

An asymptotic vanishing theorem for generic unions of multiple points

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1 Introduction

This work is devoted to the following asymptotic statement :

Theorem. 1.1 *Let X be a projective geometrically reduced and irreducible scheme over a field k of (arbitrary) characteristic p and let $\mathcal{M}, \mathcal{L}$ be line bundles on X with $\mathcal{L}$ ample. If p is positive then suppose further that X is smooth in codimension one. For fixed $m \geq 0$ there exists $d_0 = d_0(m)$, depending also on $X, \mathcal{L}, \mathcal{M}$, such that for any $d \geq d_0$ and any generic union Z of (fat) points of multiplicity $\leq m$ the canonical map*

$$H^0(X, \mathcal{M} \otimes \mathcal{L}^d) \longrightarrow H^0(Z, \mathcal{O}_Z \otimes \mathcal{M} \otimes \mathcal{L}^d)$$

has maximal rank.

Here, as usual, we call (fat) point of multiplicity m in X , any subscheme defined by $\mathcal{J}_z^m$, where $\mathcal{J}_z$ is the ideal sheaf of a point z in the smooth locus of X . The reader may prefer the following statement, which is more or less equivalent to the preceding one:

Corollary. 1.2 *Let $X, \mathcal{M}, \mathcal{L}, m$ be as above. There exists an integer ℓ such that for any generic union Z of (fat) points of multiplicity at most m and of total degree (i.e length) at least ℓ , all the canonical maps*

$$H^0(X, \mathcal{M} \otimes \mathcal{L}^d) \longrightarrow H^0(Z, \mathcal{O}_Z \otimes \mathcal{M} \otimes \mathcal{L}^d)$$

have maximal rank.

Note that the above statement applies as soon as the number of points is at least ℓ .

To simplify the presentation and highlight the essential elements, a detailed proof will only be given in the case $\mathcal{M} = \mathcal{O}_X$. The easy modifications needed to prove the general result are then outlined in §7 along with another variant.

Remark. 1.3 *The statement of the theorem is false for $p > 0$ if we allow X to be singular in codimension one. This is illustrated in the example 7.5.*

These results are already new in the case where X is the projective plane (with $\mathcal{M} = \mathcal{O}$ and $\mathcal{L} = \mathcal{O}(1)$). Indeed, even in that case, the expected vanishing theorem for generic unions of fat points [S, Ha, Hi2] is still unproven, see a survey in [G] and more recent contributions

in [Xu, ShT, CM1, CM2, M]. Reformulations of the general problem and its relation to other topics have been considered at length in [N, I1, I2, MP]. Much attention has been paid to the “homogeneous” case on $\mathbb{P}^n$, namely when all the points have the same multiplicity m : see [AC] or [Hi1] for $n = m = 2$, [A, AHi1,2,3] for $m = 2$ and n arbitrary, [Hi1] for $m = 3, n = 2, 3$, [LL2] for $m = 4, n = 2$ in perfectly adjusted cases, and finally [CM1, CM2], where they have settled completely the equal multiplicity cases $m \leq 12, n = 2$ by a new and promising method. Concerning the heterogeneous plane case, we can just mention the recent work of Th. Mignon [M], where the case of multiplicities at most four is completely elucidated, using our differential Horace lemma presented below.

In [AHi2] we developed a new technique of a differential nature for the case $m = 2$ which, in that and later papers, made it possible not only to solve some delicate low-degree cases in [AHi2,3], but also to simplify the proof for the high-degree case [AHi4][C]. The main new ingredient in the proofs of the present paper is an extension of this technique applicable to higher-order fat points ($m > 2$), see lemma 2.3 and 9.1. The new lemma does not imply the multiplicity two lemma of [AHi2], and an entirely new proof is needed.

In sections 2-7, we present the proof of the theorem. Sections 8-9 are devoted to our differential Horace lemmas. Indeed, the results presented there (see §9) are substantially more general than 2.3. While for the present asymptotic statement, 2.3 is perfectly sufficient, the full strength of 9.1 will be much more efficient for concrete cases with small n and m . The proof of 9.1 is achieved by an ideal theoretic argument. We would like to point out that our original proof of the lemmas computed the first non-zero derivative of a determinant in a way which owed much to [LL1,2].

Outline of the proof of the theorem

In the remainder of this introduction we will try to illustrate the general ideas in the proof of the main theorem in the particular case of the projective plane. We start with a given maximum multiplicity m and a sufficiently large degree d . We want to prove a maximal rank statement for a generic union Z of multiple points, which, by adding simple points we can suppose to be of total degree at least $(d+2)(d+1)/2$. Horace’s method amounts to specialising some of these points to the generic curve Γ of some intermediate degree γ . Modulo an analogous maximal rank statement on Γ which we suppose to hold inductively, our differential lemma can then be applied under certain numerical conditions (holding for large d) and we reduce to a new subscheme $\mathfrak{D}^{(1)}(Z)$ (the *derivative* of Z , see §4) and a new degree $d - \gamma$. This can be safely applied as long as the current degree, d_c , is not too small, say $d_c > \bar{d}$. But when d_c becomes smaller than or equal to $\bar{d}$, we have to backtrack in order to complete the proof. Our trick is to modify the procedure early on so as to generate in the current subscheme Z_c a sufficient number of unconstrained points of multiplicity at most $m - 1$ (of total degree at least $(\bar{d} + 2)(\bar{d} + 1)/2$). So that when the degree of the current subscheme has been lowered under $\bar{d}$, only points of multiplicity at most $m - 1$ remain. Having chosen $\bar{d}$ large enough (i.e. $\bar{d} \geq d_0(m - 1)$ in the notation of the theorem), we conclude by induction on m .

It remains to explain how we generate these free points (see §6): our differential lemma generates in $\mathfrak{D}^{(1)}(Z)$ a lot of points of multiplicity smaller than m , but all of them lie on the

exploited divisor Γ of degree γ . The trick here consists in specialising Γ to the union of two generic divisors Γ' and Γ'' of degrees γ' and γ'' , with the desired number of points specialised to say Γ' . If this number of points is sufficiently small with respect to γ' , these points suffer no constraint by being supported on a curve of degree γ' and are thus freed. Of course, the points remaining on Γ'' should not be too numerous, and we have to find numbers d_0, γ' and γ'' satisfying all the necessary inequalities.

A slight complication arises with the degree of the current divisor Γ_c . Indeed, the number of free points to be generated is computed in terms of the degree of the divisor which appears at the final stage of the procedure (this degree must be sufficiently large to comply with the induction hypothesis). On the other hand, the initial degree γ of the current divisor must be significantly larger to allow the production of enough free points. This compels us to lower the degree of the current divisor, by specialization, at each stage of the procedure (see §5).

2 The simplified differential lemma

Throughout this section, X stands for a quasi-projective variety which is geometrically reduced and irreducible, of dimension $n + 1$ over a field k of arbitrary characteristic. Since all statements are “generic” one can safely suppose k algebraically closed. The hypothesis ‘ X is smooth in codimension one if $\text{char}(k) > 0$ ’ will not come into play until the proof of the theorem in §7.

In this section, we present a weakened form of the differential lemma which we prove in §9, this form being sufficient for our main theorem. As we already outlined in the previous section, the theorem is proved by a Horace induction argument. In such an argument, specialisation techniques are used to place a certain number of points on a chosen divisor H , then the induction hypotheses are applied to the trace and the residual as defined in the

Definition. 2.1 *Let H be a Cartier divisor on X and let W be a closed subscheme of X .*

The schematic intersection

$$W'' = H \cap W$$

defined by the ideal $\mathcal{I}_{H,W''} = (\mathcal{I}_H + \mathcal{I}_W)/\mathcal{I}_H$ of $\mathcal{O}_H$ is called the trace of W on H and denoted by $\text{Tr}_H(W)$ or simply W'' if no confusion is possible.

The closed subscheme of X defined by the conductor ideal $\mathcal{I}_{W'} = (I_W : I_H)$ is called the residual of W with respect to H and denoted by $\text{Res}_H(W)$ or W' .

The canonical exact sequence

$$(1) \quad 0 \longrightarrow \mathcal{I}_{W'}(-H) \longrightarrow \mathcal{I}_W \longrightarrow \mathcal{I}_{H,W''} \longrightarrow 0$$

is called the residual exact sequence of W with respect to H .

2.1 Geometric intuition for the differential lemma

Here we try to share with the reader our intuition for our differential lemma. Suppose that X is projective and let $\mathcal{L}$ be a line bundle on X . We will keep the notation of the definition in the remainder of the discussion.

The first thing one needs to take note of is that any basic Horace type argument is based on the following trivial consequence of the residual exact sequence (1):

$$\text{if } h^0(X, \mathcal{I}_{H, W''} \otimes \mathcal{L}|_H) = 0 \text{ and } h^0(X, \mathcal{I}_{W'} \otimes \mathcal{L}(-H)) = 0 \text{ then } h^0(X, \mathcal{I}_W \otimes \mathcal{L}) = 0$$

For this to apply, one must have a priori $\deg W'' \geq h^0(H, \mathcal{L}_H)$ and $\deg W' \geq h^0(X, \mathcal{L}(-H))$. In fact to be generally applicable in an induction argument, the stronger requirement $\deg W'' = h^0(H, \mathcal{L}_H)$ is needed. We will therefore say that $h^0(H, \mathcal{L}_H)$ is the *critical degree*.

In practice one starts with some general union G of multiple points, then, by specialising them one by one to the chosen divisor H one hopes to obtain a specialisation W for which the trace has the critical degree. Since each point specialised to H increases the degree of the trace by at least the multiplicity of the point, it is not always possible to get exactly the critical degree using this process. This is the technical obstacle that the differential lemmas 9.3 and 2.3 are designed to overcome.

To see how this comes about, it is enough to consider that H is a line in the affine plane X . The ideal of a point Z of multiplicity r at the origin is then

$$(2) \quad (x, y)^r = \mathfrak{n}^r \oplus \mathfrak{n}^{r-1}y \oplus \cdots \oplus \mathfrak{n}y^{r-1} \oplus (y^r)$$

where $\mathfrak{n}$ is the ideal $(x) \subset k[x]$, and each $\mathfrak{n}^i$ is the ideal of a point of multiplicity i in H . In particular the trace corresponds to $\mathfrak{n}^r$. One can then consider that Z is formed by infinitesimally piling up the subschemes of H with ideals $\mathfrak{n}^i$. We then refer to these subschemes of H as the layers of Z . Of course only the trace given by $\mathfrak{n}^r$ is actually contained in H , the others only appear in successive infinitesimal neighbourhoods of H . Now if we consider Z as the limit of a multiple point that is translated to the origin along the y -axis, it's the layer of highest multiplicity that arrives in H (or, as might be said, arrives first) and the degree of the trace increases by r .

In the differential approach, one or more points are translated to as many distinct points supported in H . The rate of approach may differ, but all arrive at the same time. Our corollary 9.3 says that if some sequence of layers, one from each point, have degrees adding up to the critical degree, then one can consider that these arrive first and then take their union as the (differential) trace, while the subsequent remainder becomes the (differential) residual. Precisely, with respect to the ideal (2), if the layer corresponding to $\mathfrak{n}^p$ is taken at that point to be its contribution to the (differential) trace, then the (differential) residual at that point is the subscheme of the plane defined by the ideal

$$\mathfrak{n}^r \oplus \mathfrak{n}^{r-1}y \oplus \cdots \oplus \mathfrak{n}^{p+1}y^{r-p-1} \oplus \mathfrak{n}^{p-1}y^{r-p} \oplus \cdots \oplus \mathfrak{n}y^{r-2} \oplus (y^{r-1}).$$

obtained by *slicing off* the corresponding layer. If the cohomology vanishes as before when the trace and residual are replaced by the chosen differential versions, then the lemma says

that the cohomology vanishes for $\mathcal{I}_G \otimes \mathcal{L}$. The conclusion now concerns the general union G and not the specialisation W .

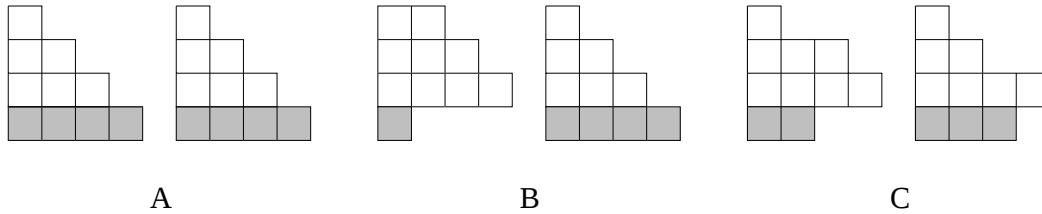


Figure 1. illustrates an example where X is the affine plane and H is a line, while the critical degree is supposed to be five. Example A shows two points of multiplicity four in the plane. The shaded region represents the trace while the unshaded region represents the residual with respect to the line H . From the standard specialisation point of view, these points are translated, one by one, to H giving a trace of degree four, then eight, so that five is unattainable. The examples B and C show two possibilities for choosing the differential traces (shaded part) so that the critical degree is obtained. The differential residuals correspond to the unshaded part.

2.2 The simplified lemma

The simplification of the following lemma with respect to that in §9, is as follows : in the process just described, instead of choosing arbitrary slices, we will systematically take the smallest non-trivial one, which is just a simple point of H . In this case, the (differential) residual falls within the bounds of the following definition.

Definition. 2.2 *Let H be a reduced Cartier divisor on X and let z be a non-singular point of H . We define the m^{th} simple residue; denoted $D_{H,m}(z)$ or $D_m(z)$ if no confusion can arise; to be the trace of z^m on $(m-1)H$;*

$$D_m(z) = z^m \cap H^{m-1}.$$

We will say that m is the multiplicity of the simple residue.

With this definition, our simplified lemma, which will be proved in §9, reads as follows.

Lemma. 2.3 *Suppose X is projective and furnished with a line bundle $\mathcal{L}$, and let H be a reduced and irreducible effective Cartier divisor on X . Let Z_0 be a zero-dimensional subscheme of X , and let a, d be positive integers. We suppose that*

$$r = h^0(H, \mathcal{L}|_H) - \deg(\text{Tr}_H(Z_0)) \geq 0$$

and that $m_1, \dots, m_r$ are positive integers satisfying

$$\deg(Z_0) + \sum_{i=1}^r \binom{m_i + n}{n+1} \geq h^0(X, \mathcal{L})$$

Let $P_1, \dots, P_r$ be generic points in X and $Q_1, \dots, Q_r$ be generic points in H . In the notation of 2.1 and 2.2, set

$$T = Z_0 \cup P_1^{m_1} \cup \dots \cup P_r^{m_r} \quad ; \quad T'_\star = Z'_0 \cup D_{m_1}(Q_1) \cup \dots \cup D_{m_r}(Q_r) \quad ; \quad T''_\star = Z''_0 \cup Q_1 \cup \dots \cup Q_r$$

Then $H^0(X, \mathcal{I}_T \otimes \mathcal{L}) = 0$ holds as soon as the following two conditions are satisfied :

$$\begin{aligned} (\text{dime}) \quad & H^0(H, \mathcal{I}_{T''_\star} \otimes \mathcal{L}|_H) = 0 \\ (\text{degue}) \quad & H^0(X, \mathcal{I}_{T'_\star} \otimes \mathcal{L}(-H)) = 0. \end{aligned}$$

Remark. 2.4 The dime and degue concern respectively the differential trace and the differential residual as discussed above.

3 Configurations and candidates

Here we introduce the general class of subschemes of X which we will be dealing with. From here on, X is projective of dimension $n + 1$ and furnished with an ample line bundle $\mathcal{O}(1)$ of degree ν . We let α_0 be the least integer such that $\mathcal{O}(a)$ is very ample for $a \geq \alpha_0$ and it will henceforth be understood that $a \geq \alpha_0$.

Definition. 3.1 Let G_a be the generic effective divisor in the linear system $|H^0(X, \mathcal{O}(a))|$. A G_a -residue or just **residue**, will be any point or simple residue (see 2.2) with support in G_a . The **multiplicity** of a residue will be its multiplicity as a point, or as a simple residue 2.2, respectively.

Given positive integers a, m an (a, m) -**configuration** will be any subscheme Z of X which is a generic union of points of multiplicity at most m in X , called the free part of Z and denoted $\text{Free}(Z)$, with a generic set of G_a -residues equally of multiplicity at most m , called the constrained part of Z and denoted $\text{Const}(Z)$.

Given a positive integer d , we say that an (a, m) -configuration Z is a (d, m, a) -**candidate** if the following two conditions hold:

$$h^0(X, \mathcal{O}(d)) \leq \deg(Z)$$

and

$$\deg(\text{Tr}_{G_a}(Z)) \leq h^0(G_a, \mathcal{O}_{G_a}(d)).$$

We consider a (d, m, a) -candidate Z to be a candidate for the property $h^0(X, \mathcal{I}_Z(d)) = 0$ and we say that Z is **winning** if this property holds.

The bound for $\deg(Z)$ in the definition of candidates is for convenience: a vanishing statement for a more general configuration will be treated by considering the candidate obtained by adding the right number of simple points.

The following easy lemma says that for large d , candidates contain sufficiently many free points.

Lemma. 3.2 *Let m and a be positive integers. For any (d, m, a) -candidate Z , we have*

$$\deg(\text{Free}(Z)) \geq \nu \frac{d^{m+1}}{(n+1)!} - O(d^n),$$

where ν is the degree of X .

For presentation purposes we introduce the

Definition. 3.3 *Given a polarised pair $(V, \mathcal{O}(1))$ and $m > 0$ we define $\mathbf{d}(V, m)$ to be the least degree (a-priori possibly infinite, and a-posteriori finite by our theorem) such that for $d \geq \mathbf{d}(V, m)$ any $(d, m, 0)$ -candidate is winning.*

4 Derivatives

In practice, when we apply lemma 2.3, we think of the condition (dime) as being satisfied. This is easily justified by an induction hypothesis on the dimension (i.e. precisely that $d(G_a, m)$ is finite). Lemma 2.3 is then a justification for replacing T by T'_* .

This leads us to introduce a formal operator $\mathfrak{D}$ sending one (a, m) -configuration to another which we call the derivative (see 4.1). Of course, we are especially interested in the case where this operator takes (d, m, a) -candidates to $(d - a, m, a)$ -candidates. In the present section, we define the derivative and show that it behaves well for large d .

Here is the idea behind the definition of a derivative. Given a (d, m, a) -candidate Z , we wish to apply our lemma 2.3 as follows. We specialize the s biggest free points of Z onto the divisor G_a , with s as large as possible. Still a few conditions (say r) are missing in G_a , and we require that r further free points be available in Z so that we may apply 2.3. In that case, the derivative of Z is the subscheme T'_* involved in the degue condition of 2.3.

Definition. 4.1 *Let Z be a (d, m, a) -candidate on X with $t = t(Z)$ free points $P_1^{m_1}, \dots, P_t^{m_t}$, where the multiplicities appear in non-decreasing order. Let $s = s(Z) \leq t$ be the greatest integer such that*

$$\deg(\text{Tr}_{G_a}(Z)) + \binom{m_1 + n - 1}{n} + \dots + \binom{m_s + n - 1}{n} \leq h^0(G_a, \mathcal{O}_{G_a}(d)).$$

and set

$$r = r(Z) = h^0(G_a, \mathcal{O}_{G_a}(d)) - \deg(\text{Tr}_{G_a}(Z)) - \binom{m_1 + n - 1}{n} - \dots - \binom{m_s + n - 1}{n}.$$

We say that Z is **derivable** with respect to G_a if

$$r + s \leq t.$$

If Z is a derivable (d, m, a) -candidate, its **derivative** with respect to G_a , denoted $\mathfrak{D}^{(1)}(Z)$, is defined to be the (a, m) -configuration

$$\begin{aligned}\mathfrak{D}^{(1)}(Z) &= P_{s+r+1}^{m_{s+r+1}} \cup \dots \cup P_t^{m_t} \cup \text{Const}(Z)' \cup \\ &\quad Q_1^{m_1-1} \cup \dots \cup Q_s^{m_s-1} \cup \\ &\quad D_{m_{s+1}}(Q_{s+1}) \cup \dots \cup D_{m_{s+r}}(Q_{s+r})\end{aligned}$$

where $Q_1, \dots, Q_{s+r}$ are generic points of G_a and the notation is that of 2.1 and 2.2.

Recall that α_0 is an integer such that $\mathcal{O}(a)$ is very ample for $a \geq \alpha_0$. What we need to know about the derivative is the following :

Lemma. 4.2 *Let $a \geq \alpha_0$ and m be positive integers. Then there exists an integer $\mathbf{der}(a, m)$ such that for any $d \geq \mathbf{der}(a, m)$ and any (d, m, a) -candidate Z on X :*

1. Z admits a derivative $\mathfrak{D}^{(1)}(Z)$;
2. for any N , if Z has either no free point of multiplicity m or at least N free points of multiplicity less than m , then so does $\mathfrak{D}^{(1)}(Z)$;
3. The degree of the trace of $\mathfrak{D}^{(1)}(Z)$ satisfies the following estimate, where, as above, $\nu = \deg(\mathcal{O}(1))$:

$$\begin{aligned}\deg \text{Tr}_{G_a}(\mathfrak{D}^{(1)}(Z)) &= \left(\frac{(m-1)a\nu}{m+n-1} \right) \frac{d^n}{n!} + O(d^{n-1}) \\ &= h^0(G_a, \mathcal{O}_{G_a}(d-a)) - \left(\frac{na\nu}{m+n-1} \right) \frac{d^n}{n!} + O(d^{n-1});\end{aligned}$$
4. $\mathfrak{D}^{(1)}(Z)$ is a $(d-a, m, a)$ -candidate;
5. if $d(G_a, m)$ is finite and $\mathfrak{D}^{(1)}(Z)$ is winning, then so is Z .

Proof.

In order to prove 1., it is enough to prove that the number of free points in Z is larger than $2h^0(G_a, \mathcal{O}_{G_a}(d))$. The latter is bounded by Cd^n for some constant C , so we may conclude by 3.2.

As for 2., it is an immediate consequence of the definition of the derivative.

For 3., let r, s, t and m_i be as in 4.1. Then

$$\frac{m+n-1}{n} \left(\sum_{i=1}^s \binom{m_i + n - 2}{n-1} \right) \geq \sum_{i=1}^s \binom{m_i + n - 1}{n} = h^0(G_a, \mathcal{O}_{G_a}(d)) - r$$

and

$$\begin{aligned}
\deg \operatorname{Tr}_{G_a} (\mathfrak{D}^{(1)}(Z)) &\leq \sum_{i=1}^s \binom{m_i + n - 2}{n} + r \binom{m + n - 1}{n} \\
&= \left(\sum_{i=1}^s \binom{m_i + n - 1}{n} + r \right) - \sum_{i=1}^s \binom{m_i + n - 2}{n - 1} \\
&\quad - r + r \binom{m + n - 1}{n} \\
&\leq h^0(G_a, \mathcal{O}_{G_a}(d)) - \frac{m-1}{m+n-1} \left(h^0(G_a, \mathcal{O}_{G_a}(d)) - r \right) \\
&\quad - r + r \binom{m + n - 1}{n} \\
&\leq \frac{n}{m-1} (h^0(G_a, \mathcal{O}_{G_a}(d))) + \binom{m + n - 1}{n}^2 \\
&= \left(\frac{(m-1)a\nu}{m+n-1} \right) \frac{d^n}{n!} + O(d^{n-1}).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
h^0(G_a, \mathcal{O}_{G_a}(d-a)) - \deg \operatorname{Tr}_{G_a} (\mathfrak{D}^{(1)}(Z)) &\geq h^0(G_a, \mathcal{O}_{G_a}(d)) - \frac{m-1}{m+n-1} h^0(G_a, \mathcal{O}_{G_a}(d)) \\
&\quad - \binom{m + n - 1}{n}^2 \\
&= \left(\frac{na\nu}{m+n-1} \right) \frac{d^n}{n!} + O(d^{n-1}).
\end{aligned}$$

For 4., we first note that, when $\mathfrak{D}^{(1)}(Z)$ is defined and $H^1(X, \mathcal{O}(d-a)) = 0$, one has

$$h^0(X, \mathcal{O}(d-a)) \leq \deg (\mathfrak{D}^{(1)}(Z)) = \deg(Z) - h^0(G_a, \mathcal{O}_{G_a}(d)).$$

This means that, for sufficiently large d , the (a, m) -configuration $\mathfrak{D}^{(1)}(Z)$ is a $(d-a, m, a)$ -candidate, since by 3., its trace on G_a has degree at most $h^0(G_a, \mathcal{O}_{G_a}(d-a))$.

For 5., using the notation of 4.1, we apply 2.3, with Z_0 the closed subscheme

$$\operatorname{Const}(Z) \cup Q_1^{m_1} \cup \dots \cup Q_s^{m_s} \cup P_{s+r+1}^{m_{s+r+1}} \cup \dots \cup P_t^{m_t}.$$

Let $W = Q_1^{m_{s+1}} \cup \dots \cup Q_r^{m_{s+r}}$. The dime of 2.3 holds for $d \geq d(G_a, m)$, while the degree of 2.3 is just the hypothesis that $\mathfrak{D}^{(1)}(Z)$ is winning, so the lemma follows from 2.3. $\square$

5 Concentrated derivatives

If theorem 1.1 were true for low degrees, then repeated applications of lemma 2.3, hence of the derivative, would suffice to prove the theorem by induction on the degree. Instead one must modify the process and try to reduce the multiplicities of the free points, thus ending the proof by induction on the multiplicity. This is done using a specialisation of the second

derivative (see 6.1): bearing in mind the semi-continuity of the cohomology, one easily sees that the (degue) of 2.3 holds if it holds for some specialisation of T'_* . A complication arises with the degree of the base divisor G_a which must be lowered during the induction on d before an induction hypothesis on m allows one to finish the proof. We get around this problem using a specialisation of the first derivative which we call a concentrated derivative. In this section we introduce this concentrated derivative and prove results analogous to those for derivatives.

Definition. 5.1 *Let d, m, a be positive integers with $a > 1$, and let Z be a derivable (d, m, a) -candidate. We define the **concentrated derivative** of Z with respect to G_a , denoted $\mathfrak{D}_c^{(1)}(Z)$, to be the $(a-1, m)$ -configuration obtained from $\mathfrak{D}^{(1)}(Z)$ by degenerating G_a to the generic union $G_1 + G_{a-1}$ and specialising all G_a -residues of $\mathfrak{D}^{(1)}(Z)$ to have generic support in G_{a-1} .*

What we need to know about the concentrated derivative is concentrated in the following:

Lemma. 5.2 *Given $m > 0$ there exists an integer $A(m)$ such that for all $a \geq A(m)$ there exists an integer $\mathbf{derc}(a, m)$ such that for any $d \geq \mathbf{derc}(a, m)$ and any (d, m, a) -candidate Z :*

1. Z admits a concentrated derivative $\mathfrak{D}_c^{(1)}(Z)$ which is a $(d-a, m, a-1)$ -candidate;
2. for any N , if Z has either no free point of multiplicity m or at least N free points of multiplicity less than m , then so does $\mathfrak{D}_c^{(1)}(Z)$;
3. if $d(G_a, m)$ is finite and $\mathfrak{D}_c^{(1)}(Z)$ is winning, then so is Z .

Proof.

For 1., let $A(m)$ be an integer a satisfying

$$\frac{m-1}{m+n-1}A(m) < A(m) - 1.$$

Then, for $a \geq A(m)$ and d sufficiently large, for any (d, m, a) -candidate Z , we have, by (4.2.3),

$$\deg \mathrm{Tr}_{G_{a-1}}(\mathfrak{D}_c^{(1)}(Z)) = \deg \mathrm{Tr}_{G_a}(\mathfrak{D}^{(1)}(Z)) \leq h^0(G_{a-1}, \mathcal{O}_{G_{a-1}}(d-a))$$

so that $\mathfrak{D}_c^{(1)}(Z)$ is a $(d-a, m, a-1)$ -candidate.

As for 2., it follows from the similar statement for the derivative, since the derivative and the concentrated derivative have the same free points.

For 3., if $\mathfrak{D}_c^{(1)}(Z)$ is winning then so is $\mathfrak{D}^{(1)}(Z)$, since the former is a specialisation of the latter. We conclude that Z is winning for $d \geq \mathbf{der}(a, m)$ by (4.2.5). $\square$

6 Special second derivative

In this section, we explain the construction which generates free points. This corresponds to a modified second derivative, which we denote by $\mathfrak{D}^{(2)}[\alpha]$, where α is an integer.

Definition. 6.1 *Let $m, a > 0$, and let Z be a twice derivable (d, m, a) -candidate. Let $r^{(2)}(Z)$ be the number of residues of $\mathfrak{D}^{(2)}(Z)$ which are points, necessarily of multiplicity at most $m - 1$. For $0 < \alpha < a$, we set*

$$r^{(2)}[\alpha](Z) = \min(h^0(X, \mathcal{O}(\alpha)) - 1, r^{(2)}(Z))$$

and define $\mathfrak{D}^{(2)}[\alpha](Z)$ to be the specialisation of the second derivative $\mathfrak{D}^{(2)}(Z)$ obtained by degenerating G_a and its residues to the generic union $G_\alpha + G_{a-\alpha}$ with $r^{(2)}[\alpha](Z)$ of the residues which are points specialised to have generic support in G_α , and all other residues specialised to $G_{a-\alpha}$.

Here is what we need to know about this construction.

Lemma. 6.2 *Given m, N , there exist integers α and $a_0 > \alpha$ such that for all $a \geq a_0$ there exists $d'_0 = d'_0(m, N, a)$ such that for $d \geq d'_0$ and any (d, m, a) -candidate Z :*

1. *Z is twice derivable and $\mathfrak{D}^{(2)}[\alpha](Z)$ is a $(d - 2a, m, a - \alpha)$ -candidate having either no free point of multiplicity m or at least N free points of multiplicity at most $m - 1$;*
2. *if $d(G_a, m)$ and $d(G_{a-\alpha}, m)$ are finite and $\mathfrak{D}^{(2)}[\alpha](Z)$ is winning, then so is Z .*

Proof. Let $\alpha = \alpha(N)$ be an integer satisfying $h^0(X, \mathcal{O}(\alpha)) > N$. For 1., let $a_0 > \alpha$ be such that for $a - \alpha > \frac{m-1}{m+n-1}a$ for $a \geq a_0$. Then for $a \geq a_0$, and $d \gg 0$, we have

$$\deg \text{Tr}_{G_a}(\mathfrak{D}^{(2)}(Z)) < h^0(G_{a-\alpha}, \mathcal{O}_{G_{a-\alpha}}(d - 2a))$$

because by (4.2.3)

$$\deg \text{Tr}_{G_a}(\mathfrak{D}^{(2)}(Z)) \leq \frac{m-1}{m+n-1}a\nu \frac{d^n}{n!} + O(d^{n-1})$$

while

$$h^0(G_{a-\alpha}, \mathcal{O}_{G_{a-\alpha}}(d - 2a)) \geq (a - \alpha)\nu \frac{d^n}{n!} + O(d^{n-1}).$$

This implies that $\mathfrak{D}^{(2)}[\alpha](Z)$ is a $(d - 2a, m, a - \alpha)$ -candidate.

Now for $d \gg 0$, by (4.2.3), we have

$$h^0(G_a, \mathcal{O}_{G_a}(d - a)) - \deg \text{Tr}_{G_a}(\mathfrak{D}^{(1)}(Z)) \geq N \binom{m+n-1}{n}$$

so that, in the notation of 4.1, $s(\mathfrak{D}^{(1)}(Z)) \geq N$. If $\mathfrak{D}^{(1)}(Z)$ has at least N free points of multiplicity m , then $\mathfrak{D}^{(2)}[\alpha](Z)$ has N free points of multiplicity $m - 1$: indeed, in that case, $r^{(2)}[\alpha](Z)$ is larger than N , and the $r^{(2)}[\alpha](Z)$ points specialised to G_α are without constraint, since any set of N points lie on an effective divisor in the linear system $|H^0(X, \mathcal{O}(\alpha))|$. Otherwise, $\mathfrak{D}^{(2)}[\alpha](Z)$ has no more free points of multiplicity m .

For 2., we observe that the second derivative $\mathfrak{D}^{(2)}(Z)$ is also a winning candidate, and conclude by applying twice (4.2.5). $\square$

7 Proof of the theorem

7.1 A proposition implying the theorem

The following proposition (which we prove below 7.2) sums up the efforts of the previous sections and, as we will now show, easily implies our theorem.

Proposition. 7.1 *Let X be a projective, geometrically reduced and irreducible variety of dimension $n+1$ over a field k of arbitrary characteristic p . If $p > 0$, suppose further that X is smooth in codimension one if $p > 0$. Let $\mathcal{O}(1)$ be an invertible ample bundle on X . Given $m > 0$, there exists $a_0(m)$ such that for any $a \geq a_0(m)$ there exists $d_0(a, m)$ such that for all $d \geq d_0(a, m)$ any (d, m, a) -candidate is winning.*

Proof of 1.1.

We first handle the case where $\mathcal{M} = \mathcal{O}$.

We take $d_0(m) = d_0(a_0(m), m)$ and consider some $d \geq d_0$ and some generic union Z of (fat) points of multiplicity $\leq m$. If the degree of Z is smaller than $h^0(\mathcal{O}(d))$, we reduce to the case with equality by adding generic simple points. Since the trace on G_{a_0} is empty, we may consider Z as a (d, m, a_0) -candidate (3.1), and conclude by 7.1.

As announced, we only gloss over the proof in the case where $\mathcal{M}$ is arbitrary.

Firstly, replacing $\mathcal{M}$ by $\mathcal{M} \otimes \mathcal{L}^b$, we may suppose that $\mathcal{M}$ is effective. Next, we can suppose as above $\deg Z \geq h^0(\mathcal{M} \otimes \mathcal{L}^d)$ and we have to prove that $H^0(X, \mathcal{I}_Z \otimes \mathcal{M} \otimes \mathcal{L}^d) = 0$. The idea of the proof is then to choose a suitable a and to apply 2.3 as in 4.2 using the generic divisor G_a^* in $|H^0(X, \mathcal{M} \otimes \mathcal{L}^a)|$. By induction on the dimension we can suppose that the dime condition holds. To prove the degue condition, we degenerate G_a^* to $M \cap G_a$, where M is in $|H^0(X, \mathcal{M})|$ and G_a is the generic divisor in $|H^0(X, \mathcal{L}^a)|$; specializing all the residues onto G_a . In this way, we get a (a, m) -configuration Z'_c and, if this is a $(d-a, m, a)$ -candidate, we can end with the particular case ($\mathcal{M} = \mathcal{O}$) since such a candidate is winning for sufficiently large d . To see that Z'_c is a $(d-a, m, a)$ -candidate for suitable large a and d , one pursues an argument analogous to 4.2.1 and one shows that for any a , and all d sufficiently large with respect to a , Z has enough free points to make 2.3 applicable. Then, as in 5.2.1, one shows that for sufficiently large a and all d sufficiently large with respect to a , the (a, m) -configuration Z'_c is a $(d-a, m, a)$ -candidate. $\square$

Remark. 7.2 *A further generalisation would be to take a fixed closed (zero-dimensional) subscheme V_0 and its union with points of multiplicity $\leq m$. The union of V_0 with sufficiently many generic simple points has maximal rank in all degrees giving the initial case for an induction on the multiplicity, while the proof of the dimension one case is virtually unchanged.*

7.2 Proof of the proposition

To prove the proposition, we argue by induction on the dimension $n + 1$. Note that in all characteristics, the generic effective divisor in a very ample linear system on a variety X of dimension > 1 is a variety which is smooth outside the singular locus of X (see [L] VII 13). Thanks to the initial cases 7.3 and 7.4 below, we may suppose that the proposition has been proven for multiplicity m in dimension n and for multiplicity $m - 1$ in dimension $n + 1$. This implies that $\mathbf{d}(G_a, m)$ is finite for all $a \geq 1$ and that there exists $a_0(m - 1)$ such that for $a \geq a_0(m - 1)$ there exists $d_0(a, m - 1)$ such that for $d \geq d_0(a, m - 1)$ any $(d, m - 1, a)$ -candidate is winning. We proceed in three steps.

First step. With the notation of 4.2 and 5.2, we define

$$b_0 = \max(A(m), a_0(m - 1)),$$

$$\Delta = \max(\mathbf{der}(b_0, m), d_0(b_0, m - 1) + b_0)$$

and

$$N = h^0(X, \mathcal{O}(\Delta + b_0 - 1)) + \binom{n + m - 1}{n},$$

and prove by induction that, for any $d \geq \Delta$, any (d, m, b_0) -candidate with either no free point of multiplicity m or at least N free points of multiplicity less than m is winning.

We start with the case $\Delta \leq d < \Delta + b_0$, and consider a (d, m, b_0) -candidate Z with either no free point of multiplicity m or at least N free points of multiplicity less than m . If $\deg(Z) \geq h^0(X, \mathcal{O}(d)) + \binom{n + m - 1}{n}$, and Z has a free point of multiplicity m , we may replace Z by the subscheme obtained by diminishing by one the multiplicity of this free point, which still has at least N free points of multiplicity less than m . In other words, we may suppose either that Z has no free point of multiplicity m , or that $\deg(Z) < h^0(X, \mathcal{O}(d)) + \binom{n + m - 1}{n}$ holds. In the latter case, there is no room for N free points of multiplicity less than m . Summing up, we can suppose that Z has no free point of multiplicity m . Thanks to $d \geq \mathbf{der}(b_0, m)$ and 4.2, Z has a first derivative $\mathfrak{D}^{(1)}(Z)$ which is a $(d - b_0, m - 1, b_0)$ -candidate. Thanks to $d - b_0 \geq d_0(b_0, m - 1)$, this candidate is winning. Thanks to $d \geq \mathbf{der}(b_0, m)$ and 4.2 again, Z is winning too.

For $d \geq \Delta + b_0$ let Z be a (d, m, b_0) -candidate having either no free point of multiplicity m , or at least N free points of multiplicity at most $m - 1$. Thanks to $d \geq \mathbf{der}(b_0, m)$ and 4.2, Z has a first derivative $\mathfrak{D}^{(1)}(Z)$ which is a $(d - b_0, m, b_0)$ -candidate having either no free point of multiplicity m or at least N free points of multiplicity at most $m - 1$. Thanks to the inductive assumption, $\mathfrak{D}^{(1)}(Z)$ is winning. Again thanks to $d \geq \mathbf{der}(b_0, m)$ and 4.2, Z is winning too.

Second step. Here we prove that for any $b \geq b_0$ there exists $\delta = \delta(b, m)$ such that for $d \geq \delta$ any (d, m, b) -candidate having either no free point of multiplicity m , or at least N free points of multiplicity at most $m - 1$ is winning.

The proof is by induction on b . The initial case $b = b_0$ is the previous step. For the induction step, we take $\delta(b) = \max(\mathbf{d}\mathbf{e}\mathbf{r}\mathbf{c}(b, m), \delta(b - 1) + b)$. The statement then follows by 5.2, which applies because $b_0 \geq A(m)$.

Final step. Here we set $a_0 = a_0(m) = \max(b_0 + \alpha(N), a_0(m, N))$ where $\alpha = \alpha(N)$ and $a_0(m, N)$ are defined in 6.2, and, for $a \geq a_0$, $d_0 = d_0(a, m) = \max(d'_0(m, N, a), \delta(a - \alpha, m) + 2a)$, and we prove the full statement, namely that, for $d \geq d_0(a, m)$, any (d, m, a) -candidate Z is winning.

Indeed, by 6.2 applied to n, N, Z is twice derivable and $\mathfrak{D}^{(2)}[\alpha](Z)$ is a $(d - 2a, m, a - \alpha)$ -candidate having either no free point of multiplicity m or at least N free points of multiplicity at most $m - 1$. Since $d - 2a \geq \delta(a - \alpha, m)$, this candidate is winning by the second step. This implies that Z itself is winning by 6.2. $\square$

7.3 The proposition in dimension one

The initial case $n = 0$ can be deduced from the following general results for curves. We first treat the characteristic zero case with the

Proposition. 7.3 *Let C be a geometrically irreducible quasi-projective curve over a field k of characteristic zero. Let $V \subset H^0(C, \mathcal{L})$ be a linear subspace of finite dimension v of global sections of the invertible sheaf $\mathcal{L}$ on C . Let $x_1, \dots, x_r$ be the generic set of r closed points of C defined over the function field K of $C \times \dots \times C$ (r factors), and let $m_1, \dots, m_r$ be positive integers. Let D be the divisor $m_1 x_1 + \dots + m_r x_r$ on $C_K = C \times_k K$. Then the canonical map*

$$V \longrightarrow H^0(C_K, \mathcal{O}_D \otimes \mathcal{L})$$

has maximal rank.

Proof. If $v \neq m = \sum_i m_i$, one can either diminish the multiplicities or add (generic) free points and suppose that $v = m$. Since the property is open, we can specialise to the case of a single point x and the divisor $D = mx$. In this case the proposition is equivalent to showing that the determinant of the canonical map

$$(3) \quad V \otimes \mathcal{O}_C \longrightarrow P^v(\mathcal{L})$$

is not identically zero, where $P^v(\mathcal{L})$ is the sheaf of v^{th} order principal parts of $\mathcal{L}$. For this we can suppose that the base field is algebraically closed and, since this map commutes with localisation and the completion at a closed point of C , it is sufficient to show that the canonical map

$$V \otimes k[[t]] \longrightarrow k[[t, x]] / ((x - t)^v)$$

$$f \mapsto f(t) + f'(t)(x - t) + f''(t) \frac{(x - t)^2}{2!} + \dots + f^{(v-1)}(t) \frac{(x - t)^{v-1}}{(v-1)!}$$

has maximal rank. Choosing a basis $f_1, \dots, f_v$ for V , the determinant of this map is just the Wronskian

$$W(f_1, \dots, f_v) = \det \left[\frac{\partial^i f_j}{\partial t^i} \right]$$

which, as is well known, has maximal rank for $f_1, \dots, f_v$ linearly independent. $\square$

We now give the initial case for smooth curves in arbitrary characteristic.

Proposition. 7.4 *Let C be a smooth, geometrically connected, projective curve of genus g over an arbitrary field. Let $\mathcal{M}, \mathcal{L}$ be line bundles on C with $\mathcal{L}$ ample, let $m > 0$ be an integer and let $d_0(m)$ be the least integer d such that $h^0(C, \mathcal{M} \otimes \mathcal{L}^d) > m(m-1)(g-1)/2$ and $\mathcal{M} \otimes \mathcal{L}^d$ is non-special. Let $x_1, \dots, x_r$ be generic points on C and let Z be the divisor $m_1x_1 + \dots + m_rx_r$ where $0 < m_i \leq m$ for $i = 1, \dots, r$. Then the canonical map*

$$H^0(C, \mathcal{M} \otimes \mathcal{L}^d) \longrightarrow H^0(C, \mathcal{O}_Z \otimes \mathcal{M} \otimes \mathcal{L}^d)$$

has maximal rank for $d \geq d_0(m)$.

Proof. Adding points if necessary, we can suppose $\deg(Z) \geq h^0(C, \mathcal{M} \otimes \mathcal{L}^d)$. By hypothesis we then have $\deg Z > m(m-1)(g-1)/2$ so that some set of g points amongst the x_i have the same multiplicity m_0 . Renumbering, we can write $m_1x_1 + \dots + m_rx_r = m_0(y_1 + \dots + y_g) + D = Z + D$, where D has support away from the y_i . Since the natural map $C^g \rightarrow \text{Pic}^g(C)$ and the power map $\text{Pic}^g(C) \rightarrow \text{Pic}^{m_0g}(C)$ are surjective, it follows that for $y_1, \dots, y_g$ generic, the sheaf $\mathcal{O}(Z)$ and hence $\mathcal{L}^d \otimes \mathcal{M} \otimes \mathcal{O}(-Z)$ is the generic sheaf in its component of the Picard scheme so that either $h^0(C, \mathcal{L}^d \otimes \mathcal{M} \otimes \mathcal{O}(-Z)) = 0$ or $h^1(C, \mathcal{L}^d \otimes \mathcal{M} \otimes \mathcal{O}(-Z)) = 0$. $\square$

This completes the proof of the cases in dimension 1. We end with the following example showing that the ‘smooth in codimension one’ hypothesis cannot be dropped in characteristic $p > 0$.

Remark. 7.5 *Let p be an odd prime and C the plane curve defined by the equation $y^2 - x^p = 0$ over an algebraically closed field of characteristic p . The tangent line at $z = (t^2, t^p)$, $t \neq 0$, is given by $y = t^p$ and has a contact of order p with C at z . It follows that for any choice $z_1, \dots, z_d$ of points on the smooth locus of C , the divisor $Z = pz_1 + \dots + pz_d$ is an effective divisor associated to $\mathcal{O}_C(d)$, whereas $h^0(C, \mathcal{O}_C(d)) = dp + 1 - (p-1)(p-2)/2 \leq dp$ for $d \geq p-2$.*

8 The formal lemma

In this section, we prove the formal part of our differential lemma, the rest of the proof being in the next section. We would like to point out that the original motivation and proof of the following results owed much to the work [LL1,2].

8.1 Preliminaries

Consider the algebra of formal functions $k[[\mathbf{x}, y]]$, where $\mathbf{x} = (x_1, \dots, x_{n-1})$, which we furnish with an ideal I of the form

$$I = I_0 \oplus I_1y \oplus \dots \oplus I_{m-1}y^{m-1} \oplus (y^m)$$

where, for $\alpha = 0, \dots, m-1$, $I_\alpha \subset k[[\mathbf{x}]]$ is an ideal. We call such ideals *vertically graded ideals*. Note that

$$(4) \quad I_0 \subset I_1 \subset \dots \subset I_{m-1}$$

An ideal

$$I_t = I_0[[t]] \oplus I_1[[t]](y - t^r) \oplus \dots \oplus I_{m-1}[[t]](y - t^r)^{m-1} \oplus ((y - t^r)^m)$$

in the algebra $k[[t, \mathbf{x}, y]]$ is called a *standard deformation* of the vertically graded ideal I . For $i \geq m$ we let $I_i = k[[\mathbf{x}]]$.

Given a function $F_0 + F_1 t + \dots$ in I_t , the functions $F_i(\mathbf{x}, y)$ must satisfy certain residual conditions. If $r = 1$ and $I = (\mathbf{x}, y)^m$, the residual condition is just that $F_i(\mathbf{x}, y)$ must vanish to the order $m - i$, and can be compared with [Xu]. This is the sense of the following statement.

Proposition. 8.1 *Let $F = \sum_{\alpha \geq 0} F_\alpha(\mathbf{x}, y)t^\alpha = \sum_{\alpha, \beta \geq 0} F_{\alpha, \beta}(\mathbf{x})t^\alpha y^\beta$ be a function in I_t . Then*

$$F_{\alpha, \beta}(\mathbf{x}) \in I_{\beta + \lfloor \frac{\alpha}{r} \rfloor}.$$

If y divides F_α for $\alpha = 0, r, 2r, \dots, pr$ then $F_0(\mathbf{x}, y)$ is in the ideal

$$I_0 y \oplus I_1 y^2 \oplus \dots \oplus I_{p-1} y^p \oplus I_{p+1} y^{p+1} \oplus \dots \oplus I_{m-1} y^{m-1} \oplus ((y^m)).$$

Proof. Write F in the following form

$$F = a_0(\mathbf{x}, t) + a_1(\mathbf{x}, t)(y - t^r) + \dots + a_{m-1}(\mathbf{x}, t)(y - t^r)^{m-1} + a_m(\mathbf{x}, t)(y - t^r)^m + \dots$$

with

$$a_i(\mathbf{x}, t) = \sum_{j \geq 0} a_{ij}(\mathbf{x}) t^j$$

hence $a_{ij}(\mathbf{x}) \in I_i$. Developping out we find

$$\begin{aligned} F_{\alpha, \beta} &= \sum_{\nu=0}^{\lfloor \frac{\alpha}{r} \rfloor} (-1)^\nu \binom{\beta + \nu}{\beta} a_{\beta + \nu, \alpha - \nu r}(\mathbf{x}) \\ &\in I_{\beta + \lfloor \frac{\alpha}{r} \rfloor} \end{aligned}$$

where $\lfloor z \rfloor$ is the greatest integer part of z . This proves the first part.

Now suppose that y divides F_α for $\alpha = \lambda r$ and $\lambda = 0, 1, \dots, p$. Then we have

$$0 = F_{\lambda r, 0} = a_{0, \lambda r} - a_{1, (\lambda-1)r} + \dots + (-1)^{\lambda-1} a_{\lambda-1, r} + (-1)^\lambda a_{\lambda, 0}$$

so that $a_{0,0} = 0$ and $a_{\lambda,0} \in I_{\lambda-1}$ for $\lambda = 1, \dots, p$ as one sees using $a_{\mu, \nu} \in I_\mu$ and (4). This gives the last part of the proposition. $\square$

8.2 The formal lemma

Throughout this subsection we will use the following notation.

For $i = 1, \dots, \ell$, let $B^{(i)} = k[[\mathbf{x}_i, y_i]]$ be an algebra of formal functions in n variables where $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,n-1})$ and let

$$I^{(i)} = I_0^{(i)} \oplus I_1^{(i)} y_i \oplus \dots \oplus I_{m_i-1}^{(i)} y_i^{m_i-1} \oplus (y_i^{m_i})$$

be a vertically graded ideal in $B^{(i)}$. Let

$$I = I^{(1)} \times \dots \times I^{(\ell)} \subset B^{(1)} \times \dots \times B^{(\ell)} = B.$$

Let $k[[\mathbf{t}]] = k[[t_1, \dots, t_\ell]]$ and let $I_{\mathbf{t}}$ in $B[[\mathbf{t}]]$ be the product of the ideals

$$I_{\mathbf{t}}^{(i)} = I_0^{(i)}[[\mathbf{t}]] \oplus I_1^{(i)}[[\mathbf{t}]](y_i - t_i) \oplus \dots \oplus I_{m_i-1}^{(i)}[[\mathbf{t}]](y_i - t_i)^{m_i-1} \oplus ((y_i - t_i)^{m_i}).$$

Let $y = (y_1, \dots, y_\ell)$ and for any linear subspace $V \subset B$, let $V_{\text{res}(y)} = \{v \in B \mid vy \in V\}$. Since y is not a zero-divisor, we get a residual exact sequence

$$(5) \quad 0 \longrightarrow V_{\text{res}(y)} \xrightarrow{y} V \longrightarrow V/V \cap (y) \longrightarrow 0$$

Proposition. 8.2 *Let $V \subset B$ be a k -linear subspace. Suppose that for $i = 1, \dots, \ell$ there exist nonnegative integers p_i such that the following two conditions are satisfied*

1. *the canonical map*

$$V/V \cap (y) \longrightarrow k[[\mathbf{x}_1]]/I_{p_1}^{(1)} \times \dots \times k[[\mathbf{x}_\ell]]/I_{p_\ell}^{(\ell)}$$

is injective

2. *The canonical map*

$$V_{\text{res}(y)} \longrightarrow B/J$$

is injective where $J = J^{(1)} \times \dots \times J^{(\ell)}$ and

$$J^{(i)} = I_0^{(i)} \oplus I_1^{(i)} y_i \oplus \dots \oplus I_{p_i-1}^{(i)} y_i^{p_i-1} \oplus I_{p_i+1}^{(i)} y_i^{p_i} \oplus \dots \oplus I_{m_i-1}^{(i)} y_i^{m_i-2} \oplus (y_i^{m_i-1})$$

Then the canonical map

$$\varphi_{\mathbf{t}} : V \otimes k[[\mathbf{t}]] \longrightarrow B_{\mathbf{t}}/I_{\mathbf{t}}$$

is (generically) injective.

Proof. We first reduce to the case where the p_i are positive.

Let us suppose for simplicity that $p_1, \dots, p_s$ are positive and $p_{s+1}, \dots, p_\ell$ are all zero. We denote by V_0 the subspace of V formed by the elements vanishing in each of the $k[[\mathbf{x}_i, y_i]]/I^{(i)}$

for $i = s + 1, \dots, \ell$. Conditions 1. and 2. of the proposition imply the corresponding conditions for V_0 when only the first s factors on the right hand side are present.

If we write $\mathbf{t}'$ for $(t_1, \dots, t_s)$, the conclusion of the proposition in the case where all p_i are positive then implies that $V_0 \otimes k[\mathbf{t}']$ injects into $B_{\mathbf{t}'} / I_{\mathbf{t}'}$. Since $\varphi_{\mathbf{t}}$ is a map of free $k[[\mathbf{t}]]$ -modules, it is enough to prove that its restriction $\varphi_{\mathbf{t}'}$ over $\text{Spec } k[[\mathbf{t}']]$ is injective. We write

$$\varphi_{\mathbf{t}'} = (\varphi'_{\mathbf{t}'}, \varphi''_{\mathbf{t}'}) : V \otimes k[\mathbf{t}'] \rightarrow B_{\mathbf{t}'} / I_{\mathbf{t}'} \times R$$

with

$$R = (B^{(s+1)} / I^{(s+1)} \times \dots \times B^{(\ell)} / I^{(\ell)}) \otimes k[[\mathbf{t}']]$$

The kernel of $\varphi''_{\mathbf{t}'}$ is $V_0 \otimes k[\mathbf{t}']$ and the restriction of $\varphi'_{\mathbf{t}'}$ to this kernel is injective, thus so is $\varphi_{\mathbf{t}'}$.

Henceforth we suppose that the p_i are positive and we let

$$h = \text{lcm}(p_1, \dots, p_\ell) = r_i p_i$$

be the least common multiple of the p_i and consider the one-parameter deformation obtained by setting $t_i = t^{r_i}$. Since the rank of $\varphi_{\mathbf{t}}$ is semi-continuous, we need only show that the canonical map

$$\varphi_t : V \otimes k[[t]] \longrightarrow B[[t]] / I_t$$

obtained by the formal base change $k[[t_1, \dots, t_\ell]] \longrightarrow k[[t]]$; $t_i \mapsto t^{r_i}$; is injective.

Let

$$F_t = (F_t^{(1)}, \dots, F_t^{(\ell)}) \in \ker \varphi_t = V_t \cap I_t,$$

where V_t is the image of $V \otimes k[[t]]$ and I_t is the image of $I_{\mathbf{t}}$ in $B[[t]]$.

In case $F_0 = 0$, we may replace F_t by F_t/t since $B[[t]]/I_t$ is a torsion free $k[[t]]$ -module. Thus we only have to prove $F_0 = 0$.

Since $F_t^{(i)} = \sum_{\alpha \geq 0} F_\alpha^{(i)}(\mathbf{x}_i, y_i) t^\alpha \in I_t^{(i)}$, where $I_t^{(i)}$ is the image of $I_{\mathbf{t}}^{(i)}$ in $B^{(i)}[[t]]$, the first part of proposition 8.1 implies

$$(F_\alpha^{(1)}(\mathbf{x}_1, 0), \dots, F_\alpha^{(\ell)}(\mathbf{x}_\ell, 0)) \in I_{p_1}^{(1)} \times \dots \times I_{p_\ell}^{(\ell)}$$

for $\alpha = 0, 1, \dots, h$. Applying hypothesis 1. of the proposition, we conclude that y divides $(F_\alpha^{(1)}(\mathbf{x}_1, y_1), \dots, F_\alpha^{(\ell)}(\mathbf{x}_\ell, y_\ell))$ for $\alpha = 0, 1, \dots, h$.

Now applying the second part of proposition 8.1 we obtain

$$F_0^{(i)}(\mathbf{x}_i, y_i) = y_i G_0^{(i)}(\mathbf{x}_i, y_i)$$

with $G_0^{(i)}(\mathbf{x}_i, y_i) \in J^{(i)}$. Letting $G_0 = (G_0^{(1)}, \dots, G_0^{(r)})$ we see that G_0 is in $V_{\text{res}(y)} \cap J$, but the second hypothesis of the proposition simply says that $V_{\text{res}(y)} \cap J = 0$, giving $F_0 = 0$ as required. $\square$

9 The differential lemma

Throughout this section, X denotes an irreducible algebraic variety, H a reduced irreducible positive Cartier divisor on X , X^0 a dense open nonsingular subscheme of X such that $H^0 := H \cap X^0$ is the nonsingular locus of H . Finally $\mathcal{I}_H$ denotes the ideal sheaf of H .

For M another k -scheme, we denote by $\text{Hom}(M, X)$ (resp. $\text{Hom}(M, X^0)$) the set of morphisms from M to X (resp. X^0) as well as, in case M is projective, the corresponding Hilbert scheme. If M is algebraic, zero-dimensional and connected, it is easy to check that the natural morphism from $\text{Hom}(M, X^0)$ to X^0 is smooth with smooth irreducible fibers. Thus $\text{Hom}(M, X^0)$ is also irreducible. Its generic point represents an embedding whose image in X we denote by M_X .

Now let M be a subscheme of $\text{Spec } k[[x_1, \dots, x_n]]$. We denote by $\text{Hom}(M, X, H)$ the set (or Hilbert scheme) of morphisms f from M to X^0 such that the ideal $f^*(\mathcal{I}_H)$ is contained in (x_n) . We call these morphisms H -morphisms from M to X , and if a H -morphism is an embedding, we say that it is a H -embedding. If M is algebraic, thus zero-dimensional and connected, it is easy to check that the natural (restriction) morphism from $\text{Hom}(M, X, H)$ to H^0 is smooth. Furthermore, its fiber $\text{Hom}(M, X, H, z)$ over a point z of H^0 is a vector space, thus smooth and irreducible. As a consequence, $\text{Hom}(M, X, H)$ is again irreducible and smooth. Its generic point is a H -embedding whose image in X we denote by $M_{X,H}$.

We say that the subscheme M of $\text{Spec } k[[x_1, \dots, x_n]]$ is a model of dimension n if its ideal is a vertically graded ideal as in §8:

$$I = I_0 \oplus I_1 x_n \oplus \dots \oplus I_m x_n^m \oplus \dots$$

where I_m is a non-decreasing sequence of ideals in $k[[x_1, \dots, x_{n-1}]]$ with $I_m = k[[x_1, \dots, x_{n-1}]]$ for large m .

For M a model of dimension n , we denote by $\text{Tr} M$ its trace on the hyperplane defined by x_n , and by $\text{Res} M$ the corresponding residual scheme, which is again a model of dimension n . We define more generally $\text{Tr}^{(p)} M$ and $\text{Res}^{(p)} M$ for any nonnegative integer p : with the notations introduced above, we set $\text{Tr}^{(p)} M := I_p$, and define $\text{Res}^{(p)} M$ to be the model corresponding to the ideal

$$I_0 \oplus I_1 x_n \oplus \dots \oplus I_{p-1} x_n^{p-1} \oplus I_{p+1} x_n^p \oplus \dots \oplus I_{q+1} x_n^q \oplus \dots$$

If $M_1, \dots, M_\ell$ are models, we say that their disjoint union $\mathbf{M}$ is a multi-model. If $\mathbf{p} = (p_1, \dots, p_\ell)$ is a multi-integer, we define $\text{Tr}^{\mathbf{p}} \mathbf{M}$ to be the disjoint union of the $\text{Tr}^{p_i} M_i$ and $\text{Res}^{\mathbf{p}} \mathbf{M}$ to be the disjoint union of the $\text{Res}^{p_i} M_i$.

The Hilbert scheme $\text{Hom}(\mathbf{M}, X^0)$ is the product $\text{Hom}(M_1, X^0) \times \dots \times \text{Hom}(M_\ell, X^0)$, thus irreducible (and smooth). Its generic point represents an embedding whose image in X we denote by $\mathbf{M}_X$. We denote by $\text{Hom}(\mathbf{M}, X, H)$ the Hilbert scheme of morphisms f from $\mathbf{M}$ to X^0 whose restrictions to the components M_i are H -morphisms. We call these morphisms H -morphisms from $\mathbf{M}$ to X , and we call H -embeddings those H -morphisms which are embeddings. The scheme $\text{Hom}(\mathbf{M}, X, H)$ is the product $\text{Hom}(M_1, X, H) \times \dots \times$

$\text{Hom}(M_\ell, X, H)$, thus irreducible (and smooth). Its generic point is a H -embedding whose image in X we denote by $\mathbf{M}_{X,H}$. We are now ready to state and prove our differential Horace lemma:

Proposition. 9.1 *Let X be, as above, a reduced projective variety of dimension n , furnished with a line bundle $\mathcal{L}$, and H a reduced irreducible positive Cartier divisor on X not contained in the singular locus of X . Let $W \subset X$ be a closed subscheme of X not containing H . We denote by W'' and W' the trace and residual of W with respect to H . Let $\mathbf{M}$ be a multi-model of dimension n . and $\mathbf{p}$ a multi-integer (of the same length). Suppose*

1. **Dime** $H^0(H, \mathcal{I}_{W'' \cup \text{Tr} \mathbf{p} \mathbf{M}_H} \otimes \mathcal{L}|_H) = 0$
2. **Degue** $H^0(X, \mathcal{I}_{W' \cup \text{Res} \mathbf{p} \mathbf{M}_{X,H}} \otimes \mathcal{L}(-H)) = 0.$

Then $H^0(X, \mathcal{I}_{W \cup \mathbf{M}_X} \otimes \mathcal{L}) = 0$.

Proof. We may suppose that k is algebraically closed. By semi-continuity, there exist rational points $z_1, \dots, z_\ell$ in H , and corresponding H -embeddings $e_i : \text{Spec } k[[x_1, \dots, x_n]] \rightarrow X$, with z_i as image of the closed point, allowing us to rewrite our first assumption as follows:

$$\text{Dime} \quad H^0(H, \mathcal{I}_{W'' \cup e_1(\text{Tr}^{p_1} M_1) \cup \dots \cup e_\ell(\text{Tr}^{p_\ell} M_\ell)} \otimes \mathcal{L}|_H) = 0.$$

Similarly, there exist rational points $z'_1, \dots, z'_\ell$ in H , and corresponding H -embeddings $e'_i : \text{Spec } k[[x_1, \dots, x_n]] \rightarrow X$, with z'_i as image of the closed point, allowing us to rewrite our second assumption as follows:

$$\text{Degue} \quad H^0(X, \mathcal{I}_{W' \cup e'_1(\text{Res}^{p_1} M_1) \cup \dots \cup e'_\ell(\text{Res}^{p_\ell} M_\ell)} \otimes \mathcal{L}(-H)) = 0.$$

Since the two conditions do not interfere with one another, we may even suppose $z_i = z'_i$. Let us now show that it's possible to obtain $e_i = e'_i$. Using induction on i , all we need to prove is the following statement: given such a point z , a vertically graded model M in $\text{Spec } k[[x_1, \dots, x_n]]$, an integer p and two non-empty open subschemes S'' in $\text{Hom}(\text{Tr}^p M, H, z)$ and S' in $\text{Hom}(\text{Res}^p M, X, H, z)$, there exists a H -embedding $e : \text{Spec } k[[x_1, \dots, x_n]] \rightarrow \text{Spec } \hat{\mathcal{O}}_{X,z} \rightarrow X$ whose restriction to $\text{Tr}^p M$ is in S'' and whose restriction to $\text{Res}^p M$ is in S' . This statement is easily proven, using the fact that the restriction maps

$$\text{Hom}(M, X, H, z) \rightarrow \text{Hom}(\text{Res}^p M, X, H, z)$$

and

$$\text{Hom}(M, X, H, z) \rightarrow \text{Hom}(\text{Tr } M, H, z) \rightarrow \text{Hom}(\text{Tr}^p M, H, z)$$

are dominant, and the fact that any element of $\text{Hom}(M, X, H, z)$ extends as a H -morphism from $\text{Spec } k[[x_1, \dots, x_n]]$ to X .

The proposition then follows by applying our formal lemma 8.2 with $V = H^0(X, \mathcal{I}_W \otimes \mathcal{L})$.

□

Remark. 9.2 *Our formal lemma can give more accurate information. For instance if X is a projective space and H a hyperplane, we may handle linear embeddings of (multi-)models in a similar way to that used for general embeddings of (multi-)models.*

We now make explicit the particular case of the proposition where the M_i are points (of various multiplicities).

Corollary. 9.3 *Let X be, as above, a reduced projective variety of dimension n , furnished with a line bundle $\mathcal{L}$, and H a reduced irreducible positive Cartier divisor on X not contained in the singular locus of X . Let W be a closed subscheme of X not containing H . Let $P_1, \dots, P_r$ be generic points of X , $Q_1, \dots, Q_r$ generic points in H and $m_1, \dots, m_r$ a sequence of positive integers. Then $H^0(\mathcal{I}_{W \cup P_1^{m_1} \cup \dots \cup P_r^{m_r}} \otimes \mathcal{L}) = 0$ if the following two conditions are satisfied (see 2.2 for the notation D_m):*

1. **Dime** $H^0(X, \mathcal{I}_{W'' \cup Q_1 \cup \dots \cup Q_r} \otimes \mathcal{L}|_H) = 0$
2. **Degree** $H^0(X, \mathcal{I}_{W' \cup D_{m_1}(Q_1) \cup \dots \cup D_{m_r}(Q_r)} \otimes \mathcal{L}(-H)) = 0$

Proof. This is just proposition 9.1 with $p_i = m_i - 1$. □

Remark. 9.4 *The lemma 2.3 is obtained by taking $H = G_a$ in the previous corollary.*

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