On the minimal free resolution for fat point schemes of multiplicity at most 3 in $\mathbb{P}^2$.

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ABSTRACT. Let Z be a fat point scheme in $\mathbb{P}^2$ supported on general points. Here we prove that if the multiplicities are at most 3 and the length of Z is sufficiently high then the number of generators of the homogeneous ideal I_Z in each degree is as small as numerically possible. Since it is known that Z has maximal Hilbert function, this implies that Z has the expected minimal free resolution.

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1. Introduction.

What is the minimal free resolution of a " general " zero-dimensional scheme $Z \subset \mathbb{P}^2$? In this paper " general " means that $\sharp(Z_{red})$ is fixed and Z_{red} is general in $\mathbb{P}^2$. We are interested in the minimal free resolution of general fat point schemes of $\mathbb{P}^2$. A fat point $mP \subset \mathbb{P}^2$ is the zero-dimensional subscheme of $\mathbb{P}^2$ with support in the point P and $(\mathcal{I}_P)^m$ as its ideal sheaf. A general fat point scheme $Z := m_1P_1 + \ldots + m_rP_r$ of $\mathbb{P}^2$, $m_1 \geq \ldots m_r \geq 0$, is a general zero-dimensional scheme such that for each $P_i \in Z_{red}$ the connected component of Z with support in P_i is the fat point m_iP_i. If $m = 1$ (resp. $m = 2$, resp. $m = 3$) we will say that mP is a simple (resp. double, resp. triple) point. We recall that length$(mP) = m(m+1)/2$ for all $m > 0$.

The Hilbert function and the minimal free resolution of plane fat point schemes have been studied quite a lot in the last years, assuming that the number of points is low, or that the multiplicities are low, or giving some other kind of restriction on the involved integers. For example, the Hilbert function is known in the equimultiple case for any r if $m = m_1 = \ldots = m_r \leq 20$ ([Hi], [C-C-M-O]) and in many other cases ([H-H-F], [H-R], [E2], [R]); it is also known if $r \leq 9$ ([N], [Ha2]) and if $m_i \leq 7$ ([M], [Y]). The graded Betti numbers for a fat point scheme $Z = m_1P_1 + \ldots + m_rP_r$ are known if $r \leq 8$ ([Ca], [F], [Ha3], [F-H-H]) and in some other cases ([H-H-F], [H-R]). For the equimultiple case, there is a general conjecture ([Ha1]), proved for $m \leq 3$; i.e., it is known that the homogeneous ideal $I(Z)$ of Z is minimally generated for $m = 1$ ([G-M]), $m = 2$ ([I]) and $m = 3$ ([G-I]).

If Z is a fat point scheme of multiplicity at most 3 in $\mathbb{P}^2$, Z has maximal Hilbert function by [M], i.e. $h^1(\mathbb{P}^2, \mathcal{I}_Z(k)) \cdot h^0(\mathbb{P}^2, \mathcal{I}_Z(k)) = 0$ for all $k \geq 0$, provided that $k \geq m_1 + m_2 + m_3$. In this paper we show that, if the length of Z is sufficiently high, the multiplication maps $\mu_k(Z) : H^0(\mathcal{I}_Z(k)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathcal{I}_Z(k+1))$ are of maximal rank for any k. The following result hence follows:

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Theorem 1.1. *Fix non-negative integers a, b, c such that $a + 3b + 6c \geq 79$, and let $Z \subset \mathbb{P}^2$ be a general union of a simple points, b double points and c triple points in $\mathbb{P}^2$. Let v be the minimal integer such that $a+3b+6c \leq (v+2)(v+1)/2$. Then Z has the expected minimal free resolution, i.e. the homogeneous ideal of Z is minimally generated by $(v+2)(v+1)/2-a-3b-6c$ forms of degree v and $\max\{0, 2a+6b+12c-v^2-2v\}$ forms of degree $v+1$.*

In the case in which there are only double points, we can handle a few low integers v and prove the following complete result:

Theorem 1.2. *Fix integers a, b such that $a \geq 0$, $b \geq 0$, and let $Z \subset \mathbb{P}^2$ be a general union of a simple points and b double points of $\mathbb{P}^2$. Let v be the minimal integer such that $a+3b \leq (v+2)(v+1)/2$. Then, for any $(a, b) \neq (0, 2), (0, 5), (1, 1), (1, 2)$, Z has the expected minimal free resolution, i.e. the homogeneous ideal of Z is minimally generated by $(v+2)(v+1)/2 - a - 3b$ forms of degree v and $\max\{0, 2a + 6b - v^2 - 2v\}$ forms of degree $v + 1$.*

We raise the following conjecture.

Conjecture 1.3. Fix integers $m > 0$ and $n \geq 2$. Then there is an integer $\alpha_{n,m}$ such that for all integers $r > \alpha_{n,m}$ the minimal free resolution of a general union $Z \subset \mathbb{P}^n$ of r multiple points with multiplicity at most m is the expected one.

For related conjectures and discussions, see also [Ha1] and [H-H-F]. Also recall the Minimal Resolution Conjecture, which is about the generic union of r simple points in $\mathbb{P}^n$, and has been proved for $n = 2$ (as said above), for $n = 3$ ([B], [B-G]), in particular cases for $n \geq 4$, and for any n and $r >> 0$ ([H-S]).

Several proofs (all of them heavily using the Horace method) of Theorem 1.1 and Theorem 1.2 might be given. We will see in section 2 that Theorem 1.2 is an obvious consequence of [I] and of the deformation of a double point to three simple points (see Remark 2.2). The proof of Theorem 1.1 is obtained as well adapting the proof of Theorem 1.1 in [G-I], but it needs a bigger effort.

Notice that there is no serious obstacle in proving Theorem 1.1 when length Z is lower and $c \neq 0$, it is essentially a question of patience; for example, in order to go down to $a + 3b + 6c \geq 56$ we only need to consider a few more cases, thanks to Lemma 3.1.5 and Lemma 3.1.6; if the fat point scheme is supported on ≤ 8 points, [F-H-H] give the Betti numbers; if a, b, c are such that $a + 3b \equiv 0 \pmod 6$, $3b \leq a$ and $\lfloor \frac{a+3b}{6} \rfloor \neq 2, 3, 5$, then [G-I] and Remark 2.2 gives the result through semicontinuity. There are low length cases where the minimal free resolution is not the expected one, for example the union of 2 or 3 or 5 triple points.

In the following we use the Horace method, introduced by A.Hirschowitz to prove this interpolation type problems, and the differential Horace method. For the first one, we send the reader to [Hi]. The second one has been introduced in [A-H] for invertible sheaves, and then extended to vector bundles (see [G-I] Proposition 2.6).

We work over an algebraically closed field $\mathbb{K}$ such that either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > 3$. In [I] and [G-I] there is the $\mathrm{char}(\mathbb{K}) = 0$ assumption, but in fact this assumption is not necessary in [I] , while in [G-I] it is used only in the proof of Lemma 2.9 where a map $t_i \mapsto t_i^{r_i}$ between formal power series rings is considered. To get the injectivity of the differential of this map at $(0, \ldots, 0)$ it is sufficient to assume $\mathrm{char}(\mathbb{K}) > r_i$ for all i. Since in our set up (fat point with multiplicity at most 3) we have $r_i \leq 3$ for all i, we hence assume $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > 3$.

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2. Preliminaries and proof of Theorem 1.2.

2.1. Let X be a 0-dimensional scheme in $\mathbb{P}^2$, and let length$X = l$; we say that X has maximal Hilbert function in degree k if $h^0(\mathcal{I}_X(k)) = max\{0, \binom{k+2}{2} - l\}$; X has maximal Hilbert function if this is true for any k.

Now assume that X has maximal Hilbert function. Then X has the expected minimal free resolution if the natural multiplication maps

$$\mu_k : H^0(\mathcal{I}_X(k)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathcal{I}_X(k+1))$$

have maximal rank for each k. Set $\Omega := \Omega^1_{\mathbb{P}^2}$. Taking the cohomology sequence of the Euler sequence in $\mathbb{P}^2$ tensored by the ideal sheaf $\mathcal{I}_X(k+1)$, we see that ker $\mu_k = H^0(\Omega(k+1) \otimes \mathcal{I}_X)$.

Set $v = v(X) = min\{k \geq 1 \mid \binom{k+2}{2} - l \geq 0\}$; then, μ_k is trivially injective for $k < v$ because $h^0(\mathcal{I}_X(k)) = 0$, and μ_k is surjective for $k > v$ by the Castelnuovo-Mumford Lemma because $h^1(\mathcal{I}_X(v)) = 0$. Hence X is minimally generated if and only if μ_v is of maximal rank.

Now let $w = w(X) = min\{k \geq 1 \mid k(k+2) - 2l \geq 0\}$ (w is the smallest integer for which the restriction map $\rho_k : H^0(\Omega(k+1)) \to H^0(\Omega(k+1)|_X)$ can be surjective). Then $2l \leq w(w+2)$ gives $l < \binom{w+2}{2}$, and by assumption X has maximal Hilbert function, hence $h^1(\mathcal{I}_X(w)) = 0$ and $h^0(\mathcal{I}_X(w)) = \binom{w+2}{2} - l > 0$. Now if $h^1(\mathcal{I}_X(k)) = 0$ then $h^1(\mathcal{I}_X(k+1)) = 0$ and we get $3h^0(\mathcal{I}_X(k)) - h^0(\mathcal{I}_X(k+1)) = 3(\binom{k+2}{2} - l) - (\binom{k+3}{2} - l) = k(k+2) - 2l$. Hence w is the smallest integer k for which the map μ_k can be surjective, without being the 0-map. From what is said above we have $v \leq w \leq v+1$, so that X is minimally generated if and only if μ_{w-1} is injective and μ_w is surjective, and this happens if and only if $h^0(\Omega(w) \otimes \mathcal{I}_X) = 0$ and $h^0(\Omega(w+1) \otimes \mathcal{I}_X) = w(w+2) - 2l$.

If we assume only that the Hilbert function of X is maximal in degree w, without any assumptions on the other degrees, then the same considerations as above show that μ_{w-1} is injective if and only if $h^0(\Omega(w) \otimes \mathcal{I}_X) = 0$, and μ_w is surjective if and only if $h^0(\Omega(w+1) \otimes \mathcal{I}_X) = w(w+2) - 2l$.

What we do in practice is to look for suitable schemes for which the map ρ_k is bijective, and from these deduce the injectivity or surjectivity for the schemes we are interested in. For arithmetical reasons (think of k odd) it is better to work in the projectivized bundle $\mathbb{P}(\Omega)$ with the canonical projection $\pi : \mathbb{P}(\Omega) \to \mathbb{P}^2$. We set $\mathcal{E}_k := \mathcal{O}_{\mathbb{P}(\Omega)}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(k)$. One has (e.g., see [I2] Lemma 2.1)

$$H^0(\Omega(k+1) \otimes \mathcal{I}_X) \cong H^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{\pi^{-1}X}), \qquad H^0(\Omega(k+1)|_X) \cong H^0(\mathcal{E}_{k+1}|_{\pi^{-1}X}).$$

If $X \subset \mathbb{P}(\Omega)$ is a 0-dimensional scheme such that $lengthX = H^0(\mathcal{E}_{k+1})$, and $H^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_X) = 0$, we say that X is k-settled.

Remark 2.2. It is immediate to see that a 2-fat point of $\mathbb{P}^2$ is the flat limit of a family whose general fiber is the general union of 3 simple points.

A 3-fat point of $\mathbb{P}^2$ is the flat limit of a family whose general fiber is the general union of one 2-fat point and 3 simple points (see [E], Proposition 4).

Hence, a 3-fat point of $\mathbb{P}^2$ is the flat limit of a family whose general fiber is the general union of 6 simple points.

Notations 2.3. We denote by R_p, with $p = 0, 1, 2, 3, 5, 8, 11$, a certain 0-dimensional scheme of length p in $\mathbb{P}(\Omega)$ which we now define. Let U be an open subset in $\mathbb{P}^2$ and $\Omega|_U \cong E_1 \oplus E_2$ a local trivialization for Ω; then, if $p \neq 0, 1$, $R_p = \eta_1 \cup \eta_2$, where η_1, η_2 have support on two distinct points A_1, A_2 in the same fiber $\pi^{-1}(P)$ where $A_i = \mathbb{P}(E_i) \cap \pi^{-1}(P)$ and $\eta_i \subset \mathbb{P}(E_i)$, $i = 1, 2$, so that length$(\eta_i \cap \pi^{-1}(P)) = 1$. If we consider affine coordinates $\{x, y, z\}$ in an affine chart of $\mathbb{P}(\Omega)$ containing R_p, we may suppose $\pi^{-1}(P) = \{x = y = 0\}$, $A_1 = (0, 0, 0)$ and $A_2 = (0, 0, 1)$; then R_p is defined as follows:
$R_0 = \emptyset$; $R_1 = \{A_1\}$ is just a point in $\mathbb{P}(\Omega)$; $R_2 = \{A_1, A_2\}$;
R_3 is made of the point $\eta_1 = A_1$ and a length 2 structure η_2 on A_2, given by an ideal of type $(x, y^2, z - 1)$;
R_5 is made of a length 2 structure η_1 on A_1, given by an ideal of type (x, y^2, z) and the first infinitesimal neighbourhood η_2 on A_2, given by an ideal of type $(x^2, xy, y^2, z - 1)$;
R_8 is made of two 4-ple structures η_1, η_2 of the same type, given by ideals of type (x^3, xy, y^2, z), $(x^3, xy, y^2, z-1)$;
R_{11} is such that η_1 is a 5-ple structure on A_1 given by an ideal of type (x^3, x^2y, y^2, z), and η_2 is given by an ideal of type $(x^3, x^2y, y^2x, y^3, z - 1)$.

So for k even R_p is two copies of a nilpotent $\eta \subset \mathbb{P}^2$ with $\eta \cong \eta_i$, while for k odd R_p is the same thing but with a nilpotent transversal to one of the components of this scheme added. Since we are interested in the schemes R_p only for the vanishing of global sections of $\mathcal{O}_{\mathbb{P}(\Omega)}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(t)$, we can consider (see Lemma 2.2 in [G-I]) that R_p is the pull back of $\eta \subset \mathbb{P}^2$ for k even, and for k odd the pull back of η with a nilpotent transversal to this scheme added. For the same reason, if $B \subset U$ is any 0-dimensional subscheme of $\mathbb{P}^2$ and we set:

$$B' := \pi^{-1}(B) \cap \mathbb{P}(E_1), \qquad B'' := \pi^{-1}(B) \cap \mathbb{P}(E_2), \qquad \widehat{B} := B' \cup B'',$$

as long as we are concerned only with the vanishing of the global section of $\mathcal{E}_k$ along $\pi^{-1}(B)$, we can substitute the last one with $\widehat{B}$.

Notations 2.4. With $Y(a, b)$ we denote in the following the generic union of a points and b double points in $\mathbb{P}^2$; we also set $\tilde{Y}(a, b) := \pi^{-1}(Y(a, b))$.
For any $k \geq 0$, let $q = q(k)$, $r = r(k)$ be positive integers such that $k(k + 2) = 6q(k) + r(k)$, with $0 \leq r \leq 5$; the possible values for r are $0, 2, 3, 5$ (see [I] Lemma 1.6).
In the following $Z(s, d, p)$ will denote the generic union in $\mathbb{P}(\Omega)$ of $\tilde{Y}(s, d)$ with R_p, where $2s+6d+p = k(k+2)$ and $0 \leq p \leq r$. Notice that $p \equiv r \pmod 2$, hence $p \neq 4$.
We set $\Delta_k = \{(s, d, p) \in \mathbb{N}^3 |\ 2s + 6d + p = k(k + 2),\ p = 0, 1, 2, 3, 5\}$.
In [I] the assertion:"$Z(0, q(k), r(k))$ is k-settled", denoted by "*A(k)*", is proved for any $k \neq 2, 3$.

Lemma 2.5. *If A(k) in [I] is true and if* $(s, d, p) \in \Delta_k$*, then* $H^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{Z(s,d,p)}) = 0$.

Proof. We write $q = q(k)$, $r = r(k)$; by assumption $2s + 6d + p = 6q + r$ with $0 \leq p \leq r \leq 5$. Writing $s = 3l + j$, $0 \leq j \leq 2$, we find $4 + r \geq 2j + p \equiv r \pmod 6$, hence $2j + p = r$.

Now observe that a double point is specialization of 3 points in the plane; R_2 is the pull back of a point of $\mathbb{P}^2$; R_3 is specialization of the union of the pull back of a point of $\mathbb{P}^2$ with R_1, which is a point in $\mathbb{P}(\Omega)$, and if $2j+p=5$, then R_5 is specialization of the general union of the pull back of j points of $\mathbb{P}^2$ with R_p $(j=1,2)$.
Hence, the scheme $Z(s,d,p)$ specializes to $Z(0,q,r)$ and we conclude by semicontinuity since by assumption $H^0(\mathcal{E}_{k+1}\otimes\mathcal{I}_{Z(0,q,r)})=0$. $\square$

Lemma 2.6. *Let a,b,k be nonnegative integers such that $(k-1)(k+1)<2a+6b\le k(k+2)$. Then, if A(k-1) is true, μ_{k-1} is injective for $Y(a,b)$;*
if A(k) is true, and if $Y(a,b)$ has maximal Hilbert function in degree k, then μ_k is surjective for $Y(a,b)$.

Proof. By assumption, $w(Y(a,b))=k$. Then, in order to prove the first and respectively the second statement, it is enough to prove (see preliminaries 2.1) $h^0(\mathcal{E}_k\otimes\mathcal{I}_{\tilde{Y}(a,b)})=0$ and respectively $h^0(\mathcal{E}_{k+1}\otimes\mathcal{I}_{\tilde{Y}(a,b)})=k(k+2)-2(a+3b)$.

Assume A(k-1) holds. We now show that there exists $(s,d,p)\in\Delta_{k-1}$ such that $Z(s,d,p)$ is contained in $\tilde{Y}(a,b)$. We write $q=q(k-1)$, $r=r(k-1)$; we are looking for s,d,p such that $2s+6d+p=(k-1)(k+1)=6q+r<2a+6b$, with $0\le r\le 5$. Set $r=2l+\epsilon$, $\epsilon=0,1$. If $q<b$, we set $d=q$, $s=0$, $p=r$; since R_r is contained in the pull back of a double point, we have $Z(0,q,r)\subset\tilde{Y}(a,b)$. If $q\ge b$, we set $d=b$, $s=3(q-b)+l$, $p=\epsilon$. Since $2(3(q-b)+l)+\epsilon<2a$, we have $s+1\le a$ and moreover if $\epsilon=1$ R_ϵ is contained in the pull back of a simple point, hence $Z(3(q-b)+l,b,\epsilon)\subset\tilde{Y}(a,b)$ both for $\epsilon=0$ and for $\epsilon=1$.

By the previous lemma $H^0(\mathcal{E}_k\otimes\mathcal{I}_{Z(s,d,p)})=0$; we conclude taking cohomology of the exact sequence: $0\to\mathcal{E}_k\otimes\mathcal{I}_{\tilde{Y}(a,b)}\to\mathcal{E}_k\otimes\mathcal{I}_{Z(s,d,p)}\to\ldots.$

Now assume A(k) holds. We now show that there exists $(s,d,p)\in\Delta_k$ such that $Z(s,d,p)$ contains $\tilde{Y}(a,b)$. We write $q=q(k)$, $r=r(k)$; we are looking for s,d,p such that $2a+6b\le 2s+6d+p=k(k+2)=6q+r$, with $0\le r\le 5$. Set $r=2l+\epsilon$, $\epsilon=0,1$. We have $6(b-q)\le r-2a\le 5$, hence $b\le q$, so we set $d=b$, $s=3(q-b)+l$, $p=\epsilon$; since $2a\le 2(3(q-b)+l)+\epsilon$, we have $a\le s$, so $Z(3(q-b)+l,b,\epsilon)\supset\tilde{Y}(a,b)$.

By the previous lemma $H^0(\mathcal{E}_{k+1}\otimes\mathcal{I}_{Z(s,d,p)})=0$; the cohomology of the exact sequence:

$$0\to\mathcal{E}_{k+1}\otimes\mathcal{I}_{Z(s,d,p)}\to\mathcal{E}_{k+1}\otimes\mathcal{I}_{\tilde{Y}(a,b)}\to\mathcal{E}_{k+1}\otimes\mathcal{I}_{\tilde{Y}(a,b),Z(s,d,p)}\to 0$$

gives $h^0(\mathcal{E}_{k+1}\otimes\mathcal{I}_{\tilde{Y}(a,b)})\le h^0(\mathcal{E}_{k+1}\otimes\mathcal{I}_{\tilde{Y}(a,b),Z(s,d,p)})=2(s-a)+6(d-b)+p=k(k+2)-(2a+6b)$. On the other hand $h^0(\mathcal{E}_{k+1}\otimes\mathcal{I}_{\tilde{Y}(a,b)})\ge h^0(\mathcal{E}_{k+1})-h^0(\mathcal{E}_{k+1}|_{\tilde{Y}(a,b)})=k(k+2)-(2a+6b)$, so we have equality. $\square$

Proof of Theorem 1.2. First let us check that $Y(a,b)$ has maximal Hilbert function (mHf for short) for $(a,b)\ne(0,2),(0,5)$. It is well known that $Y(a,b)$ has mHf for any a if $b=0$, or for any $b\ne 2,5$ if $a=0$ ([Hi]). Now let $\mathfrak{L}\ne 0$ be a linear system in $\mathbb{P}^2$; if P is a point outside of the base locus of $\mathfrak{L}$, for example a generic point of the plane, and $\mathfrak{L}(P)$ is the linear system obtained by $\mathfrak{L}$ imposing the passage through P, then dim$\mathfrak{L}$(P) $=$ dim$\mathfrak{L}-1$. Hence if we add (generic) simple points to a scheme $Y(a,b)$ with mHf we get a scheme which again has mHf. We conclude that all schemes $Y(a,b)$ with $b\ne 2,5$ have mHf. Moreover, if $a\ge 3$, and $b=2,5$, the scheme $Y(a,b)$ specializes to $Y(a-3,b+1)$ which has mHf, hence by semicontinuity $Y(a,b)$ has mHf too. It is immediate to check by hand that also in the remaining cases, i.e. $(a,b)=(1,2),(2,2),(1,5),(2,5)$ the Hilbert function is maximal.

Now we study the maps μ_k. Recall that assertion $A(k)$ is proved in [I] for $k = 1$ and $k \geq 4$. If $l = l(Y(a, b)) = a + 3b > 12$, and w is the integer such that $(w-1)(w+1) < 2a + 6b \leq w(w+2)$, then $w = w(Y(a, b)) \geq 5$ and Lemma 2.6 assures that $Y(a, b)$ is minimally generated, provided that $(a, b) \neq (0, 5)$ (see also 2.1). If $w = 4$ (i.e. $15 < 2l \leq 24$) and $v = v(Y(a, b)) = 4$ (i.e. $10 < l \leq 15$), then (see 2.1) it is enough to prove that μ_4 is surjective, and this is true by Lemma 2.6; hence $Y(a, b)$ is minimally generated also for $11 \leq l \leq 12$.

We now assume $l \leq 10$. We know that $Y(a, b)$ is minimally generated for any a if $b = 0$ ([G-M]), or for any $b \neq 2$ if $a = 0$ ([I]); the remaining cases are $\{(a, 1)\}_{a=1,\dots,7}$, $\{(a, 2)\}_{a=1,\dots,4}$ and $(1, 3)$. For $(7, 1), (4, 2), (1, 3)$ we have $h^0(\mathcal{I}_{Y(a,b)}(3)) = h^1(\mathcal{I}_{Y(a,b)}(3)) = 0$ so by the Castelnuovo-Mumford Lemma the scheme is minimally generated. The scheme $Y(6, 1)$ specializes to $Y(3, 2)$ which specializes to $Y(0, 3)$, and the last one is minimally generated, hence by semicontinuity the other two are minimally generated (notice that all schemes here have mHf, and this is why semicontinuity for $H^0(\Omega(k+1) \otimes \mathcal{I})$ is useful).

The few cases left can be recovered from [Ca]; anyway, we give their explicit description in what follows. For $(3, 1), (2, 1)$ we have $v = 2$ and $w = 3$, so it is enough to prove that μ_2 is injective, and this is true because $h^0(\mathcal{I}_{Y(3,1)}(2)) = 0$ and $h^0(\mathcal{I}_{Y(2,1)}(2)) = 1$. For $(2, 2)$ we have $v = 3$ and $w = 4$, so it is enough to prove that $H^0(\Omega(4) \otimes \mathcal{I}_{Y(2,2)}) = 0$. Let C be a conic through the four points, and L the line through the two double points; since $\Omega|_C \cong \mathcal{O}_{\mathbb{P}^1}(-3)^{\oplus 2}$, we have $H^0(\Omega(4)|_C \otimes \mathcal{I}_{Y(2,2)\cap C, C}) = 0$, hence $h^0(\Omega(4) \otimes \mathcal{I}_{Y(2,2)}) = h^0(\Omega(2) \otimes \mathcal{I}_{Y(2,0)})$ and the last one is zero because $H^0(\Omega(1)) = 0$ and $H^0(\Omega(2)|_L \otimes \mathcal{I}_{Y(2,0),L}) = 0$. The scheme $Y(5, 1)$ specializes to $Y(2, 2)$, hence also $Y(5, 1)$ is minimally generated.

For $(4, 1)$ we have $v = 3$ and $w = 3$, so we want to prove that μ_3 is surjective, or equivalently (see 2.1) that $h^0(\mathcal{E}_4 \otimes \mathcal{I}_{\tilde{Y}(4,1)}) = h^0(\Omega(4) \otimes \mathcal{I}_{Y(4,1)}) = 15 - 2l = 1$; this is in turn equivalent to $H^0(\mathcal{E}_4 \otimes \mathcal{I}_{Z(4,1,1)}) = 0$, where $Z(4, 1, 1) = \tilde{Y}(4, 1) \cup R_1$. Now we use the Horace method in $\mathbb{P}(\Omega)$, as we do, for example, in the proof of the forthcoming Lemma 3.1.4. Let C be the conic through the five points in the support of $Y(4, 1)$; then, $H^0(\mathcal{E}_4|_{\pi^{-1}C} \otimes \mathcal{I}_{Z(4,1,1)\cap\pi^{-1}C, \pi^{-1}C}) = 0$, and $H^0(\mathcal{E}_2 \otimes \mathcal{I}_{Z(1,0,1)}) = 0$ (the last one is assertion $A(1)$), so we conclude that $H^0(\mathcal{E}_4 \otimes \mathcal{I}_{Z(4,1,1)}) = 0$.

$Y(1, 1)$ is not minimally generated. In fact, let L be the line through the two points. Here $v = 2$ and $w = 2$, but μ_2 cannot be surjective since L is in the base locus of $H^0(\mathcal{I}_{Y(1,1)}(2))$, so there must be a generator of degree 3. $Y(1, 2)$ is not minimally generated. Here $v = 3$ and $w = 3$, but μ_3 cannot be surjective since the line through the two double points is in the base locus of $H^0(\mathcal{I}_{Y(1,1)}(3))$, so there must be a generator of degree 4. □

3. Proof of Theorem 1.1.

3.1 Reduction to a statement with no simple point.

3.1.1. Recall of techniques and notations from [G-I]. In the following we use, beyond the Horace method, also the differential Horace method for vector bundles, for which we refer to [G-I] Section 2 and in particular Proposition 2.6.; moreover, we'll use notations 3.1 and the ones estabilished at the beginning of the proof of 3.3 in [G-I], so we recall them briefly here.

For the definition of vertically graded subscheme with base a fixed smooth divisor see [A-H] and [G-I] 2.3. We introduce now some notations that will allow us to express ourselves as if we were working in $\mathbb{P}^2$, while our environment is actually $\mathbb{P}(\Omega)$. Let B be a 0-dimensional scheme of $\mathbb{P}^2$ with support at a point P, vertically graded with base a smooth conic C with local equation $y = 0$, and let x, y be local coordinates at P. Notice that B' and B'' (see Notations 2.3) are vertically graded with base $H = \pi^{-1}(C)$. Consider the integers a_j where, if $\mathcal{I}_{Tr^j_C(B)} := (\mathcal{I}_B : \mathcal{I}^j_C) \otimes \mathcal{O}_C$, then $Tr^j_C(B) = (x^{a_j}, y)$. We will denote $\widehat{B} = B' \cup B''$ by $\begin{pmatrix} a_s \\ \vdots \\ a_0 \end{pmatrix}$. So for example $\widehat{B}$ is denoted by (h) if $I_B = (x^h, y)$; by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ if $I_B = (x, y)^2$; by $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ if $I_B = (x^3, xy, y^2)$; by $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ if $I_B = (x^3, x^2y, y^2)$; by $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ if $I_B = (x^2, y^2)$; by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ if $I_B = (x, y)^3$.

If $\widehat{B}$ is a $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, we say that $\widehat{B}$ is "a $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ scheme over C" and write $\begin{pmatrix} 1 \\ 3 \end{pmatrix}_{overC}$. We write, for example, "$h \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ schemes general over $\mathbb{P}^2$ " to mean a union of h schemes of type $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, whose projection in $\mathbb{P}^2$ is general. Moreover, if $h, l \in \mathbb{N}$, we will use, for example, the notation "$h \begin{pmatrix} 1 \\ 2 \end{pmatrix} + l \begin{pmatrix} 1 \\ 3 \end{pmatrix}$" to denote the union of h schemes of type $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and l schemes of type $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

If a_i and b_i, $i = 1, ..., m$ are positive integers, with an abuse of notation we write $(\sum_{i=1}^m a_i b_i) = \sum_{i=1}^m a_i(b_i)$, since for our vanishing problem only the length of the scheme over C matters.

When we apply [G-I] Proposition 2.6, i.e. when we do a differential Horace, or HD, step, and we say for example that we "add over C" $\begin{pmatrix} 1 \\ [3] \end{pmatrix} + \begin{pmatrix} 1 \\ [2] \\ 3 \end{pmatrix}$, this means that we are using the "ground slide" of the vertical scheme $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and the "first floor slide" of the triple point, so that the HD trace $(\,5\,)_C$ is given by the numbers in square brackets, while the HD residue $((\,1\,) + \begin{pmatrix} 1 \\ 3 \end{pmatrix})_C$ is obtained by the eliminating the part in the square brackets.

Notations 3.1.2. With $Y(a, b, c)$ we denote in the following the generic union of a points, b double points and c triple points in $\mathbb{P}^2$; we also set $\tilde{Y}(a, b, c) := \pi^{-1}(Y(a, b, c))$.
For any $k \geq 0$, let $u = u(k)$, $\rho = \rho(k)$ be the positive integers such that $k(k+2) = 12u + \rho$, with $0 \leq \rho \leq 11$. If we write k modulo 6, we get (see [G-I] 1.2; but notice that there $u(k)$, resp.$\rho(k)$, are denoted by $q(k)$, resp.$r(k)$):

for k=6l	$u(k) = 3l^2 + l$	$\rho(k) = 0$
for k=6l+1	$u(k) = 3l^2 + 2l$	$\rho(k) = 3$
for k=6l+2	$u(k) = 3l^2 + 3l$	$\rho(k) = 8$
for k=6l+3	$u(k) = 3l^2 + 4l + 1$	$\rho(k) = 3$
for k=6l+4	$u(k) = 3l^2 + 5l + 2$	$\rho(k) = 0$
for k=6l+5	$u(k) = 3l^2 + 6l + 2$	$\rho(k) = 11.$

For a fixed k, let $Z(s, d, t, p)$ denote the generic union in $\mathbb{P}(\Omega)$ of $\tilde{Y}(s, d, t)$ with R_p, where $2s + 6d + 12t + p = k(k+2)$ and $0 \leq p \leq \rho$, $p = 0, 1, 2, 3, 5, 8, 11$.
We'll set $\Lambda_k = \{(s, d, t, p) \in \mathbb{N}^4 |\ 2s + 6d + 12t + p = k(k+2),\ 0 \leq p \leq \rho,\ p = 0, 1, 2, 3, 5, 8, 11\}$.
In the following, "$H(s, d, t, p, k)$" denotes the statement:

"If $(s, d, t, p) \in \Lambda_k$, then $H^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{Z(s,d,t,p)}) = 0$."

We want to prove that $H(s,d,t,p,k)$ holds for $k \geq 12$, and this will be done through some lemmas. Notice that if $t = 0$ the statement $H(s,d,t,p,k)$ reduces to Lemma 2.5, so we can assume $t \geq 1$.

Remark 3.1.3. Recall that $B(k)$ in [G-I] is nothing else but $H(0,0,u(k),\rho(k),k)$, and $B(k)$ is true for $k \geq 10$ ([G-I] Proposition 3.9). To be punctual about that, notice that for the remainder scheme T_8 defined in [G-I] 1.3, which is the analogous of our scheme R_8, one has the choice between two 4-ple structures η_1, η_2 of the same type, given in local coordinates x, y by an ideal of type (x^3, xy, y^2) or by an ideal of type (x^2, y^2); in the languages of vertical schemes, T_8 is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Anyway, the unique point of the proof of Theorem 1.1 in [G-I] where one chooses to use T_8 as a $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is the next to last step in the proof of Proposition 4.1, where we specialize it on (the pull back of) a smooth conic C. It is possible to choose $T_8 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ also here, specializing it as a $\begin{pmatrix} 1 \\ 1 \\ [2] \end{pmatrix}$ (in other words, we consider it as vertical scheme with respect to the y-axis instead that to the x-axis); in the last step, the residue $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ can be specialized as a (2) on C. Hence we can always assume that the remainder scheme T_8 in [G-I] is our scheme R_8.

The following lemma allows to construct a lot of well generated schemes containing 2-fat points; the proof goes exactly as the proof of Lemma 2.1 in [I], but we repeat it here for the reader's sake.

Lemma 3.1.4. *Let k be an integer ≥ 6 and R be a 0-dimensional scheme in $\mathbb{P}(\Omega)$ such that $h^0(\mathcal{E}_{k-5} \otimes \mathcal{I}_R) = 0$. Let A be the union in $\mathbb{P}(\Omega)$ of R with $\pi^{-1}(Y)$ where Y denotes the union of $2k-4$ 2-fat points in $\mathbb{P}^2$ supported at general points; then, $h^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_A) = 0$.*

Proof. Let C be a smooth conic; we denote by Z the scheme obtained by A specializing k among the 2-fat points of Y on C and consider the exact sequence:

$$0 \to \mathcal{E}_{k-1} \otimes \mathcal{I}_{Res_{\pi^{-1}C} Z} \to \mathcal{E}_{k+1} \otimes \mathcal{I}_Z \to \mathcal{E}_{k+1} \otimes \mathcal{I}_{Z \cap \pi^{-1}C, \pi^{-1}C} \to 0;$$

since on C there is a scheme of length $2k$, $H^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{Z \cap \pi^{-1}C, \pi^{-1}C}) = 0$ so that $h^0(\mathcal{E}_{k-1} \otimes \mathcal{I}_{Res_{\pi^{-1}C} Z}) = h^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_Z) \geq h^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_A)$.

The scheme $Res_{\pi^{-1}C} Z$ is the union of R and of the pull-back of $k-4$ general 2-fat points and k simple points on C. Now let C' be another smooth conic; we denote by B the scheme obtained by $Res_{\pi^{-1}C} Z$ specializing the $k-4$ 2-fat points on C' and 4 among the simple points on $C \cap C'$; consider the exact sequence:

$$0 \to \mathcal{E}_{k-3} \otimes \mathcal{I}_{Res_{\pi^{-1}C'} B} \to \mathcal{E}_{k-1} \otimes \mathcal{I}_B \to \mathcal{E}_{k-1} \otimes \mathcal{I}_{B \cap \pi^{-1}C', \pi^{-1}C'} \to 0$$

and since on C' there is a scheme of length $2k-4$, the third H^0 is 0 so that $h^0(\mathcal{E}_{k-3} \otimes \mathcal{I}_{Res_{\pi^{-1}C'} B}) = h^0(\mathcal{E}_{k-1} \otimes \mathcal{I}_B) \geq h^0(\mathcal{E}_{k-1} \otimes \mathcal{I}_{Res_{\pi^{-1}C} Z})$.

The scheme $Res_{\pi^{-1}C'} B$ is the union of R and of the pull-back of $k-4$ simple points on C and $k-4$ simple points on C'. We denote by D the scheme obtained by $Res_{\pi^{-1}C'} B$ specializing the $k-4$ C'-points on C (for details, see [I]); now consider the exact sequence:

$$0 \to \mathcal{E}_{k-5} \otimes \mathcal{I}_{Res_{\pi^{-1}C} D} \to \mathcal{E}_{k-3} \otimes \mathcal{I}_D \to \mathcal{E}_{k-3} \otimes \mathcal{I}_{D \cap \pi^{-1}C, \pi^{-1}C} \to 0$$

and since on C there is a scheme of length $2k-8$, the third H^0 is 0. Since $Res_{\pi^{-1}C}D$ is R, we finally get $0 = h^0(\mathcal{E}_{k-5} \otimes \mathcal{I}_{Res_{\pi^{-1}C}D}) = h^0(\mathcal{E}_{k-3} \otimes \mathcal{I}_D) \geq h^0(\mathcal{E}_{k-3} \otimes \mathcal{I}_{Res_{\pi^{-1}C'}B})$, that is, $h^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_A) = 0$. □

Lemma 3.1.5. *If $d \leq \frac{s}{3}$ and $k \geq 10$, then $H(s,d,t,p,k)$ is true.*

Proof. Since $d \leq \frac{s}{3}$, by Remark 2.2 the scheme $Z(s,d,t,p)$ specializes to $Z(s',0,t',p)$ where $s' = s - 3d - 6\left[\frac{s-3d}{6}\right]$, $t' = t + d + \left[\frac{s-3d}{6}\right]$.
Since $s' \leq 5$ and $p \leq \rho$, we have $10 + \rho \geq 2s' + p \equiv \rho(12)$ so that $2s' + p = \rho$, hence it is easy to see that the scheme the union of $\tilde{Y}(s',0,0)$ and of R_p specializes to R_ρ (see description of schemes R_p in Notations 2.3). So finally the scheme $Z(s',0,t',p)$ specializes to $Z(0,0,u(k),r(k))$ and we conclude by semicontinuity that $H(s,d,t,p,k)$ holds. □

Lemma 3.1.6. *The statements $H(s,d,t,p,k)$ with $d > \frac{s}{3}$ are true if both the statements $H(0,d,t,\rho(k),k)$ with any d and the statements $H(0,d,t,5,k)$ with $k \equiv 5 \pmod 6$ and with any d are true.*

Proof. Since $d > \frac{s}{3}$, Remark 2.2 allows us to say that the scheme $Z(s,d,t,p)$ specializes to $Z(\sigma,\delta,\tau,p)$ where $\sigma = s - 3\left[\frac{s}{3}\right]$, $\delta = d - \left[\frac{s}{3}\right]$, $\tau = t + \left[\frac{s}{3}\right]$; moreover, $0 \leq \sigma \leq 2$ and $\delta \geq 1$. Let $u = u(k)$, $\rho = \rho(k)$, and set $\delta = 2e + j$, $0 \leq j \leq 1$; we have: $2\sigma + 6\delta + 12\tau + p = 12(\tau + e) + 6j + 2\sigma + p = k(k+2) = 12u + \rho$, $0 \leq p \leq \rho$. Since $10 + \rho \geq 6j + 2\sigma + p \equiv \rho \pmod{12}$, we get $6j + 2\sigma + p = \rho$. Hence it is easy to see that if $(\sigma,j,p) \neq (0,1,5)$ the union of $\tilde{Y}(\sigma,j,0)$ and of R_p specializes to R_ρ, so that finally the scheme $Z(s,d,t,p)$ specializes to $Z(0,\delta-j,\tau,\rho)$, where $\delta - j \equiv 0 \pmod 2$, and we conclude by semicontinuity. If $(\sigma,j,p) = (0,1,5)$ then $\rho = 11$ so that $k \equiv 5 \pmod 6$, and it is no longer true that the union of $\tilde{Y}(\sigma,j,0)$ and of R_p specializes to R_{11}, essentially because two double points in the plane do not specialize to a triple point. □

Remark and notations 3.1.7. Now our purpose is to prove the statements $H(0,d,t,\rho(k),k)$ with any d and the statements $H(0,d,t,5,k)$ with $k \equiv 5 \pmod 6$ and any d, for $k \geq 12$.
Notice that the number d of double points in the statement $H(0,d,t,\rho(k),k)$ is necessarily even, since by assumption $6d + 12t = k(k+2) - \rho(k) \equiv 0 \pmod{12}$ (see 3.1.2).
On the other hand, the number d of double points in the statement $H(0,d,t,5,k)$ with $k \equiv 5 \pmod 6$ is necessarily odd, since (see 3.1.2 again) $k \equiv 5 \pmod 6$ if and only if $\rho(k) = 11$, and by assumption $6(d-1) + 12t = k(k+2) - 11 \equiv 0 \pmod{12}$.

In the following we set $X(d,t,k) := Z(0,d,t,\rho)$ where $6d + 12t + \rho = k(k+2)$ and $k(k+2) = 12u + \rho$, with $0 \leq \rho \leq 11$; notice that t and k, as well as d and k, determine $X(d,t,k)$, and d is always even.

With $\bar{R}_{11}$ we denote in the following the generic union of the inverse image of a double point with R_5. If $k \equiv 5 \pmod 6$, we denote by $\bar{X}(d,t,k)$ the scheme obtained by $X(d,t,k)$ substituting to R_{11} the scheme $\bar{R}_{11}$.
So finally what we want to prove is that $X(d,t,k)$ is k-settled for any $k \geq 12$ and that $\bar{X}(d,t,6l+5)$ is $(6l+5)$-settled for any $l \geq 2$.

3.2 Proof of the statement with no simple points.

3.2.1. Definition of standard step.
In the following C denotes a smooth conic.

Let $R = \left(x\begin{pmatrix}1\\2\end{pmatrix} + y\begin{pmatrix}1\\3\end{pmatrix} + z\begin{pmatrix}2\\3\end{pmatrix} + (\,e\,)\right)_{over\ C} + v\begin{pmatrix}1\\2\\3\end{pmatrix} + u\begin{pmatrix}1\\2\end{pmatrix} + R_\rho$ be a 0-dimensional scheme with length $R = h^0(\mathcal{E}_{k+1})$ and length$(R \cap \pi^{-1}C) \leq 2k$, and assume we want to prove that R is k-settled. We now define what a standard step is; the idea is that we specialize (in the sense of a differential Horace step) the maximum possible of triple points on C, and if no more triple points are available, we specialize double points on C, so that to get a scheme $\bar{R}$ with exactly $2k$ conditions on C; at this point in order to prove that R is k-settled it is enough to prove that the residual scheme is $k-2$-settled:

standard step $\mathbf{k \to k-2}$:

we "add" over C $\quad g\begin{pmatrix}1\\2\\ [3]\end{pmatrix} + n\begin{pmatrix}1\\ [2]\\3\end{pmatrix} + p\begin{pmatrix}[1]\\2\\3\end{pmatrix} + r\begin{pmatrix}1\\ [2]\end{pmatrix} + s\begin{pmatrix}[1]\\2\end{pmatrix}$ where:

$v \geq g+n+p$, $u \geq r+s$, $2x+3y+3z+e+3g+2n+p+2r+s = 2k$, $0 \leq n+p \leq 1$, $0 \leq s \leq 1$, and finally $r = s = 0$ if it is possible to find g, n, p such that $2x+3y+3z+e+3g+2n+p = 2k$.

The residue is: $\left(\,(\,x+y+2z+r+2s\,) + g\begin{pmatrix}1\\2\end{pmatrix} + n\begin{pmatrix}1\\3\end{pmatrix} + p\begin{pmatrix}2\\3\end{pmatrix}\right)_{over\ C} + (v-g-n-p)\begin{pmatrix}1\\2\\3\end{pmatrix} + (u-r-s)\begin{pmatrix}1\\2\end{pmatrix} + R_\rho$.

Notice that this construction is possible if $2x+3y+3z+e \leq 2k$ and if $v \geq g+n+p$, $u \geq r+s$.

Notation 3.2.2. We recall here Definition 3.2 given in [G-I]:

Let b, c, d, e, f, ρ, k be integers ≥ 0; with $Z(b,c,d,e,f,\rho,k)$ we denote a 0-dimensional subscheme of X, union of:

$b\begin{pmatrix}1\\2\end{pmatrix} + c(1) + d\begin{pmatrix}1\\3\end{pmatrix} + e\binom{2}{3}$ over C, and $f\begin{pmatrix}1\\2\\3\end{pmatrix} + T_r$ general over $\mathbb{P}^2$,

with the following assumptions:

$(\mathbf{0})_\mathbf{k}$ $\quad 2b+c+3d+3e \leq 2k, \quad 0 \leq d+e \leq 1,$

$(\mathbf{1})_\mathbf{k}$ $\quad 2(3b+c+4d+5e+6f)+r = k(k+2)$ (i.e., *length*$(Z(b,c,d,e,f,r,k)) = h^0(\mathcal{E}_{k+1})$)

$(\mathbf{2})_\mathbf{k}$ $\quad \rho = 0$ or $\rho = 8$ if k is even; $\rho = 3$ or $\rho = 11$ if k is odd.

In [G-I] it is proved that $Z(b,c,d,e,f,\rho,k)$ is k-settled for $k \geq 12$: see Definition 3.2 and proof of Proposition 3.9 there.

If $k = 6l+5$, i.e. if $\rho = 11$, we denote here by $\bar{Z}(b,c,d,e,f,11,k)$ the scheme obtained by $Z(b,c,d,e,f,11,k)$ substituting to R_{11} the scheme $\bar{R}_{11}$.

Lemma 3.2.3. *Let $k = 6l+5$. The schemes $\bar{Z}(b,c,d,e,f,11,k)$ are k-settled for $k \geq 12$.*

Proof. Use the proof of the fact that $Z(b,c,d,e,f,\rho,k)$ is k-settled given in [G-I], substituting to R_{11} the scheme $\bar{R}_{11}$; it is hence enough to prove the initial cases with this substitution. So it is enough to prove an analogous of Lemma 5.2 in [G-I], where 11 schemes of type $Z(b,c,d,e,f,11,7)$ are proved to be 7-settled. We'll do the same here substituting to R_{11} the scheme $\bar{R}_{11}$; so we want to prove that the schemes:
$\bar{Z}(0,9,0,1,2,11,7)$, $\bar{Z}(1,5,0,0,3,11,7)$, $\bar{Z}(1,6,0,1,2,11,7)$, $\bar{Z}(1,7,1,0,2,11,7)$,
$\bar{Z}(2,2,0,0,3,11,7)$, $\bar{Z}(2,3,0,1,2,11,7)$, $\bar{Z}(2,4,1,0,2,11,7)$, $\bar{Z}(3,5,0,0,2,11,7)$,

$\bar{Z}(4,2,0,0,2,11,7)$, $\bar{Z}(4,3,0,1,1,11,7)$, $\bar{Z}(1,1,1,0,3,11,7)$
are 7-settled. In all these cases we do 2 standard steps (see 3.2.1) and in all the 11 cases the last residue is $(\,(2)+\begin{pmatrix}1\\2\end{pmatrix})_{over\ C}+\begin{pmatrix}1/0\\2\end{pmatrix}$ or $(\,5\,)_{over\ C}+\begin{pmatrix}1/0\\2\end{pmatrix}$ (where $\begin{pmatrix}1/0\\2\end{pmatrix}$ denotes the scheme R_5). Now the second scheme specializes to the first one, since 5 points on a conic are general, so it is enough to prove that the first one is 1-settled.
We do a step $3\to 1$: we "add" over C $\begin{pmatrix}1/0\\ [2]\end{pmatrix}$ and the residue is $(\,1\,)+(\,1/0\,)$ which is 1-settled (recall that the scheme $(\,1/0\,)_{over\ C}$ is just a point of $\mathbb{P}(\Omega)$). □

Lemma 3.2.4. *If $d\le k$, $X(d,t,k)$ is k-settled for any $k\ge 12$ and $\bar{X}(d,t,k)$ is k-settled for any $k\ge 12$ and $k\equiv 5$ (mod 6).*

Proof. Let $u=u(k)$, $\rho=\rho(k)$. In [G-I] it is proved that $H^0(\mathcal{E}_{k+1}\otimes\mathcal{I}_{Z(b,0,0,0,f,\rho,k)})=0$ for $k\ge 12$, and Lemma 3.2.3 says that for $k=6l+5$, the schemes $\bar{Z}(b,0,0,0,f,11,k)$ are k-settled for $k\ge 12$. Now the scheme $Z(b,0,0,0,f,\rho,k)$ is union in $\mathbb{P}(\Omega)$ of R_ρ with the inverse image of f general triple points and of b double points whoose suports lie on a smooth conic, with $b\le k$ (this is condition $(0)_k$), so that we conclude by semicontinuity that $X(d,t,k)$ is k-settled for any $k\ge 12$. Analougously for $\bar{X}(d,t,6l+5)$. □

Lemma 3.2.5. *If $k\ge 16$ and $k<d\le k+\left[\frac{k-4}{2}\right]$, or if $k\ge 18$ and $k+\left[\frac{k-4}{2}\right]<d<2k-4$, then $X(d,t,k)$ is k-settled and if $k=6l+5$ $\bar{X}(d,t,k)$ is k-settled.*

Proof. We first prove the statement about $X=X(d,t,k)$. Let C be a smooth conic. We do an Horace step. We specialize on C k double points; now on C there is a scheme of length $2k$, so that $H^0(\mathcal{E}_{k+1}\otimes\mathcal{I}_{X\cap\pi^{-1}C,\pi^{-1}C})=0$, hence $h^0(\mathcal{E}_{k-1}\otimes\mathcal{I}_{Res_{\pi^{-1}C}X})=h^0(\mathcal{E}_{k+1}\otimes\mathcal{I}_X)$. The residual scheme $Res_{\pi^{-1}C}X$ is the generic union of $\tilde{Y}(0,d-k,t)$ with the pull back of k points on C and with R_ρ.
Now we do an HD step; this time we need to have $2k-4$ points of $\mathbb{P}^2$ in total on C, so we still need $k-4$.
case 1: $d-k\le\left[\frac{k-4}{2}\right]$:
we add on C $(d-k)\begin{pmatrix}1\\ [2]\end{pmatrix}+g\begin{pmatrix}1\\2\\ [3]\end{pmatrix}+h\begin{pmatrix}1\\ [2]\\3\end{pmatrix}+i\begin{pmatrix}[1]\\2\\3\end{pmatrix}$ where $k+2(d-k)+3g+2h+i=2k-4$ and $0\le h+i\le 1$. The HD residue is the generic union of $\tilde{Y}(0,0,t-g-h-i)$ of R_ρ and of $(d-k)(1)+g\begin{pmatrix}1\\2\end{pmatrix}+h\begin{pmatrix}1\\3\end{pmatrix}+i\begin{pmatrix}2\\3\end{pmatrix}$ on C, which in the notation of [G-I] is a generalization of $Z(g,d-k,h,i,t-g-h-i,\rho,k-4)$; in fact, it is easy to check that conditions $(1)_{k-4}$ and $(2)_{k-4}$ are automatically verified, while the condition $(0)_{k-4}$ is true for $k\ge 4$. Moreover, $t-g-h-i\ge 0$ if $k\ge 16$.
case 2: $d-k>\left[\frac{k-4}{2}\right]$:
we set $k-4=2m+l$, $l=0,1$, and we add on C $m\begin{pmatrix}1\\ [2]\end{pmatrix}+l\begin{pmatrix}[1]\\2\end{pmatrix}$ where $k+2m+l=2k-4$. The HD residue the generic union of $\tilde{Y}(0,d-k-m-l,t)$, of R_ρ and of $m(1)+l(2)$ on C.
Now we do another HD step: we need $2k-8$ points of $\mathbb{P}^2$ in total on C, so we add on C $(d-k-m-l)\begin{pmatrix}1\\ [2]\end{pmatrix}+g\begin{pmatrix}1\\2\\ [3]\end{pmatrix}+h\begin{pmatrix}1\\ [2]\\3\end{pmatrix}+i\begin{pmatrix}[1]\\2\\3\end{pmatrix}$ where $m+2l+2(d-k-m-l)+3g+2h+i=2k-8$ and $0\le h+i\le 1$. The HD

residue is the generic union of $\tilde{Y}(0,0,t-g-h-i)$, of R_ρ and of $(d-k-m-l)(1)+g\begin{pmatrix}1\\2\end{pmatrix}+h\begin{pmatrix}1\\3\end{pmatrix}+i\begin{pmatrix}2\\3\end{pmatrix}$ on C, which in the notation of [G-I] is a generalization of $Z(g,d-k-m-l,h,i,t-g-h-i,\rho,k-6)$; in fact, conditions $(1)_{k-6}$ and $(2)_{k-6}$ are automatically verified, while $(0)_{k-6}$ is true for $k\geq 16$. Moreover, it is easy to check that $t-g-h-i\geq 0$ if $k\geq 18$.

The conclusion follows by [G-I] , where, as said above, it is proved that these schemes $Z(b,c,d,e,f,r,k)$ are k-settled for $k\geq 12$.
The statement about $\bar{X}(d,t,6l+5)$ is proved exactly in the same way using Lemma 3.2.3 instead of [G-I]. $\square$

3.2.6. In the following Lemma 3.2.7 we treat the case of double and triple points with a lot of double points. Hence it is convenient to give some definitions.

Let k, n be integers with $1\leq n\leq \frac{k}{6}$ and let R be a 0-dimensional scheme in $\mathbb{P}(\Omega)$ such that $h^0(\mathcal{E}_{k+1-6n}\otimes\mathcal{I}_R)=0$. Let A be the union in $\mathbb{P}(\Omega)$ of R with the inverse image of the union of $(2k-4)+(2(k-6)-4)\ldots+(2(k-6(n-1))-4)=n(2k-6n+2)$ double points in $\mathbb{P}^2$ supported at general points; then, Lemma 3.1.4 applied n times gives $h^0(\mathcal{E}_{k+1}\otimes\mathcal{I}_A)=0$ (the condition $n\leq\frac{k}{6}$ assures that $k+1-6n\geq 1$, and also that each addend in the sum is positive, since $n\leq\frac{k+4}{6}$).
For all $k\geq 6$ and $1\leq n\leq\frac{k+4}{6}$ we set

$$\alpha(n,k):=\sum_{i=0}^{n-1}\big(2(k-6i)-4\big)=n(2k-6n+2).$$

If we fix a $k\geq 6$ the function $\alpha(n,k)$ is hence increasing as long as it is defined, and strictly increasing if $n<\frac{k+4}{6}$.
Now consider a scheme $X(d,t,k)$ with $k\geq 6$; we can set

$$\bar{n}=n(d,k):=max\{n,\ 1\leq n\leq\frac{k}{6},\ d\geq\alpha(n,k)\}\text{ if }d\geq 2k-4,$$
$$\bar{n}=n(d,k):=0\text{ if }d<2k-4.$$

Let $d\geq 2k-4$ and let m be an integer, $1\leq m\leq\bar{n}$; we have seen above that $H(0,d,t,\rho,k)$ is true if $H(0,d-\alpha(m,k),t,\rho,k-6m)$ is true. Moreover, the scheme $X(d-\alpha(m,k),t,k-6m)$ verifies

$$\alpha(\bar{n}+1-m,k-6m)>d-\alpha(m,k)\geq\alpha(\bar{n}-m,k-6m)\qquad(*)$$

so that

$$n(d-\alpha(m,k),k-6m)=n(d,k)-m\qquad(**).$$

In fact, one has $\alpha(\bar{n},k)-\alpha(m,k)=\alpha(\bar{n}-m,k-6m)$, so that the second inequality is clear. For the first one, there are two possibilities:
i) $\alpha(\bar{n}+1,k)$ is defined and $>\alpha(\bar{n},k)$, i.e. $\bar{n}+1<\frac{k+4}{6}$; then by definition $\alpha(\bar{n}+1,k)>d$ so that $\alpha(\bar{n}+1-m,k-6m)>d-\alpha(m,k)$.
ii) $\alpha(\bar{n}+1,k)$ is not defined, i.e. $\bar{n}+1>\frac{k+4}{6}$, or $\alpha(\bar{n}+1,k)=\alpha(\bar{n},k)$, i.e. $\bar{n}+1=\frac{k+4}{6}$.
Since $\bar{n}\leq\frac{k}{6}$ we have: $\bar{n}=\frac{k-1}{6}$ or $\bar{n}=\frac{k}{6}$ in the first case, $\bar{n}=\frac{k-2}{6}$ in the second. We recall that $6d+12t+\rho(k)=k(k+2)$, hence $d\leq\frac{k(k+2)-\rho}{6}=2u(k)$.
If $\bar{n}=\frac{k-2}{6}$ then $k\equiv 2\pmod 6$ so that $\rho=8$; hence, $\alpha(\bar{n},k)=\frac{(k-2)(k+4)}{6}=2u(k)$.

If $\bar{n} = \frac{k-1}{6}$ then $k \equiv 1 \pmod 6$ so that $\rho = 3$; hence, $\alpha(\bar{n}, k) = \frac{(k-1)(k+3)}{6} = 2u(k)$.
If $\bar{n} = \frac{k}{6}$ then $k \equiv 0 \pmod 6$ so that $\rho = 0$; hence, $\alpha(\bar{n}, k) = \frac{k(k+2)}{6} = 2u(k)$. Since by definition $d \geq \alpha(\bar{n}, k)$, in each of the three cases we get $d = \alpha(\bar{n}, k)$; hence the first inequality in $(*)$ becames $\alpha(\bar{n}+1-m, k-6m) > \alpha(\bar{n}-m, k-6m)$ which is true (notice that $\bar{n}+1-m \leq \bar{n}$ so that $\alpha(\bar{n}+1-m, k-6m)$ is defined).

Lemma 3.2.7. *If $k \geq 12$ then $X(d,t,k)$ is k-settled and when $k = 6l+5$ also $\bar{X}(d,t,k)$ is k-settled.*

Proof. We prove the statement about $X(d,t,k)$, since the statement about $\bar{X}(d,t,k)$ can be proved exactly in the same way.
If $d < 2k-4$, then $X(d,t,k)$ is k-settled for $k \geq 18$ by Lemma 3.2.4 and Lemma 3.2.5, and for $12 \leq k \leq 17$ by Lemma 3.3.1.
If $d \geq 2k-4$, write $k = 6h+j$, $0 \leq j \leq 5$, and let $\bar{n} = n(d,k)$. If $\bar{n} \leq h-3$, then $k - 6\bar{n} \geq 18 + j$, so that $H(0, d-\alpha(\bar{n},k), t, \rho, k-6\bar{n})$ is true by Lemma 3.2.4 and Lemma 3.2.5, since by $(*)$ $d - \alpha(\bar{n}, k) < 2(k-6\bar{n})-4$.
If $\bar{n} \geq h-2$, then we apply $h-2$ times Lemma 3.1.4, and in this way we see that $H(0,d,t,\rho,k)$ is true if $H(0, d-\alpha(h-2,k), t, \rho, 12+j)$ is true. So we conclude by Lemma 3.3.1.

3.3 Initial cases and proof of of Theorem 1.1.

Lemma 3.3.1. *If $k = 12+j$ with $0 \leq j \leq 5$ then $X(d,t,k)$ is k-settled and when $k = 17$ also $\bar{X}(d,t,k)$ is k-settled.*

Proof. If $d \leq k$ the statement is true by Lemma 3.2.4. If $k = 16, 17$ and $k < d \leq k + \left[\frac{k-4}{2}\right]$, the statement is true by Lemma 3.2.5. We are hence going to prove that the union $X(d,t,k)$ of d double and t triple points and of R_ρ verifies $H^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{X(d,t,k)}) = 0$, where $d + 2t = \frac{k(k+2)-\rho}{6} = 2u(k)$ so that d is even, and $d > k$ if $12 \leq k \leq 15$, $d > k + \left[\frac{k-4}{2}\right]$ if $k = 16, 17$. For $k = 17$ we are also going to prove that the same holds substituting to R_{11} the scheme $\bar{R}_{11}$. When $j = 0,1,3,4$ the case $t = 0$ is proved in [I], section 2 so that we'll assume $t > 0$ for $k = 12, 13, 15, 16$. Recall that at each step $l \to l-2$, which can be an Horace or a differential Horace step, the divisor is the pull back of a smoth conic C and the points of $\mathbb{P}^2$ needed on C are $2l$. All the assertions about unions of double points plus a scheme R_ρ are proved in [I] section 2.

Assume $0 \leq t \leq \left[\frac{2k}{3}\right]$. We define the following algorithm **(A)(t,k)** applying one standard step (see 3.2.1) to $X(d,t,k)$, and then another standard step to the residue (in these assumptions the standard steps are particularly simple):
step $k \to k-2$: We "add" over C $\quad t\begin{pmatrix}1\\2\\ [3]\end{pmatrix} + b\begin{pmatrix}1\\ [2]\end{pmatrix} + a\begin{pmatrix}[1]\\2\end{pmatrix}$ where $3t+2b+a = 2k$, $a = 0,1$; since the condition $3t \leq 2k$ is verified by assumption, this is possible if $d-a-b \geq 0$ is true.
The residue is $\left(t\begin{pmatrix}1\\2\end{pmatrix} + (\,b+2a\,)\right)_{over\ C} + (d-a-b)\begin{pmatrix}1\\2\end{pmatrix} + R_\rho$.

step $k-2 \to k-4$: We "add" over C $\quad h\begin{pmatrix}1\\ [2]\end{pmatrix} + i\begin{pmatrix}[1]\\2\end{pmatrix}$ where $2t+b+2a+2h+i = 2(k-2)$, $i = 0,1$; this is possible if the conditions $2t+b+2a \leq 2(k-2)$ and $d-a-b-h-i \geq 0$ are verified.

The residue is $(\,t+h+2i\,)_{over\ C} + (d-a-b-h-i)\begin{pmatrix}1\\2\end{pmatrix} + R_\rho$. We set $w = w(t,k) := t+h+2i$, and $q = q(t,k) := d-a-b-h-i$; we are reduced to prove that the scheme $(\,w\,)_{over\ C} + q\begin{pmatrix}1\\2\end{pmatrix} + R_\rho$ is $(k-4)-$ *settled.*

Assume $t > \left[\frac{2k}{3}\right]$. We define the following algorithm **(C)(t,k)** applying one standard step to $X(d,t,k)$ (again in these assumptions the standard step is very simple):
step $k \to k-2$:
we "add" over C $\quad g\begin{pmatrix}1\\2\\ [3]\end{pmatrix} + n\begin{pmatrix}1\\ [2]\\3\end{pmatrix} + p\begin{pmatrix}[1]\\2\\3\end{pmatrix}$ where $3g+2n+p = 2k$, $n+p = 0,1$ (this is always possible).
The residue is $\left(g\begin{pmatrix}1\\2\end{pmatrix} + n\begin{pmatrix}1\\3\end{pmatrix} + p\begin{pmatrix}2\\3\end{pmatrix}\right)_{over\ C} + (t-g-n-p)\begin{pmatrix}1\\2\\3\end{pmatrix} + d\begin{pmatrix}1\\2\end{pmatrix} + R_\rho$.

In the following we apply the standard step 3.2.1, (A)(t,k) and (C)(t,k) a number of times; easy calculations assure that the conditions respectively $2x+3y+3z+e \le 2k$, $v \ge g+n+p$, $u \ge r+s$ for the standard step, and $2t+b+2a \le 2(k-2)$, $d-a-b-h-i \ge 0$ for (A)(t,k) are verified (while (C)(t,k) is always possible).

case $\mathbf{k = 12}$: here $\rho = 0$; we have to treat the cases $1 \le t \le 7$.
For $1 \le t \le 7$ (A)(t,12) gives $(w,q) = (10,10)$ for $t = 6,7$, $(w,q) = (7,11)$ for $2 \le t \le 5$, $(w,q) = (4,12)$ for $t = 1$.
In all three cases we do a standard step $8 \to 6$, and the residue is $(\,3\,)_{over\ C} + 7\begin{pmatrix}1\\2\end{pmatrix}$, which specializes to 8 double points and it is 6-settled, or $(\,6\,)_{over\ C} + 6\begin{pmatrix}1\\2\end{pmatrix}$. In the last case, we do another standard step $6 \to 4$, and the residue is $(\,3\,)_{over\ C} + 3\begin{pmatrix}1\\2\end{pmatrix}$, which specializes to 4 double points and it is 4-settled.

case $\mathbf{k = 14}$: here $\rho = 8$; we have to treat the cases $0 \le t \le 10$.
If $t \le 9$ we use the first step of (A)(t,14), so that we have to prove that the residue $\left(t\begin{pmatrix}1\\2\end{pmatrix} + (\,b+2a\,)\right)_{over\ C} + (d-a-b)\begin{pmatrix}1\\2\end{pmatrix} + R_8$ is 12-settled; recall that R_8 is a scheme $\begin{pmatrix}1\\3\end{pmatrix}$.
step $12 \to 10$: We "add" over C $\quad h'\begin{pmatrix}1\\ [2]\end{pmatrix} + i'\begin{pmatrix}[1]\\2\end{pmatrix} + \begin{pmatrix}1\\ [3]\end{pmatrix}$ where $2t+b+2a+2h'+i'+3 = 2(k-2) = 24$, $i' = 0,1$; this is possible since the conditions $2t+b+2a+3 \le 24$ and $d-a-b-h'-i' \ge 0$ are verified.
The residue is $(\,t+h'+2i'+1\,)_{over\ C} + (d-a-b-h'-i')\begin{pmatrix}1\\2\end{pmatrix}$. We set $w' = w'(t) := t+h'+2i'+1$, and $q' = q'(t) := d-a-b-h'-i'$; we are reduced to prove that the scheme $(\,w'\,)_{over\ C} + q'\begin{pmatrix}1\\2\end{pmatrix}$ is $10-$*settled*; an easy calculation shows that $(w',q') = (12,16)$ for $8 \le t \le 9$, $(w',q') = (9,17)$ for $4 \le t \le 7$, $(w',q') = (6,18)$ for $0 \le t \le 3$. These three configurations specialize to the first residual scheme of (A)(t,12) (i.e. $\left(t\begin{pmatrix}1\\2\end{pmatrix} + (\,b+2a\,)\right)_{over\ C} + (d-a-b)\begin{pmatrix}1\\2\end{pmatrix}$) obtained respectively in cases $t = 1,2,4$, and we have proved that they are 10-settled.
If $t = 10$ apply (C)(10,14); it is now enough to prove that $\left(9\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix}\right)_{over\ C} + 16\begin{pmatrix}1\\2\end{pmatrix} + R_8$ is 12-settled.

step $12 \to 10$: we "add" over C $\binom{1}{[3]}$ (which is the scheme R_8) and the residue is $(\,12\,)_{over\ C} + 16\binom{1}{2}$, which is the case $(w', q') = (12, 16)$ previously treated.

case k = 16: here $\rho = 0$; we have to treat the cases $1 \le t \le 12$.
For $1 \le t \le 10$ (A)(t,16) gives $(w, q) = (12, 24)$ for $6 \le t \le 10$, $(w, q) = (9, 25)$ for $2 \le t \le 5$, $(w, q) = (6, 26)$ for $t = 1$. Now we do a standard step $12 \to 10$, and if $w = 12$, the residue is $(\,6\,)_{over\ C} + 18\binom{1}{2}$, while if $w = 9$ or $w = 6$, the residue is $(\,9\,)_{over\ C} + 17\binom{1}{2}$, and these are the cases $(w', q') = (9, 17)$ or $(6, 18)$ previously treated in $k = 14$.
For $11 \le t \le 12$ apply (C)(t,16); it is now enough to prove that $\left(10\binom{1}{2} + \binom{1}{3}\right)_{over\ C} + (t-11)\begin{pmatrix}1\\2\\3\end{pmatrix} + d\binom{1}{2}$ is 14-settled. We now do two standard steps more, $14 \to 12$ and $12 \to 10$, and the last residue is in both cases $(\,6\,)_{over\ C} + 18\binom{1}{2}$, which we have just recalled is 10-settled.

case k = 13: here $\rho = 3$; we have to treat the cases $1 \le t \le 9$.
For $1 \le t \le 8$ (A)(t,13) gives $(w, q) = (12, 12)$ for $t = 8$, $(w, q) = (9, 13)$ for $4 \le t \le 7$, $(w, q) = (6, 14)$ for $1 \le t \le 3$.
In all three cases we do a standard step $9 \to 7$, and if $w = 12$, the residue is $(\,3\,)_{over\ C} + 9\binom{1}{2} + R_3$, which specializes to 10 double points $+R_3$ and it is 7-settled.
If $w = 9$ or $w = 6$, the residue is $(\,6\,)_{over\ C} + 8\binom{1}{2} + R_3$; we do another standard step $7 \to 5$, and the residue is $(\,4\,)_{over\ C} + 4\binom{1}{2} + R_3$, which specializes to 5 double points $+R_5$ and it is 5-settled.

For $t = 9$ apply (C)(9,13); it is now enough to prove that $\left(8\binom{1}{2} + \binom{1}{3}\right)_{over\ C} + 14\binom{1}{2} + R_3$ is 11-settled. We do a standard step $11 \to 9$, and the residue is $(\,12\,)_{over\ C} + 12\binom{1}{2} + R_3$, which is one of the previous cases.

case k = 15: here $\rho = 3$; we have to treat the cases $1 \le t \le 13$.
For $1 \le t \le 10$ (A)(t,15) gives $(w, q) = (13, 19)$ for $8 \le t \le 10$, $(w, q) = (10, 20)$ for $4 \le t \le 7$, $(w, q) = (7, 21)$ for $1 \le t \le 3$. These three configurations specialize to the first residual scheme of (A)(t,13) (i.e. $\left(t\binom{1}{2} + (\,b + 2a\,)\right)_{over\ C} + (d - a - b)\binom{1}{2} + R_3$) obtained respectively in cases $t = 1, 2, 4$, and we have proved that they are 11-settled.

For $11 \le t \le 13$ apply (C)(t,15); it is now enough to prove that $\left(10\binom{1}{2}\right)_{over\ C} + (t-10)\begin{pmatrix}1\\2\\3\end{pmatrix} + d\binom{1}{2} + R_3$ is 13-settled. If $11 \le t \le 12$, this scheme specializes to the scheme used in the first step of (A)(t,13) (i.e. $\left(t\begin{pmatrix}1\\2\\3\end{pmatrix} + b\binom{1}{2} + a\binom{1}{2}\right)_{over\ C} + (d - a - b)\binom{1}{2} + R_3$) respectively in cases $t = 1$ (where $b = 11$, $a = 1$) and $t = 2$ (where $b = 10$, $a = 0$), and we have proved that they are 13-settled. If $t = 13$ we do three standard steps from 13 to 7 and the last residue is $(\,6\,)_{over\ C} + 8\begin{pmatrix}1\\2\end{pmatrix} + R_3$; in $k = 13$ we have proved that it is 7-settled.

case $\mathbf{k = 17}$: here $\rho = 11$; we have to treat the cases $0 \le t \le 14$.
We first prove that $X(d,t,17)$ is 17-settled.
For $0 \le t \le 11$ we have $d \ge 30$, and by Lemma 3.1.4 $X(d,t,17)$ is 17-settled if $X(d-30,t,11)$ is 11-settled.
If $t = 11$, $d = 30$, so that $X(d-30,t,11)$ is 11-settled by [G-I].
If $0 \le t \le 7$ (A)(t,11) applied to the scheme $X(d-30,t,11)$ gives $(w,q) = (8,6)$ for $4 \le t \le 7$, $(w,q) = (5,7)$ for $0 \le t \le 3$. Now we do another standard step, and the residue is $(\,3\,)_{over\ C} + 3\begin{pmatrix}1\\2\end{pmatrix} + R_{11}$, resp. $(\,6\,)_{over\ C} + 2\begin{pmatrix}1\\2\end{pmatrix} + R_{11}$.
step $5 \to 3$: we "add" over C $\quad r\begin{pmatrix}1\\ [2]\end{pmatrix} + s\begin{pmatrix}[1]\\2\end{pmatrix} + \begin{pmatrix}1/0\\2\\ [3]\end{pmatrix}$ with $0 \le s \le 1$ so to have 10 conditions on C, and in both cases the residue is $(\,(\,2\,) + \begin{pmatrix}1/0\\2\end{pmatrix}\,)_{over\ C} + \begin{pmatrix}1\\2\end{pmatrix}$.
step $3 \to 1$: we "add" over C $\quad \begin{pmatrix}1\\ [2]\end{pmatrix}$ and the residue is $(\,1\,) + (\,1/0\,)$ which is 1-settled (recall that the scheme $(\,1/0\,)_{over\ C}$ is just a point of $\mathbb{P}(\Omega)$).
For $8 \le t \le 10$, we do 3 standard step from 11 to 5, and the residue is $(\,3\,)_{over\ C} + 3\begin{pmatrix}1\\2\end{pmatrix} + R_{11}$ if $t = 8, 9$, which has been treated above, or $(\,(\,2\,) + \begin{pmatrix}1\\3\end{pmatrix}\,)_{over\ C} + 2\begin{pmatrix}1\\2\end{pmatrix} + R_{11}$ if $t = 10$, in which case we proceed as follows:
step $5 \to 3$: we "add" over C $\quad \begin{pmatrix}1/0\\2\\ [3]\end{pmatrix} + \begin{pmatrix}1\\ [2]\end{pmatrix}$ and the residue is $(\,(\,2\,) + \begin{pmatrix}1/0\\2\end{pmatrix}\,)_{over\ C} + \begin{pmatrix}1\\2\end{pmatrix}$ treated above.
If $12 \le t \le 14$, we do 3 standard steps $17 \to 15$, $15 \to 13$ and $13 \to 11$, and the residue is $(\,6\,)_{over\ C} + 20\begin{pmatrix}1\\2\end{pmatrix} + R_{11}$ in all the three cases; we go on with 3 other standard steps $11 \to 9$, $9 \to 7$ and $7 \to 5$, and the residue is $(\,6\,)_{over\ C} + 2\begin{pmatrix}1\\2\end{pmatrix} + R_{11}$, already treated above.
Now we want to prove that $\bar{X}(d,t,17)$ is 17-settled, so we substitute to R_{11} the scheme $\bar{R}_{11}$, and, if $t \ne 11$, we do the same steps from 17 to 5 as above; now we have to prove that the following schemes are 5-settled:
$(\,6\,)_{over\ C} + 2\begin{pmatrix}1\\2\end{pmatrix} + R_5 + \begin{pmatrix}1\\2\end{pmatrix}$;
$(\,3\,)_{over\ C} + 3\begin{pmatrix}1\\2\end{pmatrix} + R_5 + \begin{pmatrix}1\\2\end{pmatrix}$;
$(\,(\,2\,) + \begin{pmatrix}1\\3\end{pmatrix}\,)_{over\ C} + 2\begin{pmatrix}1\\2\end{pmatrix} + R_5 + \begin{pmatrix}1\\2\end{pmatrix}$. We do a standard step $5 \to 3$ and the residue is $(\,2\,)_{over\ C} + \begin{pmatrix}1/0\\2\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix}$ in the first case, $(\,5\,)_{over\ C} + \begin{pmatrix}1/0\\2\end{pmatrix}$ in the second and third case (here $\begin{pmatrix}1/0\\2\end{pmatrix}$ denotes the scheme R_5); in Lemma 3.2.3 we proved that both are 3-settled.
Now let $t = 11$. By Lemma 3.1.4 it is enough to prove that $\bar{X}(0,11,11)$ is 11-settled. In order to prove this, we do 4 standard steps from 11 to 3 and the residue is $(\,(\,2\,) + \begin{pmatrix}1\\2\end{pmatrix}\,)_{over\ C} + \begin{pmatrix}1/0\\2\end{pmatrix}$ which is proved to be 3-settled in Lemma 3.2.3. $\square$

Corollary 3.3.2. *$H(s, d, t, p, k)$ is true for $k \geq 12$.*

proof. It follows by Lemma 3.1.5, Lemma 3.1.6 and Lemma 3.2.7. □

Proof of Theorem 1.1. First let us check that in our assumptions $Y(a, b, c)$ has maximal Hilbert function (mHf for short). Notice that $Y(a, b, c)$ has mHf if and only if $h^0(\mathcal{I}_{Y(a,b,c)}(v-1)) = 0$ and $h^0(\mathcal{I}_{Y(a,b)}(v)) = \binom{v+2}{2} - l$ where $l = l(Y(a, b, c)) = a + 3b + 6c$ and $v = v(Y(a, b, c))$ (see 2.1).

By [M], a general fat point scheme $Z = m_1P_1 + \ldots + m_rP_r$, $4 \geq m_1 \geq \ldots \geq m_r \geq 0$, has mHf in any degree k such that $k \geq m_1 + m_2 + m_3$. Since for our schemes $Y(a, b, c)$ one always has $m_1 + m_2 + m_3 \leq 9$, we get that $Y(a, b, c)$ has mHf for $v \geq 10$, i.e. for $l > 55$.

Let k be the integer such that $(k-1)(k+1) < 2a + 6b + 12c \leq k(k+2)$, so that $w(Y(a, b, c)) = k$. Rephrasing what is done in the proof of Lemma 2.6 for the analogous statements with $c = 0$, it is easy to show that

i) there exists $(s, d, t, p) \in \Lambda_{k-1}$ such that $Z(s, d, t, p) \subseteq \tilde{Y}(a, b, c)$;

ii) there exists $(s', d', t', p') \in \Lambda_k$ such that $Z(s', d', t', p') \supseteq \tilde{Y}(a, b, c)$.

If $2l > 12 \cdot 14$, i.e. $l > 84$, then $k \geq 13$, hence $(s, d, t, p) \in \Lambda_{k-1}$, respectively $(s', d', t', p') \in \Lambda_k$, implies that $H(s, d, t, p, k-1)$, respectively that $H(s', d', t', p', k)$ is true, i.e. $H^0(\mathcal{E}_k \otimes \mathcal{I}_{Z(s,d,t,p)}) = 0$, respectively $H^0(\mathcal{E}_{k+1} \otimes \mathcal{I}_{Z(s',d',t',p')}) = 0$ (see Corollary 3.3.2 and 3.1.2).

So we see, exactly as in the proof of Lemma 2.6, that $\mu_{k-1}(Y(a, b, c))$ is injective and $\mu_k(Y(a, b, c))$ is surjective and we conclude that $Y(a, b, c)$ is minimally generated (see 2.1). If $k = 12$ (i.e. $143 < 2l \leq 168$) and $v = 12$ (i.e. $78 < l \leq 91$), then in order to prove that $Y(a, b, c)$ is minimally generated it is enough to prove that μ_{12} is surjective (see 2.1), and this is true again by ii) and Corollary 3.3.2; hence $Y(a, b, c)$ has the expected resolution also for $79 \leq l \leq 84$. □

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