that $\mathcal{C}$ has a section coming from $\mathcal{Q}$ and those e_i with $C \cdot e_i = 0$, and hence either that C is not fixed (contradicting $C^2 < 0$), or that $C \cdot e_i = 1$ for all i and therefore that $\mathcal{C} = \mathcal{Q}$ and $C \cdot \mathcal{Q} = C^2 < 0$.

Suppose that $C \cdot e_0 > 2$. Since $-K_X = e_0 + \mathcal{Q}$ and clearly C is not a component of $\mathcal{Q}$, we have $-C \cdot K_X \geq C \cdot e_0 \geq 3$. Hence by the adjunction formula we have $0 > C^2 = 2g - 2 - C \cdot K_X \geq 2g + 1 > 0$, contradiction.

(b) Since $-K_X$ is the class of an effective divisor, $\text{NEFF} \subset \text{EFF}$ [10]. For the rest, the case $\mathcal{F} = 0$ is clear, so assume $\mathcal{F}$ is not trivial. By Lemma II.2(c), we have $\mathcal{F}^2 \geq 0$. Since the space $e_0^\perp \subset \text{Cl}(X)$ of classes perpendicular to e_0 is negative definite, we see $\mathcal{F} \cdot e_0 > 0$. Together, this means either that $\mathcal{F} \cdot e_0 > 1$ (in which case $\mathcal{F} \cdot (-K_X) = \mathcal{F} \cdot (\mathcal{Q} + e_0) > 1$ and our result follows by [11]) or that $\mathcal{F} \cdot e_0 = 1$ and hence $\mathcal{F}$ is either of the form e_0 or $e_0 - e_i$, $i > 0$ (and again we have $\mathcal{F} \cdot (-K_X) > 1$ and our result follows by [11]).

(c) Clearly, (i) follows from (ii). To prove (ii), let $\mathcal{F}$ be in EFF. Then subtracting off fixed components, which by (b) must be curves of negative self-intersection, we obtain the free part $(\mathcal{F})_m$ of $\mathcal{F}$, which is numerically effective. Thus $(\mathcal{F})_m$ is in the cone of classes which meet $\mathcal{Q}$, each e_i , and each $e_0 - e_i - e_j$, $0 < i < j$ nonnegatively. By Proposition 1.5.3 [8], this cone is generated by classes of the form e_0 , $e_0 - e_{i_1}$, $2e_0 - e_{i_1} - e_{i_2} - e_{i_3}$, $3e_0 - e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6}$, and $de_0 - (d-1)e_{i_1} - e_{i_2} - \cdots - e_{i_{d+2}}$, $d \geq 2$, where in each expression the indices $i_1, i_2, \dots$ are nonzero and distinct. Now it is enough to check that each of these classes is a sum of classes of curves of negative self-intersection, which is straightforward. $\diamond$

Theorem III.i.2: Let X be the blowing up of essentially distinct points $p_1, \dots, p_r$ of $\mathbf{P}^2$ such that $\mathcal{Q} = 2e_0 - e_1 - \cdots - e_r$ is the class of an effective divisor, where $e_0, \dots, e_r$ is the exceptional configuration corresponding to $p_1, \dots, p_r$. Then $\mathcal{S}(\mathcal{F}, e_0) = 0$ for any numerically effective class $\mathcal{F}$.

Proof: We will induct on $e_0 \cdot \mathcal{F}$. If $e_0 \cdot \mathcal{F} = 0$ then $\mathcal{F} = 0$, and $\mathcal{S}(\mathcal{F}, e_0) = 0$ is clear. Also, the case that $r < 2$ is covered by Theorem II.8, so we may assume $r \geq 2$, and by reindexing (see Lemma II.6) we may assume $\mathcal{F} \cdot e_1 \geq \cdots \geq \mathcal{F} \cdot e_r \geq 0$. Since, if $\mathcal{F} \cdot e_r = 0$ we may as well just work on the blowing up of $\mathbf{P}^2$ at $p_1, \dots, p_{r-1}$, we may assume in fact that $\mathcal{F} \cdot e_i > 0$ for all $i > 0$. Now by explicitly checking against the cases enumerated in Lemma III.i.1(a), given the class $\mathcal{G}$ of any reduced and irreducible effective divisor of negative self-intersection, we see $(\mathcal{F} - \mathcal{Q}) \cdot \mathcal{G} \geq 0$. Thus $\mathcal{F} - \mathcal{Q}$ is numerically effective.

Since $K_X = -3e_0 + e_1 + \cdots + e_r$, we have $\mathcal{Q} = -K_X - e_0$. Clearly, given any numerically effective class $\mathcal{H}$, $(\mathcal{H} + e_0)^2 > 0$, so, by duality and Ramanujan vanishing (see the first paragraph of [19, Theorem, p. 121], which holds in all characteristics), $h^1(X, \mathcal{H} - \mathcal{Q}) = h^1(X, -(\mathcal{H} + e_0)) = 0$. Thus, taking $\mathcal{Q}$ to be an effective divisor in the class $\mathcal{Q}$, the exact sequence $0 \rightarrow \mathcal{H} - \mathcal{Q} \rightarrow \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{O}_{\mathcal{Q}} \rightarrow 0$ is exact on global sections. In particular, this follows taking $\mathcal{H}$ to be either e_0 , $\mathcal{F}$ or $\mathcal{F} + e_0$. In the former case we have $\mathcal{S}(\mathcal{F} \otimes \mathcal{O}_{\mathcal{Q}}, e_0) = \mathcal{S}(\mathcal{F} \otimes \mathcal{O}_{\mathcal{Q}}, e_0 \otimes \mathcal{O}_{\mathcal{Q}})$ by Proposition II.3(b), and the latter two cases show that Proposition II.3(a) applies. Since $\mathcal{S}(\mathcal{F} \otimes \mathcal{O}_{\mathcal{Q}}, e_0 \otimes \mathcal{O}_{\mathcal{Q}})$ vanishes by Lemma II.5, and we may, by induction, assume $\mathcal{S}(\mathcal{F} - \mathcal{Q}, e_0) = 0$, our result, $\mathcal{S}(\mathcal{F}, e_0) = 0$, now follows from the exact sequence of Proposition II.3(a). $\diamond$

Example III.i.3: We now give an example showing how to apply the results above to work out resolutions. Let L_1 and L_2 be distinct lines in $\mathbf{P}^2$ meeting at p_1 , let p_2, p_3 and p_4 be distinct points on L_1 away from p_1 , let $p_5 \neq p_1$ be a point of L_2 , and let p_6 be the point infinitely near to p_5 corresponding to the tangent direction at p_5 along L_2 . Then $Z = 3p_1 + 2p_2 + 2p_3 + p_4 + 3p_5 + 2p_6$ satisfies the proximity inequalities. Let $e_0, \dots, e_6$ be the exceptional configuration coming from the blowing up X of $\mathbf{P}^2$ at the essentially distinct

points $p_1, \dots, p_6$.

By Lemma III.i.1(a) any reduced and irreducible curve of negative self-intersection is either a component of the total transform of a line through any two of the points, a component of the exceptional curve corresponding to one of the points, or a component of the total transform of a conic through the points. Thus the classes of reduced and irreducible curves of negative self-intersection on X are: $e_0 - e_1 - \dots - e_4$, $e_0 - e_1 - e_5 - e_6$, $e_0 - e_i - e_5$ for $i \in \{2, 3, 4\}$, $e_1, e_2, e_3, e_4, e_5 - e_6$, and e_6 . If we let $\mathcal{F}_d$ denote $de_0 - 3e_1 - 2e_2 - 2e_3 - e_4 - 3e_5 - 2e_6$, then for each $d < 5$ one can find a sequence $\mathcal{C}_1, \dots, \mathcal{C}_t$ among the enumerated classes of negative self-intersection such that each of $\mathcal{F}_d \cdot \mathcal{C}_1, \dots, (\mathcal{F}_d - \mathcal{C}_1 - \dots - \mathcal{C}_{t-1}) \cdot \mathcal{C}_t$, and $(\mathcal{F}_d - \mathcal{C}_1 - \dots - \mathcal{C}_t) \cdot e_0$ is negative and hence neither $\mathcal{F}_d - \mathcal{C}_1 - \dots - \mathcal{C}_t$ nor $\mathcal{F}_d$ can be the class of an effective divisor. Thus $h^0(X, \mathcal{F}_d) = 0$ for $d < 5$.

In the case that $d = 5$, this process of subtracting off the classes of putative fixed components of negative self-intersection leads to the Zariski decomposition $\mathcal{F}_5 = \mathcal{H} + \mathcal{N}$, where $\mathcal{H} = 2e_0 - e_2 - e_3 - e_5$ is numerically effective with $h^0(X, \mathcal{F}_5) = h^0(X, \mathcal{H})$, and $\mathcal{N} = 3e_0 - 3e_1 - e_2 - e_3 - e_4 - 2e_5 - 2e_6$ with $h^0(X, \mathcal{N}) = 1$. By Lemma III.i.1(c), this process of subtracting off putative components of negative self-intersection will always either lead to a Zariski decomposition or to a determination that the original class is not the class of an effective divisor. By Lemma III.i.1(b) and Lemma II.2(c), $h^1(X, \mathcal{H}) = h^2(X, \mathcal{H}) = 0$, so $h^0(X, \mathcal{F}) = h^0(X, \mathcal{H})$ is easily computed by Riemann-Roch to be 3.

Similarly, we find: that $h^0(X, \mathcal{F}_d)$ is, respectively, 8, 14, and 23, for $6 \leq d \leq 8$; that in a Zariski decomposition of $\mathcal{F}_d$ the numerically effective part of $\mathcal{F}_d$ for $6 \leq d \leq 7$ is, respectively, $4e_0 - e_1 - e_2 - e_3 - 2e_5 - e_6$, and $5e_0 - e_1 - e_2 - e_3 - 2e_5 - e_6$; and, adding e_0 to the numerically effective parts for $5 \leq d \leq 7$ and applying h^0 , that we get, respectively, 7, 14, and 21. Thus $s(\mathcal{F}_d, e_0)$ is 0 for $d < 4$; 3, 1, 0 and 2 for $4 \leq d \leq 7$ (using Lemma II.10); and 0 by Lemma II.9 for $d > 7$.

Thus, in a resolution $0 \rightarrow F_1 \rightarrow F_0 \rightarrow I_Z \rightarrow 0$ of I_Z , we find that $F_0 = R[-5]^3 \oplus R[-6] \oplus R[-8]^2$, which allows us as discussed in Section I to determine $F_1 = R[-6]^2 \oplus R[-7] \oplus R[-9]^2$. $\diamond$

The case of points on a line is a special case of points on a conic which affords an especially nice and explicit answer, so we present this case as another example of finding a resolution using our results.

Example III.i.4: Let $p_1, \dots, p_r$ be essentially distinct points of a line in $\mathbf{P}^2$; i.e., $e_0 - e_1 - \dots - e_r$ is the class of an effective divisor on the blowing up X of $p_1, \dots, p_r$. In this case finding Zariski decompositions is straightforward, h^1 and h^2 vanish for every numerically effective class (by Lemma II.7 and Lemma II.2(c)), and $\mathcal{S}(\mathcal{H}, e_0) = 0$ for any numerically effective class $\mathcal{H}$ (by Theorem II.8 or Theorem III.i.2), which allows us to determine a resolution for any $Z = m_1 p_1 + \dots + m_r p_r$.

We now make this resolution explicit, leaving details to the reader. Let $Z = m_1 p_1 + \dots + m_r p_r$ be a nontrivial subscheme satisfying the proximity inequalities. By Lemma II.6, we may assume that $m_1 \geq \dots \geq m_r > 0$. From the sequence $m_1 \geq \dots \geq m_r > 0$ we get a Young diagram (r columns where the i th column is a column of m_i boxes, $1 \leq i \leq r$), and from this we get the conjugate sequence $\mu_1 \geq \dots \geq \mu_{m_1}$ (μ_i is the number of boxes in the i th row of the Young diagram). We also define $a_i = (i-1) + \mu_i + \dots + \mu_{m_1}$ for $1 \leq i \leq m_1$. This gives $\mu_1 + \dots + \mu_{m_1} = m_1 + \dots + m_r = a_1 \geq a_2 \geq \dots \geq a_{m_1} \geq m_1$. Then the minimal free resolution of I_Z takes the form $0 \rightarrow F_1 \rightarrow F_0 \rightarrow I_Z \rightarrow 0$, where

$$F_0 = R[-a_1] \oplus \dots \oplus R[-a_{m_1}] \oplus R[-m_1]$$

and

$$F_1 = R[-1 - a_1] \oplus \dots \oplus R[-1 - a_{m_1}].$$

In particular, we see that a minimal homogeneous generating set for I_Z has $m_1 - m_2 + 1$ generators of degree m_1 , and one generator, of degree a_i , for each $1 \leq i \leq m_2$.

From this resolution we also get a particularly nice expression (reformulating and extending that of [3, Proposition 3.3], which is for *distinct* points on a line in $\mathbf{P}^2$) for the Hilbert function of I_Z , where we follow the convention that $\binom{a}{b} = 0$ if $a < b$: For all $n \geq 0$, $h_{I_Z}(n) = \binom{n-m_1+2}{2} + \sum_{1 \leq i \leq m_1} ((\binom{n-a_i+2}{2}) - (\binom{n-a_i+1}{2}))$, or, alternatively, $h_{I_Z}(n) = ((m_1 - m_2 + 1) \binom{n-m_1+2}{2} - (m_1 - m_2) \binom{n-m_1+1}{2}) + \sum_{1 \leq i \leq m_2} ((\binom{n-a_i+2}{2}) - (\binom{n-a_i+1}{2}))$. $\diamond$

III.ii. Points on a cubic

We now consider the case of essentially distinct points $p_1, \dots, p_r$ on a smooth plane cubic C ; i.e., p_1 is a point of C , p_2 is a point of the proper transform of C , etc. In this situation, the monoid EFF of classes of effective divisors on X is controlled by $\ker(\mathrm{Cl}(X) \rightarrow \mathrm{Cl}(D))$ (which we will denote $\Lambda(X, D)$, or just Λ when X and D are clear from context), where D is the proper transform to X of C (and hence a section of $-K_X$), and $\mathrm{Cl}(X) \rightarrow \mathrm{Cl}(D)$ is the canonical homomorphism induced by the inclusion $D \subset X$. As is shown in [6], if one knows Λ , then one can determine $h^0(X, \mathcal{F})$ for any class $\mathcal{F}$ on X and one can also effect Zariski decompositions (indeed, one can determine the free part $(\mathcal{F})_m$ of $\mathcal{F}$ when $\mathcal{F}$ is the class of an effective divisor). As discussed in Section I, this reduces determining resolutions of ideals defining fat point subschemes of $\mathbf{P}^2$ supported at essentially distinct points of C to determining $s(\mathcal{F}, e_0)$ for numerically effective classes $\mathcal{F}$.

However, unlike the case of points on a line or conic, there does not seem to be a general principle for handling points on a cubic. In particular, for points on a cubic $\mathcal{S}(\mathcal{F}, e_0)$ need not vanish for every numerically effective class $\mathcal{F}$, even if its linear system of sections is fixed component free or even base point free. For example, if $p_1, \dots, p_6$ are distinct general points of $\mathbf{P}^2$, then $\mathcal{F} = 5e_0 - 2e_1 - \dots - 2e_6$ is numerically effective and its linear system of sections is fixed component and base point free, but $h^0(X, e_0) = h^0(X, \mathcal{F}) = 3$ and $h^0(X, \mathcal{F} + e_0) = 10$, so $s(\mathcal{F}, e_0) \geq 1$. For a more subtle example (one which is not evident from a simple dimension count), consider eight distinct general points $p_1, \dots, p_8$ of $\mathbf{P}^2$ and let $\mathcal{F} = t(17e_0 - 6(e_1 + \dots + e_8))$; then, for all $t > 0$, $\mathcal{F}$ is numerically effective and its linear system of sections is fixed component and base point free, but $s(\mathcal{F}, e_0) > 0$ (see [13]).

Thus we will obtain our results under certain restrictions: we will restrict either the classes $\mathcal{F}$ or the surfaces X which we consider. More specifically, we will first allow X to be any blowing up of $\mathbf{P}^2$ at $r \geq 9$ essentially distinct points $p_1, \dots, p_r$ of a smooth plane cubic, but only consider *uniform classes* (i.e., classes $\mathcal{F}$ satisfying $\mathcal{F} \cdot e_1 = \dots = \mathcal{F} \cdot e_r$). Second, we will consider arbitrary classes $\mathcal{F}$ in the case that $p_1, \dots, p_r$, $r \geq 1$, are essentially distinct points on a smooth plane cubic C but where Λ is as large as possible (i.e., Λ is the subgroup $K_X^\perp$ of classes $\mathcal{F}$ with $\mathcal{F} \cdot K_X = 0$, which in more concrete terms means p_1 is a flex of C , and for each i , p_i is the point of the proper transform of C infinitely near to p_1). We begin now by studying uniform classes.

III.ii.i. Uniform classes on a blowing up of points on a smooth plane cubic

Let $r \geq 9$, and let $p_1, \dots, p_r$ be essentially distinct points of a smooth irreducible plane cubic C . The results in this subsection allow one implicitly to compute a resolution of the ideal of any fat point subscheme of $\mathbf{P}^2$ of the form $Z = m(p_1 + \dots + p_r)$, where $m \geq 1$. For this, it is enough to consider uniform classes; i.e., those of the form $\mathcal{F}_n = ne_0 - m(e_1 + \dots + e_r)$, where X is the blowing up of the points $p_1, \dots, p_r$ and $e_0, \dots, e_r$ is the corresponding exceptional configuration. Since $-K_X = 3e_0 - e_1 - \dots - e_r$, we can write $\mathcal{F}_n$ as $te_0 - mK_X$, where $t = n - 3m$. Of course, our main interest is when n (and hence $3m + t$) is nonnegative, since otherwise $\mathcal{F}_n$ is not the class of an effective divisor.

The following Proposition recalls facts from [6] which will be helpful both in our analysis (Theorem III.ii.i.2) of $s(\mathcal{F}, e_0)$ for a uniform class $\mathcal{F}$ and also in working out complete examples of resolutions, as in Example III.ii.i.3.

Proposition III.ii.i.1: *Let X be as in the preceding two paragraphs, with $\mathcal{F} = te_0 - mK_X$ for some integers t and $m > 0$.*

- (a) *The class $\mathcal{F}$ is the class of an effective divisor if and only if $t \geq 0$.*
- (b) *Say $t \geq 0$ and $r = 9$.*
 - (i) *If $t > 0$, then the linear system of sections of $\mathcal{F}$ is fixed component free and $h^1(X, \mathcal{F}) = 0$.*
 - (ii) *If $t = 0$, then the linear system of sections of $\mathcal{F}$ has a fixed component if and only if $\mathcal{F}$ is not in Λ ; in particular, the fixed component free part $(\mathcal{F})_m$ of $\mathcal{F}$ is $-(m-s)K_X$ and we have $h^0(X, (\mathcal{F})_m) = \lambda + 1$, where s is the least nonnegative integer such that $-(m-s)K_X \in \Lambda$ and where $\lambda = 0$ if $m = s$ and otherwise $\lambda = (m-s)/l$, where l is the least positive integer such that $-lK_X \in \Lambda$.*
- (c) *Lastly, say $t \geq 0$ and $r > 9$.*
 - (i) *If $-K_X \cdot \mathcal{F} > 0$, then the linear system of sections of $\mathcal{F}$ is fixed component free and $h^1(X, \mathcal{F}) = 0$.*

- (ii) Say $-K_X \cdot \mathcal{F} \leq 0$. If $t = 0$, then $(\mathcal{F})_m = 0$, so say $t > 0$ and let s be the least nonnegative integer such that $-K_X \cdot (\mathcal{F} + sK_X) \geq 0$. If $-K_X \cdot (\mathcal{F} + sK_X) > 0$, then $(\mathcal{F})_m = \mathcal{F} + sK_X$. If $-K_X \cdot (\mathcal{F} + sK_X) = 0$, then $(\mathcal{F})_m = \mathcal{F} + sK_X$ and $h^1(X, \mathcal{F} + sK_X) = 1$ if $\mathcal{F} + sK_X \in \Lambda$, while $(\mathcal{F})_m = \mathcal{F} + (s+1)K_X$ if $\mathcal{F} + sK_X \notin \Lambda$.

Proof: (a) Since e_0 and $-K_X$ are the classes of effective divisors, $\mathcal{F}$ is the class of an effective divisor if $t \geq 0$. Conversely, note that $-K_X$ is the class of an irreducible curve of self-intersection $9 - r$, and that $-K_X \cdot \mathcal{F} = 3t + (9 - r)m$, which is negative if $t < 0$ and $r \geq 9$. Now, $-K_X$ is numerically effective for $r = 9$, but, if $t < 0$, meets $\mathcal{F}$ negatively, which implies that $\mathcal{F}$ cannot be the class of an effective divisor. For $r > 9$ and $t < 0$, if $\mathcal{F}$ is the class of an effective divisor, then $-K_X$ is a fixed component so $\mathcal{F} + K_X = te_0 - (m-1)K_X$ is the class of an effective divisor. Iterating we eventually obtain the contradiction that te_0 is the class of an effective divisor, which is absurd if $t < 0$.

Items (b,c)(i) follow by Theorems 1.1(b) and 3.1 of [6]. Items (b,c)(ii) follow by Proposition 1.2 and Theorem 3.1 of [6]. $\diamond$

We now compute $s(\mathcal{F}_n, e_0)$ for each $n > 0$. By Proposition III.ii.i.1, the fixed component free part $(\mathcal{F}_n)_m$ of $\mathcal{F}_n$ is a uniform class, hence so is $e_0 + (\mathcal{F}_n)_m$. Thus, using Proposition III.ii.i.1, it is enough by Lemma II.10 to compute $s((\mathcal{F}_n)_m, e_0)$, for which the following theorem suffices.

Theorem III.ii.i.2: Let $\mathcal{F} = (\mathcal{G})_m$ for some uniform class $\mathcal{G}$ of an effective divisor, where X is the blowing up of $r \geq 9$ essentially distinct points of a smooth plane cubic.

- (a) If $-K_X \cdot \mathcal{F} > 1$, then $\mathcal{S}(\mathcal{F}, e_0) = 0$.
- (b) If $-K_X \cdot \mathcal{F} = 1$, then $s(\mathcal{F}, e_0) = 1$.
- (c) If $-K_X \cdot \mathcal{F} = 0$, then $\mathcal{S}(\mathcal{F}, e_0) = 0$ unless either: $r = 10$, in which case $s(\mathcal{F}, e_0) = 1$; or $r = 9$, in which case $\mathcal{F} = -abK_X$ and $s(\mathcal{F}, e_0) = 3b(a-1)$, where $a = b = 0$ if $\mathcal{F} = 0$, and otherwise a is the least positive integer such that $-aK_X \in \Lambda$ and b is some nonnegative integer.

Proof: Write $\mathcal{F}$ as $te_0 - mK_X$ for some nonnegative integers t and m .

(a) Note that $-K_X \cdot (te_0 - mK_X) > 1$ implies $-K_X \cdot (te_0 - sK_X) > 1$ for all $0 \leq s \leq m$. We will induct on s , starting with the obvious fact that $\mathcal{S}(te_0, e_0) = 0$ for $t \geq 0$. So now we may assume that $0 < s \leq m$, and that $\mathcal{S}(te_0 - (s-1)K_X, e_0) = 0$. By Proposition III.ii.i.1, $h^1(X, te_0 - (s-1)K_X) = h^1(X, te_0 - (s-1)K_X + e_0) = 0$, and $h^1(X, e_0 + K_X) = h^1(X, -e_0) = h^1(\mathbf{P}^2, -e_0) = 0$, so by Proposition II.3(a,b) we have the exact sequence $\mathcal{S}(te_0 - (s-1)K_X, e_0) \rightarrow \mathcal{S}(te_0 - sK_X, e_0) \rightarrow \mathcal{S}((te_0 - sK_X) \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) \rightarrow 0$, where the leftmost term vanishes by induction, and $\mathcal{S}((te_0 - sK_X) \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) = 0$ by Proposition II.3(c), since $(te_0 - sK_X) \otimes \mathcal{O}_D$ has degree at least $2g$ and $e_0 \otimes \mathcal{O}_D$ has degree $2g + 1$, where $g = 1$ is the genus of D . Thus $\mathcal{S}(te_0 - sK_X, e_0) = 0$ follows by exactness.

(b) Under the given hypotheses, we must have $m > 0$ and $K_X^2 < 0$, hence $-K_X \cdot (\mathcal{F} + K_X) > 1$ and $\mathcal{S}(\mathcal{F} + K_X, e_0) = 0$ by (a). As in (a), we have an exact sequence $\mathcal{S}(\mathcal{F} + K_X, e_0) \rightarrow \mathcal{S}(\mathcal{F}, e_0) \rightarrow \mathcal{S}(\mathcal{F} \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) \rightarrow 0$. Applying it gives $s(\mathcal{F}, e_0) = s(\mathcal{F} \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D)$, but $h^0(D, \mathcal{F} \otimes \mathcal{O}_D) = 1$ since $\mathcal{F} \otimes \mathcal{O}_D$ has degree 1. Thus $H^0(D, \mathcal{F} \otimes \mathcal{O}_D) \otimes H^0(D, e_0 \otimes \mathcal{O}_D) \rightarrow H^0(D, (\mathcal{F} + e_0) \otimes \mathcal{O}_D)$ is injective and one easily computes $s(\mathcal{F} \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) = 1$.

(c) If $\mathcal{F} = 0$, then $\mathcal{S}(\mathcal{F}, e_0) = 0$ is clear, so assume $\mathcal{F} \neq 0$. Since $-K_X \cdot \mathcal{F} = 0$, we see $m > 0$ and $\mathcal{F} + K_X$ is numerically effective. Since $\mathcal{F} = (\mathcal{F})_m$ with $-K_X \cdot \mathcal{F} = 0$, we see $\mathcal{F} \in \Lambda$, so $\mathcal{F} \otimes \mathcal{O}_D = \mathcal{O}_D$ and $h^0(D, \mathcal{O}_D \otimes \mathcal{F}) = 1$, and as above we see $h^1(X, e_0 + K_X) = h^1(X, \mathcal{F} + K_X + e_0) = 0$. If $-K_X^2 > 0$, then also $h^1(X, \mathcal{F} + K_X) = 0$ by Proposition III.ii.i.1(c). If, however, $-K_X^2 = 0$, then using Proposition III.ii.i.1(b) we see $h^0(X, \mathcal{F} + K_X) + h^0(D, \mathcal{F} \otimes \mathcal{O}_D) = h^0(X, \mathcal{F})$, so $0 \rightarrow \mathcal{F} + K_X \rightarrow \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{O}_D \rightarrow 0$ is exact on global sections. In either case, then, by Proposition II.3(b,c) we have an exact sequence $\mathcal{R}(\mathcal{F} \otimes \mathcal{O}_D, e_0) \rightarrow \mathcal{S}(\mathcal{F} + K_X, e_0) \rightarrow \mathcal{S}(\mathcal{F}, e_0) \rightarrow \mathcal{S}(\mathcal{F} \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) \rightarrow 0$.

Since $\mathcal{F} \otimes \mathcal{O}_D = \mathcal{O}_D$, we see $\mathcal{R}(\mathcal{F} \otimes \mathcal{O}_D, e_0) = 0$ and $s(\mathcal{F} \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) = 0$. Thus we have $s(\mathcal{F}, e_0) = s(\mathcal{F} + K_X, e_0)$; now $s(\mathcal{F} + K_X, e_0)$ equals 0 by (a) if $r > 10$, and 1 by (b) if $r = 10$. If $r = 9$, then $\mathcal{F} = -mK_X$. By Proposition III.ii.i.1(b), $m = ab$, where a is the least positive integer such that $-aK_X \in \Lambda$. Likewise, $(\mathcal{F} + K_X)_m = -a(b-1)K_X$ and $h^0(X, \mathcal{F} + K_X + e_0) - h^0(X, (\mathcal{F} + K_X)_m + e_0) = 3a - 3$; by induction, we may assume $s(\mathcal{F} + K_X, e_0) = 3(b-1)(a-1)$, so $s(\mathcal{F} + K_X, e_0) = 3b(a-1)$ by Lemma II.10.

$\diamond$

Example III.ii.i.3: We now give an example using our results to obtain an explicit resolution. Consider subschemes $Z \subset \mathbf{P}^2$ of the form $Z = m(p_1 + \cdots + p_{12})$, where $p_1, \dots, p_{12}$ are essentially distinct points on a smooth cubic. Let X be the blowing up of the points, let $e_0, \dots, e_{12}$ be the associated exceptional configuration, let D be the proper transform of the cubic, and assume that the kernel Λ of $\text{Cl}(X) \rightarrow \text{Cl}(D)$ is trivial. Then the minimal free resolution of I_Z is:

$$0 \rightarrow \bigoplus_{1 \leq i \leq m} R[-3m - i - 2]^3 \rightarrow R[-3m] \oplus \left(\bigoplus_{1 \leq i \leq m} R[-3m - i - 1]^3 \right) \rightarrow I_Z \rightarrow 0.$$

We sketch a proof. Let $\mathcal{F}_n$ denote $ne_0 - m(e_1 + \cdots + e_{12})$. By Proposition III.ii.i.1, $h^0(X, \mathcal{F}_n) = 0$ for $n < 3m$, and for $n \geq 3m$ we see that $(\mathcal{F}_n)_m$ is regular, equal to 0 if $n = 3m$ and otherwise to $(n - 3m - 1)(-K_X) + (n - 3m)e_0$. Now, by Lemma II.10 and Theorem III.ii.i.2, we can compute that $\mathcal{S}(\mathcal{F}_n, e_0)$ vanishes for $n \leq 3m - 2$ or $n \geq 4m + 1$, equals 1 for $n = 3m - 1$, 0 for $n = 3m$ and 3 for $3m + 1 \leq n \leq 4m$. Thus the first syzygy module F_0 in the resolution is as claimed; now we merely need to check that the difference of the Hilbert functions of F_0 and I_Z coincides with the Hilbert function of $\bigoplus_{1 \leq i \leq m} R[-(3m + i + 2)]^3$. $\diamond$

III.ii.ii. Points infinitely near a flex

Let $p_1, \dots, p_r$ be essentially distinct points on a smooth plane cubic C , let X be the blowing up of the points with $e_0, \dots, e_r$ the corresponding exceptional configuration, and let D denote the proper transform of C on X . In this subsection we shall always assume that p_1 is a flex of C , and p_i for $i > 1$ is infinitely near to p_1 . This is equivalent to $r_0 = e_0 - e_1 - e_2 - e_3$ and $r_i = e_i - e_{i+1}$, $i > 0$, all being in the kernel Λ of $\text{Cl}(X) \rightarrow \text{Cl}(D)$. But r_0 and r_i , $i > 0$, give a basis for the subspace $K^\perp$ of $\text{Cl}(X)$ of classes perpendicular to K_X and so the requirements that p_1 be a flex of C , and p_i for $i > 1$ be points of proper transforms of C and infinitely near to p_1 are equivalent to $\Lambda = K^\perp$. Thus we may equivalently say that $p_1, \dots, p_r$ are essentially distinct points on a smooth plane cubic such that $\Lambda = K^\perp$.

Zariski decompositions and $h^0(X, \mathcal{F})$ for any $\mathcal{F} \in \text{Cl}(X)$ can be computed using [6]. Thus, as discussed in Section I, to compute a resolution for I_Z for any fat point subscheme Z supported at $p_1, \dots, p_r$ it suffices to determine $s(\mathcal{H}, e_0)$ for numerically effective classes $\mathcal{H}$, which we do below. Thus the results below allow one to work out a resolution for the ideal of any fat point subscheme supported at $p_1, \dots, p_r$.

The values of $s(\mathcal{F}, e_0)$ that we obtain are related to the structure of NEFF, so we begin by describing the cone NEFF of numerically effective divisors, for which we use the following notation: $\mathcal{H}_0 = e_0$, $\mathcal{H}_1 = e_0 - e_1$, $\mathcal{H}_2 = 2e_0 - e_1 - e_2$, and $\mathcal{H}_i = 3e_0 - e_1 - \cdots - e_i$, $i \geq 3$. In addition, we will need to recall facts about h^1 for numerically effective classes, which will also be helpful to readers moved to work out examples of resolutions of fat point ideals for fat point subschemes supported at $p_1, \dots, p_r$.

Lemma III.ii.ii.1: *Let $\mathcal{F}$ be a class on X ; then:*

- (a) $\mathcal{F} \in \text{NEFF}$ if and only if $\mathcal{F}$ is a nonnegative (integer) linear combination of $\mathcal{H}_i$, $i \geq 0$, such that $-\mathcal{F} \cdot K_X \geq 0$; and
- (b) for $\mathcal{F} \in \text{NEFF}$ we have

$$h^1(X, \mathcal{F}) = \begin{cases} 0, & \text{if } -K_X \cdot \mathcal{F} \geq 1; \\ 1, & \text{if } \mathcal{F}^2 > -K_X \cdot \mathcal{F} = 0; \\ \mathcal{F} \cdot e_1, & \text{if } \mathcal{F}^2 = -K_X \cdot \mathcal{F} = 0. \end{cases}$$

Moreover, writing $\mathcal{F} \in \text{NEFF}$ as $\mathcal{F} = \sum_i a_i \mathcal{H}_i$, the linear system of sections of $\mathcal{F}$ has a nonempty base locus if and only if: $\mathcal{F} = \mathcal{H}_8$ or $-K_X \cdot \mathcal{F} = 1$ but $\mathcal{F} \neq \mathcal{H}_8 + a_9 \mathcal{H}_9$ (in this case the base locus is a single point if $j = r$, and it is the divisor E_{j+1} if $j < r$, where j is the greatest subscript with $a_j > 0$); or $\mathcal{F} = \mathcal{H}_8 + a_9 \mathcal{H}_9$ with $a_9 > 0$ (in which case E_9 is the base locus); or $\mathcal{F} = \mathcal{H}_8 + a_9 \mathcal{H}_9 + \mathcal{H}_{10}$ (in which case $E_9 - E_{10}$ is the base locus).

Proof: (a) See [6].

(b) See [6]. We remark that $\mathcal{F}^2 = -K_X \cdot \mathcal{F} = 0$ only occurs for $\mathcal{F} = s\mathcal{H}_9$, for some $s \geq 0$, in which case $s = \mathcal{F} \cdot e_1$. $\diamond$

By Lemma III.ii.ii.1, NEFF is contained in the nonnegative subsemigroup of $\text{Cl}(X)$ generated by $\mathcal{H}_0, \dots, \mathcal{H}_r$. It turns out to be convenient to distinguish two types. Let $\mathcal{F} = \sum_i a_i \mathcal{H}_i$ be a numerically effective class (hence $a_i \geq 0$ for all i). We say $\mathcal{F}$ is of *type I* if $a_i > 0$ for some $i < 8$; otherwise, we say $\mathcal{F}$ is of *type II* (i.e., either $a_i = 0$ for $i \neq 9$ or the least index l such that $a_l \neq 0$ is 8). We first consider classes of type I.

Theorem III.ii.ii.2: *Let $\mathcal{F}$ be a type I numerically effective class on X , and, writing $\mathcal{F}$ as a nonnegative linear combination $\mathcal{F} = \sum_i a_i \mathcal{H}_i$, let j be the largest index with $a_j > 0$. Then $s(\mathcal{F}, e_0) = 0$ unless either $-K_X \cdot \mathcal{F} = 1$, or $-K_X \cdot \mathcal{F} = 0$ and $j = 10$, in which cases $s(\mathcal{F}, e_0) = 1$.*

Proof: Clearly, $\mathcal{S}(\mathcal{O}_{X_l}, e_0) = 0$, so we may assume that $\mathcal{F} \neq 0$. Let l be the least index i such that $a_i > 0$; then $l < 8$. By induction, we may assume our result is true for $\mathcal{F} - \mathcal{H}_j$. For each i , let X_i be the blowing up of $p_1, \dots, p_i$ and let D_i be the proper transform of C to X_i . By duality and the fact that $K_{X_i} + \mathcal{H}_i$ is always a multiple of e_0 , we see $h^1(X_i, e_0 - \mathcal{H}_i) = 0$, and, applying Lemma III.ii.ii.1, we see $h^1(X_j, \mathcal{F} - \mathcal{H}_j)$ and $h^1(X_j, \mathcal{F} - \mathcal{H}_j + e_0)$ vanish. Thus (suppressing the subscripts on X_j and D_j) we have the usual exact sequence $\mathcal{R}(\mathcal{F} \otimes \mathcal{O}_D, e_0) \rightarrow \mathcal{S}(\mathcal{F} - \mathcal{H}_j, e_0) \rightarrow \mathcal{S}(\mathcal{F}, e_0) \rightarrow \mathcal{S}(\mathcal{F} \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) \rightarrow 0$.

If $-K_X \cdot \mathcal{F} \geq 2$, then $\mathcal{S}(\mathcal{F} \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) = 0$ by Proposition II.3(c), and either $\mathcal{F} - \mathcal{H}_j = 0$, or $-K_X \cdot (\mathcal{F} - \mathcal{H}_j) \geq -K_X \cdot \mathcal{H}_l \geq 2$, so $\mathcal{S}(\mathcal{F} - \mathcal{H}_j, e_0) = 0$ by induction, and, applying the exact sequence, $\mathcal{S}(\mathcal{F}, e_0) = 0$.

If $-K_X \cdot \mathcal{F} < 2$, then $j > 9$ and, as in the proof of Theorem III.ii.i.2(b,c), $\mathcal{R}(\mathcal{F} \otimes \mathcal{O}_D, e_0) = 0$. Now, $\mathcal{S}(\mathcal{F} \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D)$ is 0 if $-K_X \cdot \mathcal{F} = 0$ and 1 if $-K_X \cdot \mathcal{F} = 1$, so, by exactness, $s(\mathcal{F}, e_0) = s(\mathcal{F} - \mathcal{H}_j, e_0) - K_X \cdot \mathcal{F}$ if $0 \leq -K_X \cdot \mathcal{F} \leq 1$, and applying the inductive hypothesis to $\mathcal{F} - \mathcal{H}_j$ gives the result. $\diamond$

We now consider numerically effective classes $\mathcal{F}$ of type II. Note that if $\mathcal{F} = \mathcal{H}_8 + b_9 \mathcal{H}_9 + b_{10} \mathcal{H}_{10}$ with either b_9 or b_{10} positive, then the linear system of sections of $\mathcal{F}$ has a fixed component, making this a case we do not need to consider (because then $s(\mathcal{F}, e_0) = b_9 + 1$ follows from Lemma II.10 and the following theorem). Thus the only type II classes we need to consider are $\mathcal{H}_8$, $b_9 \mathcal{H}_9$ and numerically effective classes $\sum_{i \geq 8} b_i \mathcal{H}_i$ with $b_8 > 1$.

Theorem III.ii.ii.3: *Let $\mathcal{F} = \sum_{i \geq 8} b_i \mathcal{H}_i$ be a numerically effective class on X (hence $b_i \geq 0$ for each i), and, if $\mathcal{F} \neq 0$, let j be the greatest index i such that $b_i > 0$.*

- (a) *We have $s(\mathcal{H}_8, e_0) = 1$ and $s(b_9 \mathcal{H}_9, e_0) = 0$.*
- (b) *If $b_8 > 1$, then $s(\mathcal{F}, e_0) = 1$ unless either $-K_X \cdot \mathcal{F} = 1$, or $-K_X \cdot \mathcal{F} = 0$ and $j = 10$, in which cases $s(\mathcal{F}, e_0) = 2$.*

Proof: As in the proof of Theorem III.ii.ii.2 (and again suppressing subscripts on X_j and D_j), Proposition II.3 gives an exact sequence $\mathcal{R}(\mathcal{F} \otimes \mathcal{O}_D, e_0) \rightarrow \mathcal{S}(\mathcal{F} - \mathcal{H}_j, e_0) \rightarrow \mathcal{S}(\mathcal{F}, e_0) \rightarrow \mathcal{S}(\mathcal{F} \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) \rightarrow 0$.

(a) Since $\mathcal{F} = b_9 \mathcal{H}_9$ is uniform (on the blowing up of $\mathbf{P}^2$ at $p_1, \dots, p_9$), we have $\mathcal{S}(\mathcal{F}, e_0) = 0$ by Theorem III.ii.i.2(c), as required, so assume $\mathcal{F} = \mathcal{H}_8$. Clearly, $\mathcal{S}(\mathcal{O}_X, e_0) = 0$, so, by the exact sequence with $j = 8$, $s(\mathcal{H}_8, e_0) = s(\mathcal{H}_8 \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D)$, and, as in the proof of Theorem III.ii.i.2(b), $s(\mathcal{H}_8 \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) = 1$, so $s(\mathcal{H}_8, e_0) = 1$, as required.

(b) We first show the $j = 8$ case implies the others. So suppose $j = 8$ and assume $s(b \mathcal{H}_8, e_0) = 1$ for all $b \geq 0$. Since $\mathcal{S}((b+1)\mathcal{H}_8 \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) = 0$ for all $b \geq 1$ by Proposition II.3(c), by the exact sequence above the homomorphism $\mathcal{S}(b \mathcal{H}_8, e_0) \rightarrow \mathcal{S}((b+1)\mathcal{H}_8, e_0)$ is an isomorphism for all $b > 0$.

Now suppose $j > 8$. By induction, we have an isomorphism $\mathcal{S}(b_8 \mathcal{H}_8, e_0) \rightarrow \mathcal{S}((\sum_{i \geq 8} b_i) \mathcal{H}_8, e_0)$, and it factors as $\mathcal{S}(b_8 \mathcal{H}_8, e_0) \rightarrow \mathcal{S}(\mathcal{F}, e_0) \rightarrow \mathcal{S}((\sum_{i \geq 8} b_i) \mathcal{H}_8, e_0)$, which implies $s(\mathcal{F}, e_0) \geq 1$. If $-K_X \cdot \mathcal{F} > 1$, then $s(\mathcal{F} \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) = 0$ by Proposition II.3(c), so $\mathcal{S}(\mathcal{F} - \mathcal{H}_j, e_0) \rightarrow \mathcal{S}(\mathcal{F}, e_0)$ is surjective. By induction we may assume that $s(\mathcal{F} - \mathcal{H}_j, e_0) = 1$, so $s(\mathcal{F}, e_0) \leq 1$ which with $s(\mathcal{F}, e_0) \geq 1$ gives $s(\mathcal{F}, e_0) = 1$. If, however, $-K_X \cdot \mathcal{F} = 1$, then $b_8 > 1$ implies $j > 9$, so $-K_X \cdot (\mathcal{F} - \mathcal{H}_j) > 1$, and by induction we may assume $s(\mathcal{F} - \mathcal{H}_j, e_0) = 1$. But, as above, the isomorphism $\mathcal{S}(b_8 \mathcal{H}_8, e_0) \rightarrow \mathcal{S}((\sum_{8 \leq i} b_i) \mathcal{H}_8, e_0)$ factors through $\mathcal{S}(\mathcal{F} - \mathcal{H}_j, e_0) \rightarrow \mathcal{S}(\mathcal{F}, e_0)$, so the latter is injective. The exact sequence with $s(\mathcal{F} \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) = 1$ now gives $s(\mathcal{F}, e_0) = 2$. For the case that $-K_X \cdot \mathcal{F} = 0$, we have $\mathcal{F} \otimes \mathcal{O}_D = \mathcal{O}_D$ so $\mathcal{R}(\mathcal{F} \otimes \mathcal{O}_D, e_0) = 0 = \mathcal{S}(\mathcal{F} \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D)$. Thus $s(\mathcal{F} - \mathcal{H}_j, e_0) = s(\mathcal{F}, e_0)$; but $j \geq 10$ so by what we have already done we see $s(\mathcal{F} - \mathcal{H}_j, e_0) = 1$ if $j > 10$ and $s(\mathcal{F} - \mathcal{H}_j, e_0) = 2$ if $j = 10$. This finishes the proof, modulo showing

$s(b\mathcal{H}_8, e_0) = 1$ for all $b > 0$, to which we now proceed.

We already have observed that $s(\mathcal{H}_8, e_0) = 1$ so we may assume $b \geq 2$. The pencil of cubics through $p_1, \dots, p_8$ includes our smooth cubic C and a triple line $3L$ tangent to C at the flex p_1 . If C is not supersingular in characteristic 3, then $\text{Pic}(C)$ has nontrivial 3-torsion so there is a second flex. We will denote the line on X tangent to D at this second flex by S , and we will choose projective coordinates x, y, z on $\mathbf{P}^2$ such that $z = 0$ defines L , $y = 0$ defines the line through p_1 and the second flex, and x defines the line tangent to C at the second flex (whose transform on X is thus S). By scaling x, y and z , the homogeneous form f defining C can be taken to be $y^3 + x^2z + xz^2 + axyz$, for some $a \in k$. If C is supersingular in characteristic 3, then we can take f to be $y^3 + x^2z + yz^2$ (see Proposition IV.4.21 and Example IV.4.23.1 of [14]).

By Proposition II.3(a), we have the exact sequence $\mathcal{S}(\mathcal{O}_X, b\mathcal{H}_8) \rightarrow \mathcal{S}(e_0, b\mathcal{H}_8) \rightarrow \mathcal{S}(e_0 \otimes \mathcal{O}_S, b\mathcal{H}_8) \rightarrow 0$. Clearly $\mathcal{S}(\mathcal{O}_X, b\mathcal{H}_8) = 0$, so $s(e_0, b\mathcal{H}_8) = s(e_0 \otimes \mathcal{O}_S, b\mathcal{H}_8)$. Taking y and z for projective coordinates on S , $H^0(S, e_0 + b\mathcal{H}_8)$ can be identified with the vector space with basis consisting of the monomials in y and z of degree $3b + 1$. By explicitly computing the image of $H^0(S, e_0) \otimes H^0(X, b\mathcal{H}_8) \rightarrow H^0(S, e_0 + b\mathcal{H}_8)$, we will see that $s(e_0 \otimes \mathcal{O}_S, b\mathcal{H}_8) = 1$.

Under the identification of $H^0(X, de_0)$ with $H^0(\mathbf{P}^2, de_0)$, which we regard as the space of monomials in x, y and z of degree d , we can for any class $\mathcal{G}$ on X regard $H^0(X, \mathcal{G})$ as a subspace of a space $H^0(\mathbf{P}^2, (e_0 \cdot \mathcal{G})e_0)$ of monomials of degree $e_0 \cdot \mathcal{G}$. From this point of view, $\{f, z^3\}$ is a basis for $H^0(X, \mathcal{H}_8)$. (To say this differently, f and z^3 are forms passing through $p_1, \dots, p_8$ with multiplicity at least 1 at each point, so each is an element of $H^0(\mathbf{P}^2, 3e_0)$ corresponding to an element of $H^0(X, \mathcal{H}_8)$. Since f and z^3 are linearly independent in $H^0(\mathbf{P}^2, 3e_0)$ and $h^0(X, \mathcal{H}_8) = 2$, they correspond to a basis. Of course, that f passes through each of $p_1, \dots, p_8$ with multiplicity at least 1 is obvious; to see the same for z^3 , note that $\mathcal{H}_8 = 3r_0 + 2r_1 + 4r_2 + 6r_3 + 5r_4 + 4r_5 + 3r_6 + 2r_7 + e_8$ and that each of $r_1, \dots, r_8$ and e_8 is the class of a component of E_1 , while r_0 is the class of the proper transform of $z = 0$.) Similarly, $\{f^2, f^2z^3, z^6, yz^5\}$ is a basis for $H^0(X, 2\mathcal{H}_8)$ (to see that $yz^5 \in H^0(\mathbf{P}^2, 6e_0)$ corresponds to a section of $2\mathcal{H}_8$, note that $2\mathcal{H}_8 = (e_0 - e_1) + 5r_0 + 4r_1 + 7r_2 + 10r_3 + 8r_4 + 6r_5 + 4r_6 + 2r_7$, so the y comes as a section of $e_0 - e_1$ and the z^5 from $5r_0$), $\{f^3, f^2z^3, f^2z^6, f^2yz^5, z^9, yz^8, xz^8\}$ is a basis for $H^0(X, 3\mathcal{H}_8)$ (to see that $xz^8 \in H^0(\mathbf{P}^2, 9e_0)$ corresponds to a section of $3\mathcal{H}_8$, note that $3\mathcal{H}_8 = e_0 + 8r_0 + 5r_1 + 10r_2 + 15r_3 + 12r_4 + 9r_5 + 6r_6 + 3r_7$, so the x comes as a section of e_0 and the z^8 from $8r_0$), and $\{f^4, f^3z^3, f^2z^6, f^2yz^5, f^2z^9, f^2yz^8, f^2xz^8, z^{12}, yz^{11}, xz^{11}, y^2z^{10}\}$ is a basis for $H^0(X, 4\mathcal{H}_8)$.

Also note that $\mathcal{S}(2\mathcal{H}_8, 3\mathcal{H}_8) = 0$ (one checks that $fH^0(X, 4\mathcal{H}_8) \subset H^0(X, 5\mathcal{H}_8)$ is in the image of $H^0(X, 2\mathcal{H}_8) \otimes H^0(X, 3\mathcal{H}_8)$ explicitly and that $\mathcal{S}(2\mathcal{H}_8 \otimes \mathcal{O}_D, 3\mathcal{H}_8 \otimes \mathcal{O}_D) = 0$), so $H^0(X, 5\mathcal{H}_8)$ is spanned by products of sections of $H^0(X, 2\mathcal{H}_8)$ and $H^0(X, 3\mathcal{H}_8)$. One can now check explicitly for $1 \leq b \leq 5$ that the image of the restriction homomorphism $H^0(X, b\mathcal{H}_8) \rightarrow H^0(S, b\mathcal{H}_8)$ is contained in the space spanned by $\{z^{3b}, yz^{3b-1}, \dots, y^{3b-3}z^3, (y^3 + \alpha yz^2)^b\}$, where α is 1 if C is supersingular in characteristic 3, and 0 otherwise. By Proposition III.1 of [12], $b\mathcal{H}_8$ is normally generated for $b \geq 3$, so $\mathcal{S}((b-3)\mathcal{H}_8, 3\mathcal{H}_8) = 0$ for all $b > 5$. Thus, for $b \geq 6$, $H^0(X, b\mathcal{H}_8)$ is spanned by products of forms from $H^0(X, 3\mathcal{H}_8)$ and $H^0(X, (b-3)\mathcal{H}_8)$, from which it is easy to conclude that the image of the restriction homomorphism $H^0(X, b\mathcal{H}_8) \rightarrow H^0(S, b\mathcal{H}_8)$ is contained for all b in the space spanned by $\{z^{3b}, yz^{3b-1}, \dots, y^{3b-3}z^3, (y^3 + \alpha yz^2)^b\}$. But an easy argument shows that the subspace of $H^0(S, e_0 + b\mathcal{H}_8)$ spanned by products of elements from $\{z^{3b}, yz^{3b-1}, \dots, y^{3b-3}z^3, (y^3 + \alpha yz^2)^b\}$ and $\{y, z\}$ never includes $y^{3b-2}z^2$; i.e., $s(e_0 \otimes \mathcal{O}_S, b\mathcal{H}_8) \geq 1$. To see that $s(e_0 \otimes \mathcal{O}_S, b\mathcal{H}_8) = 1$, recall that we have an exact sequence $\mathcal{S}((b-1)\mathcal{H}_8, e_0) \rightarrow \mathcal{S}(b\mathcal{H}_8, e_0) \rightarrow \mathcal{S}(b\mathcal{H}_8 \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) \rightarrow 0$ with $\mathcal{S}(b\mathcal{H}_8 \otimes \mathcal{O}_D, e_0 \otimes \mathcal{O}_D) = 0$ for $b \geq 2$ by Proposition II.3(c). Since $s(\mathcal{H}_8, e_0) = 1$ by (a), we have by induction that $s(e_0 \otimes \mathcal{O}_S, b\mathcal{H}_8) = s(b\mathcal{H}_8, e_0) \leq 1$ for all $b \geq 1$. $\diamond$

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Addendum to:

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Free Resolutions of Fat Point Ideals on $\mathbf{P}^2$

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This short note adds some details to the proof of Lemma II.5 which were not included in the proof in the originally published version of this paper. The changes are indicated by left and right indentation.

Lemma II.4: *Let X be a smooth projective rational surface and let $\mathcal{N}$ be the class of a nontrivial effective divisor N on X . If $\mathcal{N} + K_X$ is not the class of an effective divisor and $\mathcal{F}$ meets every component of $\mathcal{N}$ nonnegatively, then $h^0(N, \mathcal{F}) > 0$, $h^1(N, \mathcal{F}) = 0$, $N^2 + N \cdot K_X < -1$, and every component M of N is a smooth rational curve (of negative self-intersection, if M does not move).*

Lemma II.5: *Let X be a smooth projective rational surface, and let $\mathcal{N}$ be the class of an effective divisor N on X such that $h^0(X, \mathcal{N} + K_X) = 0$. If $\mathcal{F}$ and $\mathcal{G}$ are the restrictions to N of divisor classes $\mathcal{F}'$ and $\mathcal{G}'$ on X which meet each component of N nonnegatively, then $\mathcal{S}(\mathcal{F}, \mathcal{G}) = 0$.*

Proof: To prove the lemma, induct on the sum n of the multiplicities of the components of N . By Lemma II.4, $h^1(N, \mathcal{O}) = 0$ and every component of N is a smooth rational curve. Thus the case $n = 1$ is trivial (since then $N = \mathbf{P}^1$, and the space of polynomials of degree f in two variables tensor the space of polynomials of degree g in two variables maps onto the space of polynomials of degree $f + g$). So say $n > 1$.

As in the proof of Theorem 1.7 of [1] (or see the proof of Lemma II.6 of [10]), N has a component C such that $(N - C) \cdot C \leq 1$. Let L be the effective divisor $N - C$ and let $\mathcal{L}$ be its class. Thus we have an exact sequence $0 \rightarrow \mathcal{O}_C \otimes (-\mathcal{L}) \rightarrow \mathcal{O}_N \rightarrow \mathcal{O}_L \rightarrow 0$.

To see this, apply the snake lemma to

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{O}_X(-N) & \rightarrow & \mathcal{O}_X & \rightarrow & \mathcal{O}_N \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{O}_X(-L) & \rightarrow & \mathcal{O}_X & \rightarrow & \mathcal{O}_L \rightarrow 0 \end{array}$$

to see that the kernel of $\mathcal{O}_N \rightarrow \mathcal{O}_L$ is just the cokernel of $\mathcal{O}_X(-N) \rightarrow \mathcal{O}_X(-L)$, which is just $\mathcal{O}_C \otimes \mathcal{O}_X(-L)$, which we may write as $\mathcal{O}_C(-L)$.

Now, $-L \cdot C \geq -1$, and both $\mathcal{F}'$ and $\mathcal{G}'$ meet C nonnegatively. We may assume $\mathcal{F}' \cdot C \geq \mathcal{G}' \cdot C$, otherwise reverse the roles of $\mathcal{F}'$ and $\mathcal{G}'$. Since $C = \mathbf{P}^1$, we see that $h^1(C, \mathcal{O}_C \otimes (\mathcal{F}' - \mathcal{L}))$, $h^1(C, \mathcal{O}_C \otimes (\mathcal{G}' - \mathcal{L}))$ and $h^1(C, \mathcal{O}_C \otimes (\mathcal{F}' + \mathcal{G}' - \mathcal{L}))$ all vanish. An argument similar to that used to prove Proposition II.3(a, b) now shows that we have an exact sequence $\mathcal{S}(\mathcal{O}_C \otimes (\mathcal{F}' - \mathcal{L}), \mathcal{O}_C \otimes \mathcal{G}') \rightarrow \mathcal{S}(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{S}(\mathcal{O}_L \otimes \mathcal{F}, \mathcal{O}_L \otimes \mathcal{G}) \rightarrow 0$.

What is actually clear here is that we have $\mathcal{S}(\mathcal{O}_C \otimes (\mathcal{F}' - \mathcal{L}), \mathcal{G}) \rightarrow \mathcal{S}(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{S}(\mathcal{O}_L \otimes \mathcal{F}, \mathcal{G}) \rightarrow 0$. Since $h^1(C, \mathcal{O}_C \otimes (\mathcal{G}' - \mathcal{L})) = 0$, we know $\mathcal{G} \rightarrow \mathcal{O}_L \otimes \mathcal{G}$ is surjective on global sections, and hence that $\mathcal{S}(\mathcal{O}_L \otimes \mathcal{F}, \mathcal{G})$ is the same as $\mathcal{S}(\mathcal{O}_L \otimes \mathcal{F}, \mathcal{O}_L \otimes \mathcal{G})$. What needs additional justification here is that $\mathcal{O}_N \otimes \mathcal{G}' \rightarrow \mathcal{O}_C \otimes \mathcal{G}'$ is surjective on global sections, so that we can conclude that $\mathcal{S}(\mathcal{O}_C \otimes (\mathcal{F}' - \mathcal{L}), \mathcal{G})$ is the same as $\mathcal{S}(\mathcal{O}_C \otimes (\mathcal{F}' - \mathcal{L}), \mathcal{O}_C \otimes \mathcal{G}')$.

Now, $\mathcal{N} + K_X$ is not the class of an effective divisor, and the same will remain true if we replace N by any subscheme of N obtained by subtracting off irreducible components of N . Thus any such resulting subscheme M of N has the property, like N itself, that there is a component D of M such that $(M - D) \cdot D \leq 1$. If M is just N with the reduced induced scheme structure, then by induction on the number of components of M it follows (using Lemma II.4) that any two components of N are smooth rational curves that are either disjoint or meet transversely at a single point, and no sequence $B_1, \dots, B_i$ of distinct components exists such that $B_i \cdot B_1 > 0$ and $B_j \cdot B_{j+1} > 0$ for

$1 \leq j < i$ (in particular, no three components meet at a single point, and the components of M form a disjoint union of trees).

First assume that N is reduced; i.e. that $N = N_{red}$. Then C is not a component of $N - C$. Choose a section σ_C of $\mathcal{O}_C \otimes \mathcal{G}'$, and for each of the other components B of N , choose a section σ_B of $\mathcal{O}_B \otimes \mathcal{G}'$ such that σ_B does not vanish at any of the points where B meets another component of N . (This is possible since B is smooth and rational, so $\mathcal{O}_B \otimes \mathcal{G}'$ is $\mathcal{O}_{\mathbf{P}^1}(d)$ for some $d \geq 0$, so a section can always be chosen which does not vanish at any of a given finite set of points of B .) Since N has no cycles and the components meet transversely, it is clear that starting from σ_C one can patch together the sections σ_B to get a section σ of $\mathcal{G}$ which restricts to σ_C . Thus $\mathcal{O}_N \otimes \mathcal{G}' \rightarrow \mathcal{O}_C \otimes \mathcal{G}'$ is surjective on global sections.

Now assume that N is not reduced. Let M be the union of the components of N which have multiplicity greater than 1 (taken with the same multiplicities as they have in N) together with those multiplicity 1 components of N that meet one of these. No multiplicity 1 component B of M satisfies $B \cdot (M - B) \leq 1$, so there must be a component B of multiplicity more than 1 that does, and hence we also have $B \cdot (N - B) \leq 1$ for some component B of N of multiplicity more than 1. Now from this and $0 \rightarrow \mathcal{O}_B(-N + B) \otimes \mathcal{G} \rightarrow \mathcal{O}_N \otimes \mathcal{G}' \rightarrow \mathcal{O}_J \otimes \mathcal{G}' \rightarrow 0$, where $J = N - B$, we see $h^1(B, \mathcal{O}_B(-N + B) \otimes \mathcal{G}) = 0$, so $\mathcal{O}_N \otimes \mathcal{G}' \rightarrow \mathcal{O}_J \otimes \mathcal{G}'$ is surjective on global sections. But J still has C as a component, because either C has multiplicity 1 in N (and hence $C \neq B$), or C has multiplicity more than 1 in N (and so even if $B = C$, C remains a component of $N - B = J$). By induction on the number of components, we conclude that $\mathcal{O}_N \otimes \mathcal{G}' \rightarrow \mathcal{O}_{N_{red}} \otimes \mathcal{G}'$ is surjective on global sections. But C is still a component of N_{red} , and $\mathcal{O}_{N_{red}} \otimes \mathcal{G}' \rightarrow \mathcal{O}_C \otimes \mathcal{G}'$ is surjective on global sections from above, hence so is $\mathcal{O}_N \otimes \mathcal{G}' \rightarrow \mathcal{O}_C \otimes \mathcal{G}'$.

Since $\mathcal{S}(\mathcal{O}_L \otimes \mathcal{F}, \mathcal{O}_L \otimes \mathcal{G}) = 0$ by induction, it suffices to show $\mathcal{S}(\mathcal{O}_C \otimes (\mathcal{F}' - \mathcal{L}), \mathcal{O}_C \otimes \mathcal{G}') = 0$. If $C \cdot (\mathcal{F}' - \mathcal{L}) \geq 0$, then the latter is 0 (as in the previous paragraph). Otherwise, we must have $0 = \mathcal{F}' \cdot C = \mathcal{G}' \cdot C$ and $C \cdot L = 1$, so $\mathcal{O}_C(-1) = \mathcal{O}_C \otimes (\mathcal{F}' - \mathcal{L})$ and $\mathcal{O}_C = \mathcal{O}_C \otimes \mathcal{G}'$, which means $h^0(\mathcal{O}_C, \mathcal{O}_C \otimes (\mathcal{F}' + \mathcal{G}' - \mathcal{L})) = 0$ and hence again $\mathcal{S}(\mathcal{O}_C \otimes (\mathcal{F}' - \mathcal{L}), \mathcal{O}_C \otimes \mathcal{G}') = 0$. $\diamond$