

Resolutions of Fat Point Ideals involving 8 General Points of $\mathbf{P}^2$

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Abstract: The main result, Theorem 1, provides an algorithm for determining the minimal free resolution of fat point subschemes of $\mathbf{P}^2$ involving up to 8 general points of arbitrary multiplicities; the resolutions obtained hold for any algebraically closed field, independent of the characteristic. The algorithm works by giving a formula in nice cases, and a reduction to the nice cases otherwise. The algorithm, which does not involve Gröbner bases, is very fast. Partial information is also obtained in certain cases with $n > 8$.

1. Introduction

Determining the Hilbert function and minimal free resolution of ideals defining n general fat points of $\mathbf{P}^2$ is a difficult problem that has attracted the attention of numerous researchers over the years. For $n > 9$, the problem remains unsolved in general. For $n \leq 9$, Nagata [N] resolved the problem for Hilbert functions, while for $n \leq 5$, Catalisano [C] resolved the problem for minimal free resolutions, extended (using different methods) by Fitchett [F2] to $n = 6$ and by Harbourne [H6] to $n = 7$. In this paper, we now extend this work to $n = 8$. Our results are based on studying linear systems on blow ups of $\mathbf{P}^2$ at the n points. For $n \leq 8$, these surfaces are Del Pezzo surfaces, and hence the semigroup of classes of effective divisors is finitely generated. It will be very difficult to extend our approach to the case of $n = 9$; for one thing, the surfaces obtained for $n \geq 9$ are no longer Del Pezzos, for another, the semigroup of effective divisor classes is no longer finitely generated.

We now recall the notion of fat points. Consider n distinct points $p_1, \dots, p_n$ of $\mathbf{P}^2$. Given nonnegative integers m_i , a fat point subscheme $Z = m_1p_1 + \dots + m_np_n$ is that subscheme defined by the homogeneous ideal $I_Z = I_1^{m_1} \cap \dots \cap I_n^{m_n}$ in the homogeneous coordinate ring $R = k[\mathbf{P}^2]$ of $\mathbf{P}^2$ (over any algebraically closed field k), where I_j is the ideal generated by all forms vanishing at p_j .

Because Z has codimension 2 and is arithmetically Cohen-Macaulay, the minimal free graded resolution of I_Z is of the form $0 \rightarrow F_1 \rightarrow F_0 \rightarrow I_Z \rightarrow 0$. To determine F_0 it then suffices to compute for each t the dimension $\nu_{t+1}(Z)$ of the cokernel of the multiplication map $\mu_t(Z) : (I_Z)_t \otimes R_1 \rightarrow (I_Z)_{t+1}$, since F_0 is simply $\bigoplus_{i \geq 0} R[-i]^{\nu_i}$. If we know the Hilbert function h_Z of I_Z (i.e., the dimension $h_Z(t) = \dim(I_Z)_t$ for each degree t of the homogeneous component $(I_Z)_t$ of I_Z of degree t), exactness of the resolution then allows us to determine the Hilbert function of F_1 and thus (since F_1 is free) F_1 itself. More explicitly, since in general $\nu_t(Z) = \dim(\text{Tor}_0(I_Z, k))_t$ and $F_1 = \bigoplus_{i \geq 0} R[-i]^{s_i}$ where $s_i = \dim(\text{Tor}_1(I_Z, k))_i$, we can tensor the Koszul complex for k by I_Z to compute these Tor's, and we find $\nu_i - s_i = \Delta^3 h_Z(i)$, where Δ is the difference operator (hence $\Delta h_Z(i) = h_Z(i) - h_Z(i-1)$, for example).

Instead of working with I and its components, by a standard translation, one can instead work on the surface X obtained by blowing up the n distinct points $p_1, \dots, p_n$ of $\mathbf{P}^2$, in which case one then has the corresponding birational morphism $\pi : X \rightarrow \mathbf{P}^2$. Throughout this paper, this is what X will denote.

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We now recall the standard translation referred to above. With respect to $\pi : X \rightarrow \mathbf{P}^2$, let L be the total transform to X of a line on $\mathbf{P}^2$, and let $E_i = \pi^{-1}(p_i)$ for $1 \leq i \leq n$. We call the divisors $L, E_1, \dots, E_n$ an *exceptional configuration*; their classes $[L], [E_1], \dots, [E_n] \in \text{Cl}(X)$ form a basis for the divisor class group $\text{Cl}(X)$ of X , with intersections given by $E_i \cdot E_j = -\delta_{ij}$ (where δ_{ij} denotes Kronecker's delta), $L \cdot E_i = 0$ for all i , and $L^2 = 1$. For $Z = m_1 p_1 + \dots + m_n p_n$, under natural identifications we have $(I_Z)_t = H^0(X, \mathcal{O}_X(F_t))$, hence $h_Z(t) = h^0(X, \mathcal{O}_X(F_t))$, where $F_t = tL - m_1 E_1 - \dots - m_n E_n$. Moreover, the map $\mu_t(Z) : (I_Z)_t \otimes R_1 \rightarrow (I_Z)_{t+1}$ given by multiplication corresponds under these identifications to the natural map $\mu_{F_t} : H^0(X, \mathcal{O}_X(F_t)) \otimes H^0(X, \mathcal{O}_X(L)) \rightarrow H^0(X, \mathcal{O}_X(\mathcal{O}_X(F_t) \otimes \mathcal{O}_X(L))) = H^0(X, \mathcal{O}_X(F_{t+1}))$. Thus, although in this paper we will be concerned with μ_F for an arbitrary divisor F , our results have an immediate application to computing resolutions of fat point ideals. We also note that as long as $h^0(X, \mathcal{O}_X(H))$ is known for arbitrary divisors H , to compute the dimension of the cokernel of μ_F for some divisor F , it is just as good to compute the dimension of the kernel of μ_F or the rank of μ_F , depending on convenience.

In addition to assuming that we have a birational morphism $X \rightarrow \mathbf{P}^2$ (i.e., that X is *basic*, that is, obtained by blowing up $n \geq 0$ points p_i of $\mathbf{P}^2$), we will assume that there is a particular smooth, irreducible anticanonical divisor D_X on X (i.e., the points blown up lie on a smooth cubic), and that X satisfies the following properties:

- (A1) the only integral curves besides possibly D_X of negative self-intersection on X are exceptional curves (i.e., smooth rational curves of self-intersection -1), and
- (A2) $h^1(X, \mathcal{O}_X(F)) = 0$ for any effective, numerically effective divisor F (F being *numerically effective* means that $F \cdot H \geq 0$ for every effective divisor H).

In these circumstances we will say that X is a *good* surface.

For example, any blow up X of $\mathbf{P}^2$ at $n \leq 8$ general points is good: (A2) holds by Theorem 8 of [H3], and (A1) holds by adjunction since the anticanonical class $-K_X$ is ample (see the first two paragraphs of the proof of Theorem 1 of [H3]). For another example, say that a basic surface X with a smooth, irreducible anticanonical divisor D_X is *injective* if the canonical map $\text{Pic}(X) \rightarrow \text{Pic}(D_X)$ is injective. (Although we will sometimes use D to denote an arbitrary divisor, we will throughout this paper use D_X to denote a smooth, irreducible anticanonical divisor.) Such a surface X is good: (A2) holds by Theorem III.1 of [H4], while (A1) holds by adjunction (any integral curve C of negative self-intersection which is neither an exceptional curve nor D_X must by adjunction have $C \cdot D_X = 0$ and hence $C = 0$ since $\text{Pic}(X) \rightarrow \text{Pic}(D_X)$ is injective).

We note that the condition that $\text{Pic}(X) \rightarrow \text{Pic}(D_X)$ is injective holds in all characteristics if the points blown up are sufficiently general points of a smooth plane cubic curve, if the ground field k is sufficiently large. It cannot hold, however, if k is the algebraic closure of a finite field. On the other hand, assuming the points p_i are general points of a smooth plane cubic curve, our result for the rank of μ_F for a particular F or for the minimal free resolution of the ideal I_Z for a particular Z , holds over any algebraically closed field k (this is because our result for a particular F or Z holds for some set of n $\bar{K}$ -points of a smooth plane cubic k -curve $C \subset \mathbf{P}^2$, where $\bar{K}$ is the algebraic closure of a sufficiently large extension K of k , but the k -points of C^n are a dense subset of the $\bar{K}$ -points).

We will denote the set of reduced, irreducible curves C on X with $h^0(X, \mathcal{O}_X(C)) = 1$ by Γ_X . By Theorem III.1 of [H4], (A1) implies that Γ_X includes the set of exceptional curves on X , and D_X if $D_X^2 \leq 0$, but nothing else.

Moreover, as shown in section I.4 of [H6] (*mutatis mutandis*, since $-K_X$ need not always be the class of an effective divisor in [H6]), given that X is good, one can in a completely effective manner determine the fixed components and dimension of any complete linear system on X . Our main interest, then, is, in addition, to determine the rank of μ_F for an arbitrary divisor F .

Assuming (A1) and (A2) above, our approach is to reduce the problem of computing the rank of μ_F for an arbitrary divisor F to that of doing so for a special class of divisors. A similar approach was taken in [H6], in which a reduction was made to ample F . Unfortunately, the class of ample divisors is still too coarse to get a nice answer in general (note that the exceptional case in Theorem 1(c)(ii) is ample), even for $n = 8$ points, so here we present a refined reduction to a particular special set of divisors F . Although for arbitrary n we cannot always handle the resulting divisors by present methods, for $n \leq 8$ using ad hoc methods we always can, thereby achieving a complete determination. In terms of fat points, this gives a complete determination of the minimal free resolution for the ideal of any fat points subscheme involving up to 8 general points of $\mathbf{P}^2$, essentially by reduction to nice cases where a formula applies.

To state our main result, we introduce some terminology and notation. For convenience, we will often write simply $h^0(X, F)$ for $h^0(X, \mathcal{O}_X(F))$. Also, we say a divisor F on X is *monotone* provided $F \cdot E_1 \geq$

$F \cdot E_2 \geq \dots \geq F \cdot E_n$. We now define quantities λ and Λ for each curve C in Γ_X . For $C = E_i$ for any i , let $\lambda_C = \Lambda_C = 0$. Otherwise, let m_C be the maximum of $C \cdot E_1, \dots, C \cdot E_n$, define Λ_C to be the maximum of m_C and of $(C \cdot L) - m_C$, define λ'_C to be the minimum of m_C and of $(C \cdot L) - m_C$, and, if C is a smooth rational curve, define λ_C to be λ'_C , and otherwise define λ_C to be the maximum of λ'_C and 2. For example, if C is anticanonical, then $\lambda'_C = 1$ and $\lambda_C = \Lambda_C = 2$.

Here now is our main result, Theorem 1. For general n , it determines the rank of μ_F for certain (sufficiently nice) divisors F , which is sometimes enough to determine resolutions, as shown in Section 4. For $n \leq 8$, Theorem 1 gives a reduction for arbitrary F to the nice case, thereby in principle providing an algorithm for computing the rank of μ_F for any divisor F . Briefly, the algorithm works as follows. By applying Lemma 4 with Lemma 3(a), we can compute $h^0(X, F)$ for any divisor F (see section 4 for examples, or [H1], [H4] and [H6] for more comprehensive background). By Lemma 3, Γ_X is finite for $n = 8$, so cases (a) and (b) of Theorem 1 are easy to implement. If cases (a) or (c) occur, we are done, while case (b) need be applied only if $h^0(X, F) > 0$ since μ_F is clearly injective if $h^0(X, F) = 0$. In case (b) with $h^0(X, F) > 0$, repeated applications of (b) gives a divisor F for which either $h^0(X, F) = 0$ or which satisfies the conditions either of (a) or (c). (It is not hard to implement the algorithm explicitly; see [H9], which includes an explicit Macaulay script implementing this algorithm.)

Theorem 1: *Let X be a good surface with exceptional configuration $L, E_1, \dots, E_n$. Let F be a monotone divisor on X .*

- (a) *If $F \cdot C \geq \Lambda_C$ for all $C \in \Gamma_X$, then μ_F has maximal rank (i.e., is either injective or surjective), hence $\dim(\text{cok}(\mu_F)) = \max(0, h^0(X, F + L) - 3h^0(X, F))$.*
- (b) *If $F \cdot C < \lambda_C$ for some $C \in \Gamma_X$, then $\dim(\ker(\mu_F)) = \dim(\ker(\mu_{F-C}))$.*
- (c) *If $n = 8$ but neither case (a) nor case (b) holds, then either*
 - (i) *$F \cdot (L - E_1 - E_2) = 0$, in which case the dimension of $\text{cok}(\mu_F)$ is $h^1(X, F - (L - E_1)) + h^1(X, F - (L - E_2))$, or*
 - (ii) *$[F]$ is $[3L - E_1 - \dots - E_7] + r[8L - 3E_1 - \dots - 3E_7 - E_8]$ for some $r \geq 1$, in which case $\dim(\text{cok}(\mu_F)) = r$ and $\dim(\ker(\mu_F)) = r + 1$, or*
 - (iii) *μ_F has maximal rank, hence $\dim(\text{cok}(\mu_F)) = \max(0, F \cdot L + F \cdot K_X - F^2)$.*

Proof: Part (a) is Corollary 5. (The proof here depends on a criterion for μ_F to be surjective which involves vanishing of certain h^1 's, in the form of $q^*(F)$ and $l^*(F)$ defined below; see Lemma 2(b). Being defined in terms of h^1 's involving F , intuitively we can expect $q^*(F)$ and $l^*(F)$ to vanish if F is sufficiently nice, which is precisely what our hypotheses guarantee.) Part (b) is Proposition 6. (This part generalizes the fact that μ_F and μ_{F-C} have kernels of the same dimension if F is effective and C is an integral curve with $F \cdot C < 0$, which is obvious since in this situation C is a fixed component of $|F|$.) Most of the work here is in proving (c), which follows from Proposition 13 and Proposition 15. (The proof of this part amounts to an analysis of all possibilities not dealt with by (a) or (b).) $\diamond$

2. The Nice Case and the Reduction

We regard case (a) of Theorem 1 as the nice case because there we have the rank of μ_F directly. Case (b) of Theorem 1 is the reduction case. In this section we prove cases (a) and (b) of Theorem 1. Given a basic surface X with exceptional configuration $L, E_1, \dots, E_n$ and a monotone divisor F on X , we will denote $h^0(X, F - E_1)$ by $q(F)$, $h^1(X, F - E_1)$ by $q^*(F)$, $h^0(X, F - (L - E_1))$ by $l(F)$ and $h^1(X, F - (L - E_1))$ by $l^*(F)$.

Lemma 2: *Let X be obtained by blowing up distinct points of $\mathbf{P}^2$, with exceptional configuration $L, E_1, \dots, E_n$. Let F be a monotone divisor on X .*

- (a) *Then $\dim(\ker(\mu_F)) \leq q(F) + l(F)$; in particular, μ_F is injective if $q(F) = 0 = l(F)$.*
- (b) *If F is effective and $h^1(X, F) = 0$, then $\dim(\text{cok}(\mu_F)) \leq q^*(F) + l^*(F)$; in particular, μ_F is surjective, if, in addition, $l^*(F) = 0 = q^*(F)$.*

Proof: (a) See Lemma 4.1 of [H8] (which assumes $F \cdot E_i > 0$ for all i , but it is easy to check that the result holds even if $F \cdot E_i \leq 0$ for some i).

(b) Clearly, L is numerically effective. Thus, $F \cdot L \geq 0$, since F is effective. Now, from $h^1(X, F) = 0$ and $0 \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_X(F + L) \rightarrow \mathcal{O}_L(F + L) \rightarrow 0$, we see $h^1(X, F + L)$ vanishes also and we compute $h^0(X, F + L) - 3h^0(X, F) = 2 + F \cdot L - 2h^0(X, F)$. Similarly, $l^*(F) - l(F) = F \cdot (L - E_1) + 1 - h^0(X, F)$ and $q^*(F) - q(F) = F \cdot E_1 + 1 - h^0(X, F)$, so $(l^*(F) - l(F)) + (q^*(F) - q(F)) = h^0(X, F + L) - 3h^0(X, F)$. Therefore,

$\dim(\text{cok}(\mu_F)) = \dim(\ker(\mu_F)) + h^0(X, F + L) - 3h^0(X, F) \leq l(F) + q(F) + h^0(X, F + L) - 3h^0(X, F) = l^*(F) + q^*(F)$, as required. $\diamond$

We will need to refer to the following result.

Lemma 3: *Let C be a curve on the blow-up X of $\mathbf{P}^2$ at 8 general points, with exceptional configuration $L, E_1, \dots, E_8$.*

- (a) *Then, up to permutation of the indices, C is an exceptional curve if and only if $[C]$ is one of the following: $[E_8]$, $[L - E_1 - E_2]$, $[2L - E_1 - \dots - E_5]$, $[3L - 2E_1 - E_2 - \dots - E_7]$, $[4L - 2E_1 - 2E_2 - 2E_3 - E_4 - \dots - E_8]$, $[5L - 2E_1 - \dots - 2E_6 - E_7 - E_8]$, or $[6L - 3E_1 - 2E_2 - \dots - 2E_8]$.*
- (b) *And, up to permutation of the indices, C is a smooth rational curve with $C^2 = 0$ if and only if $[C]$ is one of the following: $[L - E_1]$, $[2L - E_1 - \dots - E_4]$, $[3L - 2E_1 - E_2 - \dots - E_6]$, $[4L - 2E_1 - 2E_2 - 2E_3 - E_4 - \dots - E_7]$, $[4L - 3E_1 - E_2 - \dots - E_8]$, $[5L - 3E_1 - 2E_2 - 2E_3 - 2E_4 - E_5 - \dots - E_8]$, $[5L - 2E_1 - \dots - 2E_6 - E_7]$, $[6L - 3E_1 - 3E_2 - 2E_3 - \dots - 2E_6 - E_7 - E_8]$, $[7L - 3E_1 - \dots - 3E_4 - 2E_5 - 2E_6 - 2E_7 - E_8]$, $[7L - 4E_1 - 3E_2 - 2E_3 - \dots - 2E_8]$, $[8L - 3E_1 - \dots - 3E_7 - E_8]$, $[8L - 4E_1 - 3E_2 - \dots - 3E_5 - 2E_6 - 2E_7 - 2E_8]$, $[9L - 4E_1 - 4E_2 - 3E_3 - \dots - 3E_7 - 2E_8]$, $[10L - 4E_1 - \dots - 4E_4 - 3E_5 - \dots - 3E_8]$, and $[11L - 4E_1 - \dots - 4E_7 - 3E_8]$.*

Proof: Under the action on $\text{Cl}(X)$ by the Weyl group (which is generated by permutation of the indices and by the action of quadratic Cremona transformations centered at any three of the 8 points; see [H1] or [H2]), any class of an effective divisor F is in the orbit of a class F' of the form $[dL + a_1E_1 + \dots + a_8E_8]$, where $d \geq 0$, $3d + a_1 + \dots + a_8 \geq 0$ (since, as mentioned in the introduction, $-K_X$ is ample), $d + a_1 + a_2 + a_3 \geq 0$ and $a_1 \leq a_2 \leq \dots \leq a_8$. If F is reduced and irreducible, then F' is either $[E_8]$ or $a_8 \leq 0$. In the latter case, F' is by Lemma 1.4 of [H1] a nonnegative sum of the classes of L , $L - E_1$, $2L - E_1 - E_2$, $3L - E_1 - E_2 - E_3$, $\dots$, $3L - E_1 - \dots - E_8$. But any such sum has nonnegative self-intersection. Thus classes of exceptional curves are in the orbit of $[E_8]$; these are the classes listed in (a). And the only such sums with self-intersection 0 are the positive multiples of $[L - E_1]$, hence only $[L - E_1]$ itself represents the class of a reduced and irreducible divisor with self-intersection 0. Thus the classes to be listed in (b) comprise the orbit of $[L - E_1]$ under the action of the Weyl group; it is easy to check that the list in (b) is (up to permutations) the complete orbit. $\diamond$

Given a surface X , denote by EFF_X (or just EFF) the subsemigroup of the divisor class group $\text{Cl}(X)$ of X of classes of effective divisors, and let NEFF denote the cone of classes of numerically effective divisors.

Lemma 4: *Let X be a good surface obtained by blowing up n points of $\mathbf{P}^2$.*

- (a) *If $2 \leq n < 8$, then EFF is generated by the classes of exceptional curves, while for $n \geq 8$, EFF is generated by the classes of exceptional curves and by $-K_X$.*
- (b) *Let $F \in \text{Cl}(X)$. If $2 \leq n \leq 8$, then F is in NEFF if and only if $F \cdot C \geq 0$ for every exceptional curve C , while if $9 \leq n$, then F is in NEFF if and only if both $-F \cdot K_X \geq 0$ and $F \cdot C \geq 0$ for every exceptional curve C .*
- (c) *EFF contains NEFF , and $h^1(X, F) = h^2(X, F) = 0$ and $h^0(X, F) = (F^2 - F \cdot K_X)/2 + 1$ for any $F \in \text{NEFF}$.*

Proof: (a) By (A1), any effective divisor is (up to linear equivalence) a nonnegative sum of D_X , exceptional curves and an element of NEFF . But by Corollary 3.2 and Lemma 1.4, both of [H1], any element of NEFF is (with respect to some exceptional configuration $L, E_1, \dots, E_n$) a sum of $[L]$, $[L - E_1]$, $[2L - E_1 - E_2]$, $-K_X$ and classes of exceptional curves. But $n \geq 2$, so $[L] = [L - E_1 - E_2] + [E_1] + [E_2]$, $[L - E_1] = [L - E_1 - E_2] + [E_2]$ and $[2L - E_1 - E_2] = 2[L - E_1 - E_2] + [E_1] + [E_2]$ are sums of classes of exceptional curves, so NEFF is generated by the classes of exceptional curves and by the class $-K_X$ of D_X . Moreover, for $n = 2, 3, 4, 5, 6$ or 7 , $-K_X$ is, respectively, the following sums of classes of exceptional curves: $3[L - E_1 - E_2] + 2[E_1] + 2[E_2]$, $[L - E_1 - E_2] + [L - E_1 - E_3] + [L - E_2 - E_3] + [E_1] + [E_2] + [E_3]$, $[L - E_1 - E_2] + [L - E_3 - E_4] + [L - E_1 - E_2] + [E_1] + [E_2]$, $[L - E_1 - E_2] + [L - E_3 - E_4] + [L - E_1 - E_5] + [E_1]$, $[L - E_1 - E_2] + [L - E_3 - E_4] + [L - E_5 - E_6]$, or $[2L - E_1 - E_2 - \dots - E_5] + [L - E_6 - E_7]$.

(b) This follows immediately from (a), except in the case that $n = 8$. For $n = 8$, we have $-2K_X = [3L - 2E_1 - E_2 - \dots - E_7] + [3L - E_2 - \dots - E_7 - 2E_8]$, hence if $F \cdot E \geq 0$ for all exceptional curves E , then also $-F \cdot K_X \geq 0$, and our result follows here too.

(c) By Proposition 4 of [H3], $F^2 \geq 0$ and $h^2(X, F) = 0$ for any $F \in \text{NEFF}$. Since we assume $-K_X$ is effective, we also have $-F \cdot K_X \geq 0$. But by Riemann-Roch, $h^0(X, F) \geq (F^2 - F \cdot K_X)/2 + 1$, so $h^0(X, F) \geq 1$,

hence $F \in \text{EFF}$. Now $h^1(X, F) = 0$ for any $F \in \text{NEFF}$ by assumption (A2) and the rest is immediate from Riemann-Roch. $\diamond$

Here is the proof of Theorem 1(a).

Corollary 5: *Let X be a good surface with exceptional configuration $L, E_1, \dots, E_n$, and let F be a monotone divisor on X . If $F \cdot C \geq \Lambda_C$ for all $C \in \Gamma_X$, then μ_F has maximal rank.*

Proof: If $h^0(X, F) = 0$, then clearly μ_F is injective and so has maximal rank. So assume $h^0(X, F) > 0$. Since both $\Lambda_C \geq 0$ and $F \cdot C \geq \Lambda_C$ for all $C \in \Gamma_X$ and hence for all curves C of negative self-intersection, we see F is numerically effective.

If $n \leq 5$, then μ_F is surjective simply because F is numerically effective (Theorem 3.1.2, [H7]). So we may assume that $n \geq 6$.

Since F is numerically effective, then $h^1(X, F) = 0$ by (A2) and $F \cdot E_i \geq 0$ for all i , and so $(F - E_1) \cdot E_i \geq 0$ for all i . If C is an exceptional curve but not E_i for any i , then $\Lambda_C \geq C \cdot E_1$ and hence $F \cdot C \geq \Lambda_C$ implies $(F - E_1) \cdot C \geq 0$. If $6 \leq n \leq 8$, then $F - E_1$ is numerically effective by Lemma 4, hence $q^*(F) = h^1(X, F - E_1) = 0$ by (A2). If $n \geq 9$, then $D_X \in \Gamma_X$, so in addition we have $(F - E_1) \cdot D_X \geq \Lambda_{D_X} - 1 \geq 1$, and again we see $F - E_1$ is numerically effective, and $q^*(F) = h^1(X, F - E_1) = 0$.

Now consider $l^*(F)$. If $(F - (L - E_1)) \cdot E_i < 0$ for some i , then clearly $i = 1$ and $F \cdot E_1 = 0$. Since F is monotone we see $F \cdot E_i = 0$ for all i , so F is (up to linear equivalence) a nonnegative multiple of L , for which it is easy to see that μ_F has maximal rank. Thus we may assume $(F - (L - E_1)) \cdot E_i \geq 0$ for all i . For any other exceptional curve C we have $F \cdot C \geq \Lambda_C$, so $(F - (L - E_1)) \cdot C \geq \Lambda_C - (L - E_1) \cdot C \geq 0$ too. Thus, as above, $F - (L - E_1)$ is numerically effective if $n \leq 8$, so $l^*(F) = h^1(X, F - (L - E_1)) = 0$, while if $n \geq 9$, we have in addition that $(F - (L - E_1)) \cdot D_X \geq \Lambda_{D_X} - 2 \geq 0$ so again $l^*(F) = h^1(X, F - (L - E_1)) = 0$. The result now follows by Lemma 2(b). $\diamond$

We now prove part (b) of Theorem 1.

Proposition 6: *Let F be a divisor on a good surface X having exceptional configuration $L, E_1, \dots, E_n$. If $F \cdot C < \lambda_C$ for some $C \in \Gamma_X$, then $\dim(\ker(\mu_F)) = \dim(\ker(\mu_{F-C}))$.*

Proof: If $h^0(X, \mathcal{O}_X(F)) = 0$, then, of course, neither $\mathcal{O}_X(F)$ nor $\mathcal{O}_X(F - C)$ have any global sections, so both $\ker(\mu_F)$ and $\ker(\mu_{F-C})$ vanish. Therefore, we may assume F is effective.

If C is a fixed component of $|F|$ for some $C \in \Gamma_X$, then clearly the canonical injection $\mathcal{O}_X(F - C) \rightarrow \mathcal{O}_X(F)$ induces an isomorphism both of global sections and of kernels of μ . So now we may assume that $|F|$ is fixed component free, hence $F \cdot C \geq 0$ for all $C \in \Gamma_X$.

Since $\lambda_{D_X} = 2$, if $F \cdot D_X < \lambda_{D_X}$, then it must be that $F \cdot D_X$ is 0 or 1. If 0, then $\mathcal{O}_X(F)$ is in the kernel of $\text{Pic}(X) \rightarrow \text{Pic}(D_X)$, and since this is injective, we see $F = 0$, in which case both $\ker(\mu_F)$ and $\ker(\mu_{F-D_X})$ again vanish. If 1, then $h^0(D_X, \mathcal{O}_{D_X}(F)) = 1$ so $\mu_{\mathcal{O}_{D_X}(F)} : H^0(D_X, \mathcal{O}_{D_X}(F)) \otimes H^0(X, L) \rightarrow H^0(D_X, \mathcal{O}_{D_X}(F + L))$ has the same kernel as the map $H^0(X, L) \rightarrow H^0(D_X, \mathcal{O}_{D_X}(L))$ given by restriction, which is easily seen to be injective. From the exact sequence $0 \rightarrow \ker(\mu_{F-D_X}) \rightarrow \ker(\mu_F) \rightarrow \ker(\mu_{\mathcal{O}_{D_X}(F)})$ induced by $0 \rightarrow \mathcal{O}_X(F - D_X) \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_{D_X}(F) \rightarrow 0$ we see that $\dim(\ker(\mu_F)) = \dim(\ker(\mu_{F-D_X}))$.

Finally, if $F \cdot C < \lambda_C$ for some $C \in \Gamma_X$ and $C \neq D_X$, then C is an exceptional curve. As above, the exact sequence $0 \rightarrow \mathcal{O}_X(F - C) \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_C(F) \rightarrow 0$ induces an exact sequence $0 \rightarrow \ker(\mu_{F-C}) \rightarrow \ker(\mu_F) \rightarrow \ker(\mu_{F \cdot C, C})$, where $\mu_{F \cdot C, C}$ denotes the map

$$H^0(C, \mathcal{O}_X(F) \otimes \mathcal{O}_C) \otimes H^0(X, L) \rightarrow H^0(C, \mathcal{O}_X(F + L) \otimes \mathcal{O}_C).$$

By Theorem 3.1 of [F1], $\ker \mu_{F \cdot C, C}$ is injective if $F \cdot C < \lambda_C$, so $\dim(\ker(\mu_F))$ equals $\dim(\ker(\mu_{F-C}))$. $\diamond$

3. The Main Theorem, Part (c)

To prove Part (c) of Theorem 1, we need some additional background, which we now develop. For the purpose of stating the next result, given sheaves $\mathcal{F}$ and $\mathcal{L}$ on a scheme Y , we denote the kernel of

$$H^0(Y, \mathcal{F}) \otimes H^0(Y, \mathcal{L}) \rightarrow H^0(Y, \mathcal{F} \otimes \mathcal{L})$$

by $R(\mathcal{F}, \mathcal{L})$ and the cokernel by $S(\mathcal{F}, \mathcal{L})$ (taking Y to be understood).

Proposition 7: *Let Y be a closed subscheme of projective space, let $\mathcal{F}$ and $\mathcal{L}$ be coherent sheaves on Y and let $\mathcal{C}$ be the sheaf associated to an effective Cartier divisor C on Y . If the restriction homomorphisms $H^0(Y, \mathcal{F}) \rightarrow H^0(C, \mathcal{F} \otimes \mathcal{O}_C)$ and $H^0(Y, \mathcal{F} \otimes \mathcal{L}) \rightarrow H^0(C, \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{O}_C)$ are surjective, then we have an exact sequence*

$$\begin{aligned} 0 \rightarrow R(\mathcal{F} \otimes \mathcal{C}^{-1}, \mathcal{L}) &\rightarrow R(\mathcal{F}, \mathcal{L}) \rightarrow R(\mathcal{F} \otimes \mathcal{O}_C, \mathcal{L}) \rightarrow \\ S(\mathcal{F} \otimes \mathcal{C}^{-1}, \mathcal{L}) &\rightarrow S(\mathcal{F}, \mathcal{L}) \rightarrow S(\mathcal{F} \otimes \mathcal{O}_C, \mathcal{L}) \rightarrow 0. \end{aligned}$$

Proof: This is a snake lemma argument; see [M]. $\diamond$

Example 8: The preceding result will often be applied in situations in which we have an exact sequence $0 \rightarrow \mathcal{O}_X(F - C) \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_C(F) \rightarrow 0$, where C is a curve on a surface X with an exceptional configuration $L, E_1, \dots, E_n$, and F is a divisor on X with $h^1(X, F - C) = 0$ and $h^1(L, F - C + L) = 0$. In such a situation, we can apply Proposition 7 with $Y = X$, $\mathcal{F} = \mathcal{O}_X(F)$, $\mathcal{C} = \mathcal{O}_X(C)$ and $\mathcal{L} = \mathcal{O}_X(L)$. For example, let X be a blow up of $\mathbf{P}^2$ at 8 general points, with exceptional configuration $L, E_1, \dots, E_8$. Recall D_X is a smooth, irreducible anticanonical divisor on X . For future reference we would like to show that μ_{D_X+tL} and μ_{2D_X+tL} have maximal rank for all integers t . First say that $t = 0$. Consider the exact sequence $0 \rightarrow \mathcal{O}_X(2D_X - E) \rightarrow \mathcal{O}_X(2D_X) \rightarrow \mathcal{O}_E(2) \rightarrow 0$, where E is the exceptional curve whose class is $[6L - 3E_1 - 2E_2 - \dots - 2E_8]$. Since $2D_X - E = E_1$, clearly μ_{2D_X-E} is injective, and by Lemma 12 below, $\mu_{2,E}$ is also injective, hence applying Proposition 7 it follows $\dim(\ker(\mu_{2D_X})) = 0$; i.e., μ_{2D_X} has maximal rank, and an easy calculation now shows that $\dim(\text{cok}(\mu_{2D_X})) = 0$ too. From Proposition 7 and the exact sequence $0 \rightarrow \mathcal{O}_X(D_X) \rightarrow \mathcal{O}_X(2D_X) \rightarrow \mathcal{O}_{D_X}(2D_X) \rightarrow 0$ it also follows that $\dim(\ker(\mu_{D_X})) = 0$. It is easier to see that μ_{D_X+tL} and μ_{2D_X+tL} have maximal rank for all $t \neq 0$, since $h^0(X, D_X + tL) = 0$ and $h^0(X, 2D_X + tL) = 0$ for $t < 0$ (in which case μ_{D_X+tL} and μ_{2D_X+tL} are clearly injective), while for $t \geq 0$ we have $h^1(X, D_X + tL) = 0$ and $h^1(X, 2D_X + tL) = 0$, in which case we apply the general fact that μ_{H+L} is surjective if $h^1(X, H + tL) = 0$ for $t \geq 0$. (Indeed, $h^1(X, H) = 0$ ensures $H^0(X, H + L) \rightarrow H^0(L, H + L)$ is surjective, and hence $S(\mathcal{O}_L(L), \mathcal{O}_X(H + L)) = S(\mathcal{O}_L(L), \mathcal{O}_L(H + L))$). Then applying Proposition 7 with $Y = X$, $C = L$, $\mathcal{F} = \mathcal{O}_X(L)$ and $\mathcal{L} = \mathcal{O}_X(H + L)$ we see μ_{H+L} is onto—just note that $S(\mathcal{O}_X(L), \mathcal{O}_X(H + L)) = 0$ since in this situation it is easy to check that both $S(\mathcal{O}_X, \mathcal{O}_X(H + L)) = 0$ and $S(\mathcal{O}_L(L), \mathcal{O}_L(H + L)) = 0$.)

Proposition 9: *Let F be a monotone divisor on a good surface X with exceptional configuration $L, E_1, \dots, E_n$. If $F \cdot (L - E_1 - E_2) = 0$, then*

$$\dim(\ker(\mu_F)) = h^0(X, F - (L - E_1)) + h^0(X, F - (L - E_2)).$$

If in addition F is effective and $h^1(X, F) = 0$, then

$$\dim(\text{cok}(\mu_F)) = h^1(X, F - (L - E_1)) + h^1(X, F - (L - E_2)).$$

Proof: If $h^0(X, F) = 0$, then certainly $\dim(\ker(\mu_F)) = 0$ and both $h^0(X, F - (L - E_1))$ and $h^0(X, F - (L - E_2))$ are also 0. Otherwise, this is Proposition II.2(e) and Remark II.3, both of [H6]. $\diamond$

We recall that a nonzero element of a free abelian group is *primitive* if it is not a multiple greater than 1 of another element of the group.

Lemma 10: *Let F be a divisor on the blow-up X of $\mathbf{P}^2$ at $n \leq 8$ general points, and suppose that $F \cdot C \geq -1$ for all exceptional curves C on X . Then either $h^1(X, F) = 0$, or $F = rH + K_X$ with $r \geq 2$, where $[H]$ is primitive and H is smooth, rational, and numerically effective with $H^2 = 0$ (in which case $h^1(X, F) = r - 1$).*

Proof: Since $F \cdot C \geq -1$ for all exceptional curves C , $C \cdot (F - K_X) \geq 0$ for all exceptional curves C , which implies $F - K_X = D$ is numerically effective by Lemma 4. Therefore $h^1(X, F) = h^1(X, D + K_X) = h^1(X, K_X - (D + K_X)) = h^1(X, -D)$, with the center equality due to Serre duality. By Ramanujam vanishing [R] (see [T] for a characteristic p version, or see Theorem 2.8 of [H5]), $h^1(X, -D) = 0$ if $D^2 > 0$. If $D^2 = 0$, we still have $D \cdot (-K_X) > 0$ since, for $n \leq 8$, $-K_X$ is ample. Then by an easy calculation applying Lemma 1.4 of [H1] we see that the only possibility is for $[D]$ to be in the orbit of $r[L - E_1]$ under the action of the Weyl group (see the proof of Lemma 3) on $\text{Cl}(X)$. I.e., $[D] = r[H]$ for some r , where $[H]$ is primitive and H is a smooth rational curve with $H^2 = 0$. Now $h^1(X, -D) = r - 1$ follows by induction via $0 \rightarrow \mathcal{O}_X((-r - 1)H) \rightarrow \mathcal{O}_X(-rH) \rightarrow \mathcal{O}_H \rightarrow 0$. $\diamond$

We will need to refer to the main result of [H6]:

Theorem 11: Let F be a monotone, numerically effective divisor on the blow up X of $\mathbf{P}^2$ at 7 general points, $L, E_1, \dots, E_7$ being the corresponding exceptional configuration. Let t_F denote the number of indices i such that $F \cdot E_i = -F \cdot K_X$ and let $\gamma_F = \max(0, h^0(X, F + L) - 3h^0(X, F))$ (this is the “expected,” maximal rank dimension of the cokernel of μ_F). Then $\dim(\text{cok}(\mu_F)) = \max(t_F, \gamma_F)$ unless F is, up to linear equivalence, either $0, B, B - K_X - E_4, B - 2K_X - E_4 - E_5, B - 3K_X - E_4 - E_5 - E_6, B - 4K_X - E_4 - E_5 - E_6 - E_7, G$ or $G - K_X - E_7$, where $B = 4L - 2E_1 - 2E_2 - 2E_3 - E_4 - \dots - E_7$ and $G = 5L - 2E_1 - \dots - 2E_6 - E_7$, in which case μ_F is injective and $\dim(\text{cok}(\mu_F)) = \gamma_F$.

Proof: This is Theorem I.6.1 of [H6]. $\diamond$

Lemma 12: Let C be a smooth rational curve on the blow-up X of $\mathbf{P}^2$ at 8 general points with exceptional configuration $L, E_1, \dots, E_8$, and assume $C \cdot L = d$. Let $0 \leq t$, and consider the map

$$\mu_{t,C} : H^0(X, \mathcal{O}_C(t)) \otimes H^0(X, \mathcal{O}_X(1)) \rightarrow H^0(C, \mathcal{O}_C(t+d)),$$

given by restriction and multiplication on simple tensors.

- (a) If $C^2 = -1$ (i.e., C is an exceptional curve on X), then $\mu_{t,C}$ has maximal rank.
- (b) If $C^2 = 0$, then $\mu_{t,C}$ has maximal rank, except (up to permutation of the indices) in the following cases:
 - (i) $[C] = [4L - 3E_1 - E_2 - \dots - E_8]$ and $t = 1$, or
 - (ii) $[C] = [8L - 3E_1 - \dots - 3E_7 - E_8]$ and $t = 3$.

For both (i) and (ii), the rank of $\mu_{t,C}$ is one short of maximal rank.

Proof: It is easy to see that the kernel of the surjective sheaf map $\mathcal{O}_C \otimes H^0(X, L) \rightarrow \mathcal{O}_C(d)$ is $\mathcal{O}_C(-a) \oplus \mathcal{O}_C(-b)$, for some nonnegative a and b with $a + b = d$. If we assume $a \leq b$, then it turns out (see the proof of Theorem 3.1 of [F1]) that $a \geq \min\{d - m_C, m_C\}$, and thus when $d - m_C$ and m_C differ by at most one, a is the smaller of $d - m_C$ and m_C and b is the larger. Moreover, when a and b differ by at most one, then for each t either $h^0(C, \mathcal{O}_C(t-a) \oplus \mathcal{O}_C(t-b)) = 0$ or $h^1(C, \mathcal{O}_C(t-a) \oplus \mathcal{O}_C(t-b)) = 0$, hence $\mu_{t,C}$ has maximal rank for every t .

Part (a) now follows by checking Lemma 3(a) to see that $d - m_C$ and m_C always differ by at most 1 if C is an exceptional curve.

The same proof via Lemma 3(b) also works for (b), except for the following four cases, for which $d - m_C$ and m_C differ by more than one: $4L - 3E_1 - E_2 - \dots - E_8$, $8L - 3E_1 - \dots - 3E_7 - E_8$, $10L - 4E_1 - \dots - 4E_4 - 3E_5 - \dots - 3E_8$, and $11L - 4E_1 - \dots - 4E_7 - 3E_8$.

To handle $C = 4L - 3E_1 - E_2 - \dots - E_8$, let $D = 4L - 3E_1 - E_2 - \dots - E_7$ (so $D \cdot C = 1$ and $D - C = E_8$), and consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(D - C) \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_C(1) \rightarrow 0.$$

We can compute $\dim(\text{cok}(\mu_{E_8}))$ trivially (μ_{E_8} is bijective) and μ_D has a one-dimensional kernel and cokernel by Theorem 11, so by Proposition 7, $\dim(\ker(\mu_{1,C})) = 1$. Since the kernel of $\mu_{t,C}$ is $\mathcal{O}_C(t-a) \oplus \mathcal{O}_C(t-b)$, we see

$$h^0(C, \mathcal{O}_C(1-a) \oplus \mathcal{O}_C(1-b)) = 1,$$

which with $a + b = 4$ gives $a = 1$ and $b = 3$. We see that $\mu_{t,C}$ has maximal rank unless $t = 1$, in which case the rank is one short of maximal.

To handle $C = 8L - 3E_1 - \dots - 3E_7 - E_8$, let $D = 8L - 3E_1 - \dots - 3E_7$ (so again $D \cdot C = 1$ and $D - C = E_8$). Consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(3D - C) \rightarrow \mathcal{O}_X(3D) \rightarrow \mathcal{O}_C(3) \rightarrow 0.$$

Note that $3D - C = 2D + E_8$. The inclusion $\ker(\mu_{2D}) \subset \ker(\mu_{2D+E_8})$ is clearly an isomorphism since it is induced by the canonical injection $\mathcal{O}_X(2D) \rightarrow \mathcal{O}_X(2D + E_8)$ which gives an isomorphism on global sections. But by Theorem 11, μ_{2D} and μ_{3D} have kernels of dimensions 0 and 1, respectively, so by Proposition 7, $\dim(\ker(\mu_{3,C}))$ is at least 1 dimensional. Thus $a \leq 3$, but we know $a + b = 8$ and a is at least $\min\{d - m, m\} = 3$, so in fact $a = 3$ and $b = 5$, which gives the desired result.

To handle $C = 10L - 4E_1 - \dots - 4E_4 - 3E_5 - \dots - 3E_8$, note that (a, b) must be either $(4, 6)$ or $(5, 5)$. We just need to compute the dimension of the kernel of $\mu_{4,C}$ to tell which. From the exact sequence

$$0 \rightarrow \mathcal{O}_X(-2K_X) \rightarrow \mathcal{O}_X(-2K_X + C) \rightarrow \mathcal{O}_C(4) \rightarrow 0,$$

and the fact that μ_{-2K_X} is bijective (see Example 8), it suffices to compute $\dim(\ker(\mu_{-2K_X+C}))$. Let $D = 6L - 3E_1 - 2E_2 - \cdots - 2E_8$ and look at:

$$0 \rightarrow \mathcal{O}_X(-2K_X + C - D) \rightarrow \mathcal{O}_X(-2K_X + C) \rightarrow \mathcal{O}_D(2) \rightarrow 0.$$

Using the injectivity of $\mu_{2,D}$ from part (a), we reduce to μ_{-2K_X+C-D} . Now let $E = 6L - 2E_1 - 3E_2 - 2E_3 - \cdots - 2E_8$, look at

$$0 \rightarrow \mathcal{O}_X(-2K_X + C - D - E) \rightarrow \mathcal{O}_X(-2K_X + C - D) \rightarrow \mathcal{O}_E(2) \rightarrow 0,$$

and note $-2K_X + C - D - E = 4L - E_1 - E_2 - 2E_3 - 2E_4 - E_5 - \cdots - E_8$. Permuting indices gives $F = 4L - 2E_1 - 2E_2 - E_3 - \cdots - E_8$, which has $l(F) = q(F) = 0$, hence $\mu_{-2K_X+C-D-E}$ and thus $\mu_{4,C}$ is injective by Lemma 2; i.e., a must be bigger than 4, so $a = 5$. Again, the desired result follows.

To handle $C = 11L - 4E_1 - \cdots - 4E_7 - 3E_8$, note that we know that (a, b) is either $(4, 7)$ or $(5, 6)$ and as before, the dimension of the kernel of $\mu_{4,C}$ tells which. Let $D = 6L - 3E_1 - 2E_2 - \cdots - 2E_8$ and $E = 6L - 2E_1 - 3E_2 - 2E_3 - \cdots - 2E_8$ as above, and let $F = 2L - E_3 - \cdots - E_7$. Following a process similar to above, we find that $\dim(\ker(\mu_{-2K_X+C-D-E-F})) = \dim(\ker(\mu_{4,C}))$, but $-2K_X + C - D - E - F = -K_X$ and we know $\dim(\ker(\mu_{-K_X})) = 0$ (see Example 8). Thus $a > 4$ and hence $a = 5$, as needed. $\diamond$

Proposition 13: *Let X be a good surface with exceptional configuration $L, E_1, \dots, E_8$. Let F be a monotone divisor on X such that $F \cdot E \geq \lambda_E$ for all exceptional curves E but $F \cdot C < \Lambda_C$ for some exceptional curve C . Then either*

- (i) $F \cdot (L - E_1 - E_2) = 0$, in which case

$$\dim(\text{cok}(\mu_F)) = h^1(X, F - (L - E_1)) + h^1(X, F - (L - E_2)), \text{ or}$$

- (ii) μ_F has maximal rank, or

- (iii) $F \cdot E_1 = \cdots = F \cdot E_7$ (i.e., F is nearly uniform), F is numerically effective and $F \cdot C = 2$, where $[C] = [5L - 2E_1 - \cdots - 2E_6 - E_7 - E_8]$.

Proof: By Lemma 4, we see that F is effective and numerically effective, and hence (by (A1)) $h^1(X, F)$ vanishes. From Lemma 3 and monotonicity of F we can assume that $[C]$ is one of $[L - E_1 - E_2]$, $[3L - 2E_1 - E_2 - \cdots - E_7]$, or $[5L - 2E_1 - \cdots - 2E_6 - E_7 - E_8]$, since for all other cases $\lambda_C = \Lambda_C$. We also may as well assume that μ_F fails to have maximal rank.

If $[C]$ is $[L - E_1 - E_2]$, then $F \cdot (L - E_1 - E_2) = 0$ and our result follows by Proposition 9, so now we may assume that $F \cdot (L - E_1 - E_2) > 0$.

If $[C]$ is $[3L - 2E_1 - E_2 - \cdots - E_7]$, then $F \cdot C = 1$ and, by Lemma 10, either $q^*(F) = h^1(X, F - E_1) = 0$, or $F = rH + K_X + E_1$ for some $r \geq 2$, where $[H]$ is primitive and H is smooth, rational, and numerically effective with $C \cdot H = 0$ and $H^2 = 0$.

In the latter case, since $F \cdot (L - E_1 - E_2) > 0$, F is monotone and $[H]$ is, up to permutation of the indices, one of the classes listed in Lemma 3(b), we see that $[H]$ can only be $[8L - 4E_1 - 3E_2 - \cdots - 3E_5 - 2E_6 - 2E_7 - 2E_8]$. Thus $[F] = [5L - 2E_1 - \cdots - 2E_5 - E_6 - E_7 - E_8] + (r - 1)[H]$, which for the purposes of induction we denote F_r . By a calculation, $q(F_1) = 0$ and $l(F_1) = 0$, so μ_{F_1} is injective by Lemma 2. Now consider the exact sequence $0 \rightarrow \mathcal{O}_X(F_{r-1}) \rightarrow \mathcal{O}_X(F_r) \rightarrow \mathcal{O}_H(2) \rightarrow 0$. By Lemma 12, $\mu_{2,H}$ has maximal rank (and hence here must be injective), so applying Proposition 7 and inducting, we see that μ_{F_r} is injective for all $r \geq 1$, contradicting our assumption that μ_F fails to have maximal rank.

On the other hand, suppose $q^*(F) = 0$. We still have $F \cdot C = 1$ and $F \cdot E \geq \lambda_E$ for all exceptional curves E . If $F \cdot E_1 > F \cdot E_2$, then $F - (L - E_1)$ is monotone, and $(F - (L - E_1)) \cdot E \geq 0$ for all exceptional curves E unless $E = 5L - 2(E_1 + \cdots + E_6) - E_7 - E_8$ (or one obtained from this class by permuting the indices) and $F \cdot E = 2$, in which case we at least have $(F - (L - E_1)) \cdot E \geq -1$. As before, by applying Lemma 10, either $l^*(F) = 0$ or $F = (L - E_1) + rH + K_X$ for some $r \geq 2$ with $[H]$ primitive and H smooth, rational, and numerically effective with $H^2 = 0$. If $C \cdot H = 0$, then $F \cdot C = 0$ (contradicting $F \cdot C = 1$), while if $C \cdot H \geq 1$, then $F \cdot C \geq r \geq 2$ (contradicting $F \cdot C = 1$). Thus we see $l^*(F) = 0$ in addition to $q^*(F) = 0$, so μ_F is surjective by Lemma 2, contradicting our assumption that μ_F fails to have maximal rank.

If, however, $F \cdot E_1 = F \cdot E_2$, then by checking each possible exceptional curve E we see that either:

- (1) $(F - (L - E_1)) \cdot E \geq -1$ for all exceptional curves E , or

- (2) $(F - (L - E_1)) \cdot E = -2$ for $[E] = [3L - 2E_2 - E_3 - \cdots - E_8]$, hence $F \cdot E = F \cdot C = 1$ and so $F \cdot E_1 = \cdots = F \cdot E_8$, or
- (3) $(F - (L - E_1)) \cdot E = -2$ for $[E] = [5L - E_1 - 2E_2 - \cdots - 2E_7 - E_8]$, hence $F \cdot E = 2$ and, since, for $[E'] = [5L - 2E_1 - \cdots - 2E_6 - E_7 - E_8]$, we have $2 = F \cdot E \geq F \cdot E' \geq \lambda_{E'} = 2$ by monotonicity, we see $F \cdot E_1 = \cdots = F \cdot E_7$.

In case 1, by applying Lemma 10, we see as before that $l^*(F) = 0$, since having $F = (L - E_1) + rH + K_X$ again contradicts $F \cdot C = 1$. Thus μ_F is surjective by Lemma 2, contrary to assumption. In case 2, $F = tL - m(E_1 + \cdots + E_8)$ for some t and m . Since F is numerically effective, we must have $m \geq 0$ and $F \cdot (6L - 3E_1 - 2(E_2 + \cdots + E_8)) \geq 0$, so $t \geq 17m/6$. Thus $m/2 = 17m/2 - 8m \leq F \cdot C = 1$, so $m \leq 2$, in which case by Example 8 we know that μ_F for the particular F we have here will have maximal rank, contrary to assumption. Case 3 is just case (iii) of Proposition 13.

Now we may assume that $F \cdot (3L - 2E_1 - E_2 - \cdots - E_7) > 1$, and we consider the case that $[C] = [5L - 2E_1 - \cdots - 2E_6 - E_7 - E_8]$ with $F \cdot C = 2$. Note now that $(F - E_1) \cdot E \geq 0$ for all exceptional curves E so $F - E_1$ is, by Lemma 4, numerically effective and effective, and hence $q^*(F) = 0$.

If $(F - (L - E_1)) \cdot E < -1$ for some exceptional curve E , then E can be taken to be $5L - E_1 - 2E_2 - \cdots - 2E_6 - 2E_7 - E_8$ and $F \cdot E_1 = \cdots = F \cdot E_7$. Otherwise, we may assume $F \cdot E \geq -1$ for all exceptional curves E , hence by Lemma 10 either $l^*(F) = 0$ (and μ_F is surjective, contrary to assumption), or $F = rH + K_X + L - E_1$ where $r \geq 2$, and $[H]$ is primitive and H is smooth, rational and numerically effective with $C \cdot H = 0$ and $H^2 = 0$.

In the latter case, keeping in mind that F is monotone (which means, since $r \geq 2$, that H must be monotone too), that $F \cdot C = 2$ and that $F \cdot (3L - 2E_1 - E_2 - \cdots - E_7) > 1$, from Lemma 3 we see that $[H]$ must be either $[10L - 4E_1 - \cdots - 4E_4 - 3E_5 - \cdots - 3E_8]$ or $[11L - 4E_1 - \cdots - 4E_7 - 3E_8]$. In each case μ_F ends up having maximal rank. The argument in each case is similar; here are the details for the latter case.

So let $[H]$ be $[11L - 4E_1 - \cdots - 4E_7 - 3E_8]$ and $[F] = [9L - 4E_1 - 3E_2 - \cdots - 3E_7 - 2E_8] + (r - 1)[H]$; we will denote $(9L - 4E_1 - 3E_2 - \cdots - 3E_7 - 2E_8) + tH$ by F_t , for $t \geq 0$. Consider the exact sequence $0 \rightarrow \mathcal{O}_X(F_{t-1}) \rightarrow \mathcal{O}_X(F_t) \rightarrow \mathcal{O}_H(5) \rightarrow 0$. By Lemma 2, μ_{F_0} is surjective, and by Lemma 12, $\mu_{5,H}$ is too. Applying Proposition 7 and inducting, we see that μ_{F_t} is surjective for all $t \geq 0$, contradicting our assumption that μ_F fails to have maximal rank. $\diamond$

To complete the proof of Theorem 1, it still remains to analyze numerically effective monotone divisors F which are nearly uniform, have $F \cdot E \geq \lambda_E$ for all exceptional curves E and have $F \cdot C = 2$, where $[C] = [5L - 2E_1 - \cdots - 2E_6 - E_7 - E_8]$. It will be useful to determine all nearly uniform monotone numerically effective classes; this is what we do now. To simplify notation, we will denote the class $[dL - aE_1 - \cdots - aE_7 - bE_8]$ by the triple (d, a, b) .

Proposition 14: *If X is the blow-up of $\mathbf{P}^2$ at eight general points with exceptional configuration $L, E_1, \dots, E_8$, the classes $(1, 0, 0)$, $(3, 1, 0)$, $(3, 1, 1)$, $(8, 3, 0)$, $(8, 3, 1)$, $(11, 4, 3)$, and $(17, 6, 6)$ generate the cone of monotone numerically effective nearly uniform classes.*

Proof: Since $(1, 0, 0)$, $(3, 1, 0)$, $(3, 1, 1)$, $(8, 3, 0)$, $(8, 3, 1)$, $(11, 4, 3)$, and $(17, 6, 6)$ are all numerically effective, monotone and nearly uniform, any nonnegative $\mathbf{Z}$ -linear combination is also numerically effective, monotone and nearly uniform.

Conversely, let $F = (d, a, b)$ be a nearly uniform class which is monotone and numerically effective. Since F is monotone, we have $a \geq b$, and since F is numerically effective, we have $F \cdot E_8 \geq 0$, $F \cdot (3L - 2E_1 - E_2 - \cdots - E_7) \geq 0$, $F \cdot (5L - 2E_1 - \cdots - 2E_6 - E_7 - E_8) \geq 0$, and $F \cdot (6L - 3E_1 - 2 - \cdots - 2E_8) \geq 0$; i.e., we have $a \geq b$, $b \geq 0$, $3d - 8a \geq 0$, $5d - 13a - b \geq 0$, and $6d - 15a - 2b \geq 0$.

It is not hard to check that the rational solution set to these inequalities is the cone $\Xi(\mathbf{Q})$ given by all nonnegative rational linear combinations of $(1, 0, 0)$, $(8, 3, 0)$, $(8, 3, 1)$, $(11, 4, 3)$, and $(17, 6, 6)$: each of these classes satisfies all of the inequalities, but $a = b$ for $(17, 6, 6)$ and $(1, 0, 0)$, $b = 0$ for $(1, 0, 0)$ and $(8, 3, 0)$, $3d - 8a = 0$ for $(8, 3, 0)$ and $(8, 3, 1)$, $5d - 13a - b = 0$ for $(8, 3, 1)$ and $(11, 4, 3)$, and $6d - 15a - 2b = 0$ for $(11, 4, 3)$ and $(17, 6, 6)$. Thus we see that the cone of monotone numerically effective nearly uniform classes is just the cone $\Xi = \Xi(\mathbf{Z})$ of integer lattice points in $\Xi(\mathbf{Q})$.

We now show that Ξ is in fact the set of nonnegative $\mathbf{Z}$ -linear combinations of $(1, 0, 0)$, $(8, 3, 0)$, $(8, 3, 1)$, $(11, 4, 3)$, $(17, 6, 6)$, $(3, 1, 0)$ and $(3, 1, 1)$. Let $\langle \dots \rangle$ denote the cone generated over $\mathbf{Z}$, and let $\langle \dots \rangle_{\mathbf{Q}}$ denote the cone generated over $\mathbf{Q}$. It is easy to see that $\Xi(\mathbf{Q})$ is the union of the rational cones

$$\Xi_1 = \langle (11, 4, 3), (8, 3, 1), (8, 3, 0) \rangle_{\mathbf{Q}},$$

$$\Xi_2 = \langle (11, 4, 3), (17, 6, 6), (8, 3, 0) \rangle_{\mathbf{Q}},$$

and

$$\Xi_3 = \langle (1, 0, 0), (17, 6, 6), (8, 3, 0) \rangle_{\mathbf{Q}}.$$

First, consider Ξ_1 . Since

$$\begin{vmatrix} 11 & 8 & 8 \\ 4 & 3 & 3 \\ 3 & 1 & 0 \end{vmatrix} = 1,$$

every integer lattice point which is a rational linear combination of $(11, 4, 3)$, $(8, 3, 1)$, and $(8, 3, 0)$ is in fact a $\mathbf{Z}$ -linear combination; i.e., $\Xi \cap \Xi_1 = \langle (11, 4, 3), (8, 3, 1), (8, 3, 0) \rangle$.

Suppose $(d, a, b) \in \Xi \cap \Xi_2$. Then the integer triple (d, a, b) is $\alpha(17, 6, 6) + \beta(8, 3, 0) + \gamma(11, 4, 3)$ for some nonnegative rational numbers α, β and γ , and we have

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 17 & 8 & 11 \\ 6 & 3 & 4 \\ 6 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} d \\ a \\ b \end{bmatrix} = \begin{bmatrix} 3 & -8 & -1/3 \\ 2 & -5 & -2/3 \\ -6 & 16 & 1 \end{bmatrix} \begin{bmatrix} d \\ a \\ b \end{bmatrix}.$$

Thus $3d - 8a - b/3 = \alpha$, $2d - 5a - 2b/3 = \beta$, and $-6d + 16a + b = \gamma$, so γ is an integer and α and β (in lowest terms, of course) have denominators of 1 or 3. Thus any element of $\Xi \cap \Xi_2$ is of the form $A + B$, where $A \in \langle (11, 4, 3), (17, 6, 6), (8, 3, 0) \rangle$ and $B = \alpha'(17, 6, 6) + \beta'(8, 3, 0)$ with $\alpha', \beta' \in \{0, 1/3, 2/3\}$. But the only such B which are integer triples are $(1/3)(17, 6, 6) + (2/3)(8, 3, 0) = (11, 4, 2) = (8, 3, 1) + (3, 1, 1)$ and $(2/3)(17, 6, 6) + (1/3)(8, 3, 0) = (14, 5, 4) = (11, 4, 3) + (3, 1, 1)$. This argument shows that all elements of $\Xi \cap \Xi_2$ are contained in the rational cone $\langle (11, 4, 3), (17, 6, 6), (8, 3, 0), (8, 3, 1), (3, 1, 1) \rangle$.

Finally, suppose $(d, a, b) = \alpha(1, 0, 0) + \beta(8, 3, 0) + \gamma(17, 6, 6) \in \Xi \cap \Xi_3$; then $6\gamma \in \mathbf{Z}$, $3\beta + 6\gamma \in \mathbf{Z}$, and $\alpha + 8\beta + 17\gamma \in \mathbf{Z}$. Thus we may assume α and γ are multiples of $1/6$ and β is a multiple of $1/3$. Thus any element of $\Xi \cap \Xi_3$ is of the form $A + B$, where $A \in \langle (1, 0, 0), (17, 6, 6), (8, 3, 0) \rangle$ and $B = \alpha'(1, 0, 0) + \beta'(8, 3, 0) + \gamma'(17, 6, 6)$ with $\alpha', \gamma' \in \{0, 1/6, \dots, 5/6\}$ and $\beta' \in \{0, 1/3, 2/3\}$. By direct check, for every integer triple B we have $B \in \langle (1, 0, 0), (17, 6, 6), (8, 3, 0), (3, 1, 1), (3, 1, 0) \rangle$.

Thus $(1, 0, 0)$, $(3, 1, 0)$, $(3, 1, 1)$, $(8, 3, 0)$, $(8, 3, 1)$, $(11, 4, 3)$, and $(17, 6, 6)$ generate the cone of monotone, numerically effective nearly uniform classes on a blow-up of $\mathbf{P}^2$ at eight general points. $\diamond$

We now analyze those classes falling into case (iii) of Proposition 13.

Proposition 15: *Let F be a monotone, numerically effective, nearly uniform divisor class such that $F \cdot E \geq \lambda_E$ for all exceptional curves E and $F \cdot (5L - 2E_1 - \dots - 2E_6 - E_7 - E_8) = 2$. Then either*

- (a) F is $(3, 1, 0) + r(8, 3, 1)$ for some $r \geq 0$, in which case $\dim(\text{cok}(\mu_F)) = r$ and $\dim(\ker(\mu_F)) = r + 1$,
or
- (b) μ_F has maximal rank.

Proof: By Proposition 14, we know

$$F \in \langle (1, 0, 0), (3, 1, 0), (3, 1, 1), (8, 3, 0), (8, 3, 1), (11, 4, 3), (17, 6, 6) \rangle.$$

Since $F \cdot (5L - 2E_1 - \dots - 2E_6 - E_7 - E_8) = 2$, one deduces that F must be of the form $H + r(8, 3, 1) + s(11, 4, 3)$ for some nonnegative integers r and s , where H is one of $(3, 1, 0)$, $(6, 2, 2)$, $(16, 6, 0)$, $(20, 7, 7)$, $(25, 9, 6)$, or $(34, 12, 12)$. (Note that we need not consider the possibility $H = (11, 4, 1)$ since $(11, 4, 1)$ has been accounted for by taking $H = (3, 1, 0)$ with $r = 1$ and $s = 0$.) For $F = H + r(8, 3, 1) + s(11, 4, 3)$, in order for $F \cdot E \geq \lambda_E$ for all exceptional curves E (see Lemma 3), it is easy to check that the following additional restrictions are necessary:

- If $H = (6, 2, 2)$, then $r > 0$, hence we can replace $H = (6, 2, 2)$ by $H = (6, 2, 2) + (8, 3, 1) = (14, 5, 3)$ and remove the requirement that $r > 0$.
- If $H = (16, 6, 0)$, then $s > 0$, so we replace $H = (16, 6, 0)$ by $H = (27, 10, 3)$.
- If $H = (20, 7, 7)$, then $r > 1$, so we replace $H = (20, 7, 7)$ by $H = (36, 13, 9)$.
- If $H = (34, 12, 12)$, then $r > 2$, so we replace $H = (34, 12, 12)$ by $H = (58, 21, 15)$.

Thus F must be of the form $H + r(8, 3, 1) + s(11, 4, 3)$ for some nonnegative integers r and s , where H is one of $(3, 1, 0)$, $(14, 5, 3)$, $(27, 10, 3)$, $(36, 13, 9)$, $(25, 9, 6)$, or $(58, 21, 15)$. We consider each possibility for H in turn, beginning with $H = (3, 1, 0)$.

So $F = H + r(8, 3, 1) + s(11, 4, 3)$, with $H = (3, 1, 0)$. We first consider the case that $s = 0$. Note that $\text{cok}(\mu_H) = 0$ by Theorem 11. Now, $F \cdot (8, 3, 1) = 3$ so for $r \geq 1$ we have the exact sequence $0 \rightarrow \mathcal{O}_X(F - D) \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_D(3) \rightarrow 0$ with $D = (8, 3, 1)$. By Lemma 12 and an easy calculation, $\mu_{3,D}$ has 1-dimensional kernel and cokernel. Applying Proposition 7 to the foregoing exact sequence with $r = 1$ (and $s = 0$) we see that the induced map $\ker(\mu_F) \rightarrow \ker(\mu_{3,D})$ is surjective. Since the restriction of $\mathcal{O}_X(D)$ to D is trivial, the image of the map $H^0(X, H + rD) \rightarrow H^0(D, \mathcal{O}_D(3))$ and hence of $H^0(X, H + rD) \otimes H^0(X, L) \rightarrow H^0(D, \mathcal{O}_D(3)) \otimes H^0(X, L)$ is the same for all $r \geq 1$. Thus $\ker(\mu_F) \rightarrow \ker(\mu_{3,D})$ is surjective for all $r \geq 1$. We therefore see that the exact sequence

$$\begin{aligned} 0 \rightarrow \ker \mu_{F-D} &\rightarrow \ker \mu_F \rightarrow \ker \mu_{3,D} \\ &\rightarrow \text{cok} \mu_{F-D} \rightarrow \text{cok} \mu_F \rightarrow \text{cok} \mu_{3,D} \rightarrow 0, \end{aligned}$$

coming from Proposition 7 is exact separately on kernels and cokernels. It now follows for $s = 0$ and all $r \geq 0$ that $\dim(\ker(\mu_F)) = r + 1$ and $\dim(\text{cok}(\mu_F)) = r$, as claimed in part (a).

Now assume $F = H + s(11, 4, 3)$; we find $F \cdot (11, 4, 3) = 5$ so we have the exact sequence $0 \rightarrow \mathcal{O}_X(F - D) \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_D(5) \rightarrow 0$ where this time we take $D = (11, 4, 3)$. By Lemma 12, $\mu_{5,D}$ has maximal rank and one easily checks that $\mu_{5,D}$ therefore is surjective. Applying Proposition 7 and inducting on s we see that $\text{cok}(\mu_F) = 0$.

Finally, consider $F = H + s(11, 4, 3) + r(8, 3, 1)$ with $s > 0$; we find $F \cdot (8, 3, 1) = 3 + s$ so we have the exact sequence $0 \rightarrow \mathcal{O}_X(F - D) \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_D(3 + s) \rightarrow 0$ where this time we take $D = (8, 3, 1)$. By Lemma 12, $\mu_{3+s,D}$ is surjective. Applying Proposition 7 and inducting on r we see that $\text{cok}(\mu_F) = 0$ for all $r \geq 0$ and $s > 0$.

Now let $H = (14, 5, 3)$. Note that $(14, 5, 3) = (3, 1, 0) + (11, 4, 3)$. Thus $F = H + r(8, 3, 1) + s(11, 4, 3)$ is just $(3, 1, 0) + r(8, 3, 1) + (s + 1)(11, 4, 3)$, and our preceding analysis shows that $\text{cok}(\mu_F) = 0$ for $F = (3, 1, 0) + r(8, 3, 1) + (s + 1)(11, 4, 3)$.

The remaining possibilities for H reduce in a similar way to the case $H = (3, 1, 0)$ treated above: $(27, 10, 3)$ is $(3, 1, 0) + 3(8, 3, 1)$; $(36, 13, 9)$ is $(3, 1, 0) + 3(11, 4, 3)$; $(25, 9, 6)$ is $(3, 1, 0) + 2(11, 4, 3)$; and $(58, 21, 15)$ is $(3, 1, 0) + 5(11, 4, 3)$. In each instance the reader will easily verify that either (a) or (b) of the statement of Proposition 15 is obtained. $\diamond$

4. Examples

In this section we show by example how our results give minimal free resolutions for fat point subschemes $Z = m_1 p_1 + \cdots + m_n p_n$ of $\mathbf{P}^2$ with $n \leq 8$, where the points p_i are assumed to be general. We also give two examples for $n > 8$ general points on a smooth plane cubic curve, one showing that our results sometimes determine resolutions even though $n > 8$ and one showing that sometimes they do not.

If we are interested in a resolution of I_Z for $Z = m_1 p_1 + \cdots + m_n p_n$ with $n < 8$ we might as well assume $n = 8$ and simply set some multiplicities m_i equal to 0; i.e., having $n < 8$ is no different from having $n = 8$. Now, for our first example, consider $Z = 54(p_1 + \cdots + p_8)$. Recall that $h_Z(t)$ is the Hilbert function of I_Z in degree t ; thus $h_Z(t) = \dim((I_Z)_t)$. Let X be the surface obtained by blowing up the points p_i , with the corresponding exceptional configuration being $L, E_1, \dots, E_8$. Note that $D = 17L - 6(E_1 + \cdots + E_8)$ is numerically effective by Lemma 4. Denote $tL - 54(E_1 + \cdots + E_8)$ by H_t . Since $H_t \cdot D = 17t - 54 \cdot 6 \cdot 8$ is negative for $t < 153$, we see $h^0(X, H_t) = 0$ for $t < 153$. Since $H_t \cdot E \geq 0$ for all exceptional curves E when $t \geq 153$, we know H_t is numerically effective for all $t \geq 153$ and hence that $h^1(X, H_t) = 0$ and $h^0(X, H_t) = ((H_t)^2 - K_X \cdot H_t)/2 + 1 = \binom{t+2}{2} - 8 \binom{54+1}{2}$ for $t \geq 153$. In other words, $h_Z(t) = 0$ for $t < 153$, and $h_Z(t) = \binom{t+2}{2} - 8 \binom{54+1}{2}$ for $t \geq 153$.

By vanishing of $h_Z(t)$ for $t < 153$ we see $\nu_t = 0$ for $t < 153$, and, by vanishing of $h^1(X, H_t)$ for $t \geq 153$ and by the general fact at the end of Example 8 we see $\nu_{t+2} = 0$ for all $t \geq 153$. Since it is obvious that $\nu_{153} = h_Z(153) = 55$, we are left with finding ν_{154} . But $H_{153} \cdot E = 0 < \lambda_E$ for $E = 6L - 3E_1 - 2E_2 - \cdots - 2E_8$, and likewise for $E = 6L - 2E_1 - 3E_2 - 2E_4 - \cdots - 2E_8, \dots$, so by Theorem 1(b), $\mu_{H_{153}}$ has the same kernel as the maps corresponding to $H_{153} - (6L - 3E_1 - 2E_2 - \cdots - 2E_8)$, $H_{153} - (6L - 3E_1 - 2E_2 - \cdots - 2E_8) - (6L - 2E_1 - 3E_2 - 2E_4 - \cdots - 2E_8)$, etc., and we find eventually that $\mu_{H_{153}}$ has the same kernel as μ_H with $H = 9L - 3E_1 - \cdots - 3E_8$. By Theorem 1(a), μ_H has maximal rank, and since $h^0(X, H) = 7$ and $h^0(X, H + L) = 18$ by Lemma 4, we see that μ_H must be surjective with 3-dimensional

kernel. Thus $\mu_{H_{153}}$ has 3-dimensional kernel, and from $h^0(X, H_{153}) = 55$ and $h^0(X, H_{154}) = 210$, we see ν_{154} equals 48 (rather than the maximal rank value of 45).

Using the relation $\nu_t - s_t = \Delta^3 h_Z(t)$, we find that s_t is 0 for $t < 154$ or $t > 155$, $s_{154} = 3$ and $s_{155} = 99$. Thus the minimal free resolution of I_Z is $0 \rightarrow F_1 \rightarrow F_0 \rightarrow I_Z \rightarrow 0$ where $F_0 = R^{55}[-153] \oplus R^{48}[-154]$ and $F_1 = R^3[-154] \oplus R^{99}[-155]$.

Consider now another example. Suppose $Z = mp_1 + \cdots + mp_n$ for $n \geq 9$ general points of a smooth cubic curve in $\mathbf{P}^2$. We show our results recover the resolution of I_Z , which is known in this case (see section 3.2.1 of [H7]). So let $H_t = tL - m(E_1 + \cdots + E_n)$ with $m > 0$ and let D_X as usual be a smooth section of $-K_X$, where X is obtained from $\mathbf{P}^2$ by blowing up the points p_i . If $t < 3m$, then $H_t \cdot (-K_X) < 0$, so $h^0(X, H_t) = h^0(X, H_t + K_X)$, but we still have $(H_t + K_X) \cdot (-K_X) < 0$ and iterating we eventually find that $h^0(X, H_t) = h^0(X, H_t + mK_X) = 0$ (the last equality follows since $L \cdot (H_t + mK_X) < 0$). Thus $h_Z(t) = 0$ for $t < 3m$, hence $\mu_t(Z)$ has maximal rank for $t < 3m$. For $t = 3m$, then $|H_t| = \{mD_X\}$, so $\mu_t(Z)$ has maximal rank. For $t > 3m$, then $H_t = (t - 3m)L - mK_X$, so $E \cdot H_t = (t - 3m)E \cdot L - mE \cdot K_X > E \cdot L > \Lambda_E$ for every exceptional curve E and either $H_t \cdot (-K_X) \geq 2 = \Lambda_{D_X}$ (hence $\mu_t(Z)$ has maximal rank by Theorem 1(a)) or $H_t \cdot (-K_X) < 2 = \Lambda_{D_X}$ (hence μ_{H_t} and $\mu_{H_t + K_X}$ have kernels of the same dimension by Theorem 1(b), and iterating, for some l we eventually obtain $H_t + lK_X$ falling under case (a) of Theorem 1). Thus in any case we can compute the rank of μ_{H_t} for every t , which makes it easy to work out the resolution for any particular m and n .

Finally, consider $Z = 156p_1 + 121(p_2 + \cdots + p_7) + 104p_8 + 78p_9$. Again, let $H_t = tL - (156E_1 + 108(E_2 + \cdots + E_7) + 104E_8 + 78E_9)$. It turns out that $[27L - 12E_1 - 9(E_2 + \cdots + E_7) - 8E_8 - 6E_9]$ is the class $[E]$ of an exceptional curve E , and that $[H_t] = [(t - 351)L + 13E]$. From this it is easy to check that μ_{H_t} is injective for $t \leq 351$ (since $h^0(X, H_t) = 0$ for $t < 351$ while $h^0(X, H_{351}) = 1$) and (by Example 8, since $h^1(X, H_t) = 0$ for $t > 351$) surjective for $t > 352$. However, for $t = 352$ we have $H_t \cdot C \geq \Lambda_C$ for all $C \in \Gamma_X$ except $C = E$, for which we have $\lambda_E = 12 < H_t \cdot E = 14 < 15 = \Lambda_E$, and hence Theorem 1 does not apply and, indeed, the rank of μ_{H_t} is not known.

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