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The ideal resolution for generic 3-fat points in $\mathbb{P}^2$

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Abstract

In this paper we prove that the union Y of the second infinitesimal neighbourhoods of n generic points in $\mathbb{P}^2$ is minimally generated for $n \neq 2, 3, 5$, i.e., the maps $\sigma_k : H^0(\mathscr{I}_Y(k)) \otimes H^0(\mathscr{O}_{\mathbb{P}^2}(1)) \to H^0(\mathscr{I}_Y(k+1))$ are of maximal rank. This, together with the maximality of the Hilbert function, gives the graded Betti numbers for the ideal.

MSC: 14F05; 13D02; 14C99

0. Introduction

It is still an open problem to determine the dimension of a vector space I_d of homogeneous polynomials of degree d vanishing with their derivatives of order $< m_i$ at n generic points P_i of a projective space, even in the case of the projective plane $\mathbb{P}^2 = \mathbb{P}^2_\kappa$ (here we always work on an algebraically closed field κ of characteristic 0), or in the so called "homogeneous case" $m_1 = \cdots = m_n$.

In other words, it is still unknown what is the Hilbert function of the scheme defined by the homogeneous ideal $I_Y = \oplus_d I_d$, where $I_y = \mathfrak{p}_1^{m_1} \cap \cdots \cap \mathfrak{p}_n^{m_n}$ and each $\mathfrak{p}_i$ is the ideal of a point $P_i \in \mathbb{P}^2$, even for a generic choice of the points P_i. The scheme Y is usually called a scheme of "fat points" (we will call the scheme associated to $\mathfrak{p}_i^m$ an "m-fat point").

Only partial results are known, e.g. when $n < 10$, (see [17]), when $m_1 = \cdots = m_n \leqslant 20$ (see [7,8]), or when max $\{m_1, \ldots, m_n\} \leqslant 4$ (see [15,21,28]). Moreover, the Hilbert

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function of Y is asymptotically known (see [1]), and it confirms the conjecture stated in [15,17,22]. Results for high values of the m_i's can be found in [10] and [4].

Here we are interested in another problem, on the same line of thought: which are the degrees of a minimal set of generators for the ideal I_Y? Since Y is 0-dimensional of codimension 2, this is the same as asking which are the graded Betti numbers of a minimal free resolution of the ideal I_Y. Such resolutions are known when $n \leqslant 8$, for all m_i's, see [11], and for $n \leqslant 9$ and $m_1 = \cdots = m_n$, see [18]. Other cases are treated in [12].

B. Harbourne has conjectured (in [18]) that for all $n > 9$ and all $m = m_1 = \cdots = m_n$ all the multiplication maps: $\sigma_k : I_k \otimes (\kappa[x_0, x_1, x_2])_1 \to I_{k+1}$ (or, equivalently: $\sigma_k : H^0(\mathscr{I}_Y(k)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathscr{I}_Y(k+1))$, where $\mathscr{I}_Y$ is the ideal sheaf associated to I_Y), have maximal rank, in other words, the degree k part of the ideal generates, by multiplication with linear forms, as much as possible of the degree $k+1$ part. If Y has maximal Hilbert function, this amounts to saying that the ideal I_Y has the same "good resolution" as the ideal of a scheme of $n\binom{m+1}{2}$ generic (reduced) points; it has been conjectured that any set of t generic points in $\mathbb{P}^r$ has a very simple resolution (Minimal Resolution Conjecture, see [27]).

The MRC has been proved for $r = 2, 3, 4$ (any number of points) and for particular values of t, r, e.g. see [2,3,6,13,14,27,30]; or for any r and $t \gg 0$ (see [23]), but the MRC has also been proved to be false in all its generality in [9], where counterexamples were described.

For the case of fat points see also [19].

The second author proved Harbourne's conjecture for $m = 2$ in [26], showing that the only exception to rank maximality of the σ_k's are for $n = 2$, and to the MRC for $n = 2, 5$ (in those cases Y does not have maximal Hilbert function). In this paper we prove the same kind of result for $m = 3$ (see Theorem 1.1 and Remark 1.7); here the exceptions are for $n = 2, 3, 5$: cases 2,5 are as in [26], while the situation for $n = 3$ is different; here Y has maximal Hilbert function, but the presence of a fixed component in the initial degree α of I_Y implies that σ_α is not of maximal rank.

As in [24] and [26], we work in $\mathbb{P}(\Omega)$ (where $\Omega = \Omega_{\mathbb{P}^2}$) instead of working only in $\mathbb{P}^2$; an essential tool to construct the induction procedure which we use in the proof of Theorem 1.1 is "la methode d'Horace différentiel", as developed in [1], which we extend here so that it works also for points in $\mathbb{P}(\Omega)$ which are not generic but every two of them lie on a fiber.

The statement of the theorem, the general setting and the reduction to an induction procedure are explained in Section 1. The extension of "la methode d'Horace différentiel" is in Section 2, while the general induction is in Section 3 and the initial cases are in Sections 4 and 5.

1. The main result

1.0. Notations. Let $P_1, \ldots, P_n$ be generic points of $\mathbb{P}^2$; in the following $Y = Y(n)$ will denote the 0-dimensional scheme with support on the P_i's defined by the ideal $I = \mathfrak{p}_1^3 \cap \cdots \cap \mathfrak{p}_n^3$, where $\mathfrak{p}_i$ is the homogeneous (prime) ideal of the point P_i in $\kappa[x_0, x_1, x_2]$.

In the following we set $\Omega := \Omega_{\mathbb{P}^2}$.

What we want to prove is the following:

1.1. Theorem. *Let $P_1,\dots,P_n$ and $Y=Y(n)$ be as above. Then if $n \neq 2,3,5$ the natural maps*

$$\sigma_k : H^0(\mathscr{I}_Y(k)) \otimes H^0(\mathscr{O}_{\mathbb{P}^2}(1)) \to H^0(\mathscr{I}_Y(k+1))$$

have maximal rank for each k.

The theorem follows by Propositions 1.6 and 3.9 (see also Definition 1.4).

Notice that we are interested in those values of k for which $h^0(\mathscr{I}_Y(k)) > 0$, otherwise σ_k is trivially injective, and we know (see [21]) that for $n \neq 2,5$ the scheme Y has maximal Hilbert function, i.e. that $h^0(\mathscr{I}_Y(k))h^1(\mathscr{I}_Y(k)) = 0$ (in other words, when $h^0(\mathscr{I}_Y(k)) > 0$, Y imposes $6n$ independent conditions to curves of degree k, since each triple point has length 6).

For any projective scheme $Z \subset \mathbb{P}^r$, let $\alpha(Z) = \min\{k \mid h^0(\mathscr{I}_Z(k)) > 0\}$, i.e. $\alpha(Z)$ is the initial degree of I_Z; when Z is 0-dimensional and has maximal Hilbert function, it is well known that I_Z is generated at most in degrees $\alpha(Z), \alpha(Z)+1$ (by Castelnuovo-Mumford lemma, see [29], σ_k is surjective for all $k > \alpha(Z)$, while σ_k is trivially injective (it is the 0-map) for all $k < \alpha(Z)$). Hence there is only one possible "critical" value of k, namely $k = \alpha(Z)$.

Now we introduce the following invariant:

$$v = v(Y(n)) = \min\left\{k \geqslant 1 \mid \frac{(k+1)(k+2)}{2} - 6n \geqslant 0\right\}.$$

in other words, v is the smallest k for which the restriction map:

$$\rho_k : H^0(\mathscr{O}_{\mathbb{P}^2}(k)) \to H^0(\mathscr{O}_{Y(n)}(k))$$

can be surjective; if $n \neq 2,5$, since $Y(n)$ is of maximal rank, v is actually the smallest k for which ρ_k is onto, and $\alpha = v$ except when $H^0(\mathscr{I}_{Y(n)}(v)) = H^1(\mathscr{I}_{Y(n)}(v)) = 0$.

Since $Y(n)$ is of maximal rank, the map $\sigma_k = \sigma_k(Y(n)) : H^0(\mathscr{I}_{Y(n)}(k)) \otimes H^0(\mathscr{O}_{\mathbb{P}^2}(1)) \to H^0(\mathscr{I}_{Y(n)}(k+1))$ is trivially injective for all $k < v(Y(n))$, and σ_k is surjective for all $k \geqslant v(Y(n)) + 1$.

Let us give an outline of the proof of Theorem 1.1 before going into all its details. A natural way to study the maps σ_k is to look at the exact sequence obtained by tensoring the Euler sequence $0 \to \Omega \to \mathscr{O}_{\mathbb{P}^2}(-1) \otimes H^0(\mathscr{O}_{\mathbb{P}^2}(1)) \to \mathscr{O}_{\mathbb{P}^2} \to 0$ by the ideal sheaf $\mathscr{I}_Y(k+1)$, and then taking its cohomology sequence:

$$\begin{aligned} 0 &\to H^0(\Omega(k+1) \otimes \mathscr{I}_Y) \to H^0(\mathscr{I}_Y(k)) \otimes H^0(\mathscr{O}_{\mathbb{P}^2}(1)) \xrightarrow{\sigma_k} H^0(\mathscr{I}_Y(k+1)) \\ &\to H^1(\Omega(k+1) \otimes \mathscr{I}_Y) \to H^1(\mathscr{I}_Y(k)) \otimes H^0(\mathscr{O}_{\mathbb{P}^2}(1)) \end{aligned}$$

where $H^1(\mathscr{I}_Y(k))=0$ when $k=v=v(Y(n))$, since in our assumptions Y has maximal Hilbert function.

If we were able to find for each $Y=Y(n)$ a scheme T such that its ideal sheaf $\mathscr{I}_T$ satisfies:

$$H^0(\Omega(v+1)\otimes\mathscr{I}_T)=H^1(\Omega(v+1)\otimes\mathscr{I}_T)=0$$

and such that either $\mathscr{I}_Y\subset\mathscr{I}_T$ (i.e. $T\subset Y$), or $\mathscr{I}_T\subset\mathscr{I}_Y$ (i.e. $Y\subset T$), we would be done, since the first inclusion would imply $H^0(\Omega(v+1)\otimes\mathscr{I}_Y)=0$ (Y imposes more conditions to the global sections of $\Omega(v+1)$), while the second would imply $H^1(\Omega(v+1)\otimes\mathscr{I}_Y)=0$ (the conditions imposed by Y are independent since those imposed by T are).

Notice that $h^0(\Omega(k+1))=k(k+2)$, so the idea is, for each k, to look for a T imposing exactly $k(k+2)$ independent conditions to the global sections of $\Omega(k+1)$. The first attempt in order to find such a T could be to try with schemes made by generic 3-fat points and one point with a suitable multiple structure contained in a 3-fat point, so that such scheme can either contain or be contained in our Y. Note that, since Ω is a rank 2 locally free sheaf, the degree of T should be $k(k+2)/2$ and each triple point imposes 12 conditions to the global sections of $\Omega(k+1)$. Hence consider:

1.2. Definition. For every $k\geqslant 0$, let $q(k)$ and $r(k)$ be the integers such that:

$$h^0(\Omega(k+1))=k(k+2)=12q(k)+r(k),\quad \text{with } 0\leqslant r(k)\leqslant 11.$$

So the candidate for T should be given by a scheme of $q(k)$ 3-fat points plus a structure of degree $r(k)/2$ on a $(q(k)+1)$th point. Unfortunately, this is impossible for trivial reasons: for k odd, $k(k+2)$ is odd too.

The way around this inconvenience is as follows: instead of working in $\mathbb{P}^2$ with the rank 2 bundle $\Omega(k)$, we shall work in the threefold $X=\mathbb{P}(\Omega)$, with the invertible sheaf $\mathscr{E}_k=\mathscr{O}_X(1)\otimes\pi^*\mathscr{O}_{\mathbb{P}^2}(k)$, where $\pi:X=\mathbb{P}(\Omega)\to\mathbb{P}^2$ is the canonical projection. We define a scheme $T(k)$ in X of type $T(k)=\pi^{-1}(Y(q(k)))\cup T_{r(k)}$, where $Y(q(k))$ is given by $q(k)$ generic 3-fat points in $\mathbb{P}^2$ and $T_{r(k)}$ is a 0-dimensional scheme in X of degree $r(k)$, which has support on $\pi^{-1}(P_{q(k)+1})$, the preimage of another point in $\mathbb{P}^2$.

Now let us get to the tools in order to prove our result. If we set $k=6t+\tau$, $0\leqslant\tau\leqslant 5$, and consider $q(k)$ and $r=r(k)$ as in 1.2, we get Table 1.

Table 1

k	$q(k)$	$r(k)$
$6t$	$3t^2+t$	0
$6t+1$	$3t^2+2t$	3
$6t+2$	$3t^2+3t$	8
$6t+3$	$3t^2+4t+1$	3
$6t+4$	$3t^2+5t+2$	0
$6t+5$	$3t^2+6t+2$	11

Hence the values that we have to consider for $r(k)$ are only 0,3,8,11. Now let us define the schemes $T(k) \subset X = \mathbb{P}(\Omega)$ we were talking about:

1.3. Definition. We will set:

$$T(k) = \pi^{-1}(Y(q(k))) \cup T_{r(k)},$$

where $P_1, \ldots, P_{q(k)}$, $P_{q(k)+1}$ are generic points in $\mathbb{P}^2$ and $T_{r(k)} \subset X$, the *remainder scheme*, is a 0-dimensional scheme of degree $r(k)$; if $r(k) \neq 0$, $T_{r(k)} = \eta_1 \cup \eta_2$, where η_1, η_2 have support on two distinct points A, B in the fiber $F = \pi^{-1}(P_{q(k)+1})$ and η_i, $i = 1, 2$, is contained in a section of $\mathcal{O}_X(1)$, hence length $(\eta_i \cap F) = 1$. If we consider affine coordinates $\{x, y, z\}$ in an affine chart of X containing $T(k)$, we may suppose $F = \{x = y = 0\}$, $A = (0, 0, 0)$ and $B = (0, 0, 1)$; $T_{r(k)}$ is defined as follows:

$$T_0 = \emptyset;$$

T_3 is made by the simple point $\eta_1 = A \in F$ and a double structure η_2 on $B \in F$, given by an ideal of type $(x, y^2, z-1)$;

T_8 is made by two 4-ple structures η_1, η_2 of the same type, and we have the two following possibilities: either the η_i's are given by ideals of type (x^2, y^2, z), $(x^2, y^2, z-1)$, or by ideals of type (x^3, xy, y^2, z), $(x^3, xy, y^2, z-1)$;

T_{11} is such that η_1 is a 5-ple structure on A given by an ideal of type (x^3, x^2y, y^2, z), and η_2 is given by an ideal of type $(x^3, x^2y, y^2x, y^3, z-1)$.

We recall that: $\pi_*(\mathscr{E}_k \otimes \mathscr{I}_{\pi^{-1}(Y)}) \cong \Omega(k) \otimes \mathscr{I}_Y$, and $\pi_*(\mathscr{E}_{k|\pi^{-1}(Y)}) \cong \Omega(k)|_Y$ (e.g, see [25, Lemma 2.1]); hence $H^0(\mathscr{E}_k \otimes \mathscr{I}_{\pi^{-1}(Y)}) \cong H^0(\Omega(k) \otimes \mathscr{I}_Y)$ and $H^0(\mathscr{E}_{k|\pi^{-1}(Y)}) \cong H^0(\Omega(k)|_Y)$.

1.4. Definition. In the following, for $k \geqslant 0$, "$\mathbf{B}(k)$" will denote the statement:

"The scheme $T(k)$ satisfies $H^0(\mathscr{E}_{k+1} \otimes \mathscr{I}_{T(k)}) = 0$".

Notice that $\mathbf{B}(k)$ is equivalent to saying that the restriction map ρ is an isomorphism, where

$$0 \to H^0(\mathscr{E}_{k+1} \otimes \mathscr{I}_{T(k)}) \to H^0(\mathscr{E}_{k+1}) \xrightarrow{\rho} H^0(\mathscr{E}_{k+1|T(k)})$$

$$\to H^1(\mathscr{E}_{k+1} \otimes \mathscr{I}_{T(k)}) \to H^1(\mathscr{E}_{k+1})$$

in fact $\mathbf{B}(k)$ implies that ρ is injective, but $h^0(\mathscr{E}_{k+1}) = h^0(\Omega(k+1)) = k(k+2)$, and also

$$h^0(\mathscr{E}_{k+1|T(k)}) = h^0(\mathscr{E}_{k+1}|_{\pi^{-1}(Y(q(k)))}) + h^0(\mathscr{E}_{k+1}|_{T_{r(k)}})$$

$$= h^0(\Omega(k+1)|_{Y(q(k))}) + r(k) = 12q(k) + r(k) = k(k+2),$$

hence ρ is bijective.

This implies also that $H^1(\mathscr{E}_{k+1} \otimes \mathscr{I}_{T(k)}) = 0$, since $h^1(\mathscr{E}_{k+1}) = h^1(\Omega(k+1)) = 0$ for $k \geqslant 0$ (e.g. see [20] ex.III.8.1 and ex.III.8.4).

The following proposition will show that the proof of our main result reduces to proving suitable statements $\mathbf{B}(k)$.

1.5. Proposition. *Let n,k be positive integers and let $Y = Y(n)$. If $\mathbf{B}(k)$ holds, we have*:

(i) *when $n > q(k)$, σ_k is injective for Y;*
(ii) *when $n \leqslant q(k)$ and $n \neq 2,5$, σ_k is surjective for Y.*

Proof. (i) Since $n > q(k)$, we have

$$T(k) = \pi^{-1}(Y(q(k))) \cup T_{r(k)} \subseteq \pi^{-1}(Y),$$

so we get the exact sequence (where $\mathscr{I}_{T(k),\pi^{-1}(Y)}$ is the ideal sheaf of $T(k)$ in $\pi^{-1}(Y)$):

$$0 \to \mathscr{E}_{k+1} \otimes \mathscr{I}_{\pi^{-1}(Y)} \to \mathscr{E}_{k+1} \otimes \mathscr{I}_{T(k)} \to \mathscr{E}_{k+1} \otimes \mathscr{I}_{T(k),\pi^{-1}(Y)} \to 0$$

from which, since $H^0(\mathscr{E}_{k+1} \otimes \mathscr{I}_{T(k)}) = 0$, we get $H^0(\mathscr{E}_{k+1} \otimes \mathscr{I}_{\pi^{-1}(Y)}) \cong H^0(\Omega(k+1) \otimes \mathscr{I}_Y) = 0$, i.e. that σ_k is injective for Y.

(ii) Since $n \leqslant q(k)$, we have

$$\pi^{-1}(Y) \subseteq T(k) = \pi^{-1}(Y(q(k))) \cup T_{r(k)}$$

and the exact sequence:

$$0 \to \mathscr{E}_{k+1} \otimes \mathscr{I}_{T(k)} \to \mathscr{E}_{k+1} \otimes \mathscr{I}_{\pi^{-1}(Y)} \to \mathscr{E}_{k+1} \otimes \mathscr{I}_{\pi^{-1}(Y),T(k)} \to 0$$

from which, since $h^0(\mathscr{E}_{k+1} \otimes \mathscr{I}_{T(k)}) = h^1(\mathscr{E}_{k+1} \otimes \mathscr{I}_{T(k)}) = 0$, we get

$$\begin{aligned} h^0(\mathscr{E}_{k+1} \otimes \mathscr{I}_{\pi^{-1}(Y)}) &= h^0(\mathscr{E}_{k+1} \otimes \mathscr{I}_{\pi^{-1}(Y),T(k)}) \\ &= h^0(\mathscr{E}_{k+1} \otimes \mathscr{O}_{T(k)}) - h^0(\mathscr{E}_{k+1} \otimes \mathscr{O}_{\pi^{-1}(Y)}) \\ &= 12q(k) + r(k) - 12n, \end{aligned}$$

hence $h^0(\Omega(k+1) \otimes \mathscr{I}_Y) = h^0(\mathscr{E}_{k+1} \otimes \mathscr{I}_{\pi^{-1}(Y)}) = 12q(k) + r(k) - 12n = k(k+2) - 12n$.

On the other hand, $n \leqslant q(k)$, so an easy computation shows that $6n \leqslant \binom{k+2}{2}$. Since Y has maximal Hilbert function, we have

$$\begin{aligned} 3h^0(\mathscr{I}_Y(k)) - h^0(\mathscr{I}_Y(k+1)) &= 3\left(\binom{k+2}{2} - 6n\right) - \left(\binom{k+3}{2} - 6n\right) \\ &= k(k+2) - 12n. \end{aligned}$$

Hence $h^0(\Omega(k+1) \otimes \mathscr{I}_Y) = 3h^0(\mathscr{I}_Y(k)) - h^0(\mathscr{I}_Y(k+1))$, so that the map σ_k is surjective. □

1.6. Proposition. *If $\mathbf{B}(k)$ holds for $k \geqslant 10$, then Theorem* 1.1 *holds.*

Proof. Let n be a fixed integer > 10; since $q(10) = 10$, there exists w such that $q(w-1) < n \leqslant q(w)$, with $w \geqslant 11$. Since by assumption $\mathbf{B}(w-1)$ is true, $\sigma_{w-1}(Y(n))$ is injective (Proposition 1.5), hence $\sigma_k(Y(n))$ is injective too for all $k \leqslant w-1$.

Moreover, $\mathbf{B}(k)$ is true for all $k \geqslant w$, hence $\sigma_k(Y(n))$ is surjective for $k \geqslant w$ (Proposition 1.5).

Finally $\mathbf{B}(10)$ true implies that $\sigma_k(Y(10))$ is surjective for $k \geqslant 10$, since $q(10)=10$; then, $r(10)=0$ says that $\sigma_{10}(Y(10))$ is also injective, so that $\sigma_k(Y(10))$ is injective for all $k \leqslant 10$. Since [18] proves Theorem 1.1 for $n \leqslant 9$, this completes the proof. □

1.7. Remark. Recall that, when $n \neq 2,3$ or 5, the resolution of $Y = Y(n)$ is equal to the resolution of $6n$ general points, namely, if $\alpha = \alpha(Y)$, and $d = \binom{\alpha+2}{2} - 6n$, then

$$0 \to \mathcal{O}_{\mathbb{P}^2}^{\alpha+1-d}(-\alpha-2) \oplus \mathcal{O}_{\mathbb{P}^2}^{\{2d-\alpha-2\}^+}(-\alpha-1)$$
$$\to \mathcal{O}_{\mathbb{P}^2}^{\{\alpha+2-2d\}^+}(-\alpha-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{d}(-\alpha) \to \mathscr{I}_Y \to 0$$

where $\{a\}^+ = \max\{a,0\}$.

In case $n=2,3$ or 5, the resolution of the ideal sheaf $\mathscr{I}_Y$ is actually different from the one above (see [5]). Namely:

$$n=2 \quad 0 \to \mathcal{O}_{\mathbb{P}^2}(-7) \oplus \mathcal{O}_{\mathbb{P}^2}(-6) \oplus \mathcal{O}_{\mathbb{P}^2}(-5) \to \mathcal{O}_{\mathbb{P}^2}(-6) \oplus \mathcal{O}_{\mathbb{P}^2}(-5) \oplus \mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathscr{I}_Y \to 0.$$

$n = 3$. Notice that in this case $Y = Y(3)$ does have maximal Hilbert function; the resolution is:

$$0 \to \mathcal{O}_{\mathbb{P}^2}^3(-7) \oplus \mathcal{O}_{\mathbb{P}^2}^2(-6) \to \mathcal{O}_{\mathbb{P}^2}^3(-6) \oplus \mathcal{O}_{\mathbb{P}^2}^3(-5) \to \mathscr{I}_Y \to 0.$$

$$n=5 \quad 0 \to \mathcal{O}_{\mathbb{P}^2}^3(-9) \oplus \mathcal{O}_{\mathbb{P}^2}^2(-8) \to \mathcal{O}_{\mathbb{P}^2}^2(-8) \oplus \mathcal{O}_{\mathbb{P}^2}^3(-7) \oplus \mathcal{O}_{\mathbb{P}^2}(-6) \to \mathscr{I}_Y \to 0.$$

2. Horace différential for $\Omega_{\mathbb{P}^2}$

2.1. Notations. If A is a closed subscheme of $\mathbb{P}^2$ or, respectively, of $X = \mathbb{P}(\Omega)$, $\mathscr{I}_A$ will denote $\mathscr{I}_{A,\mathbb{P}^2}$ or, respectively, $\mathscr{I}_{A,X}$.

Let U be an open subset in $\mathbb{P}^2$ such that $\Omega|_U \cong E \oplus G$, $E \cong G \cong \mathcal{O}_U$; let $Y \subset U$ be a closed subscheme of U. We set:

$$Y' := \pi^{-1}(Y) \cap \mathbb{P}(E), \quad Y'' := \pi^{-1}(Y) \cap \mathbb{P}(G), \quad \hat{Y} := Y' \cup Y''$$

(hence π gives isomorphisms $Y' \cong Y \cong Y''$).

2.2. Lemma. *Let Y be a 0-dimensional scheme in $\mathbb{P}^2$, then $H^0(\mathscr{E}_k \otimes \mathscr{I}_{\pi^{-1}(Y)}) \cong H^0(\mathscr{E}_k \otimes \mathscr{I}_{\hat{Y}})$.*

Proof. That is obvious if Y is reduced, since $\mathscr{E}_k$ is $\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibers F, $F \cong \mathbb{P}^1$ and $\hat{Y}$ consists of 2 points on each point of Y.

In general we have:

$$0 \to \mathscr{I}_{\pi^{-1}(Y)} \to \mathscr{I}_{\hat{Y}} \to \mathscr{I}_{\hat{Y},\pi^{-1}(Y)} \to 0.$$

By tensoring with $\mathscr{E}_k$ and taking cohomology, we get:

$$0 \to H^0(\mathscr{E}_k \otimes \mathscr{I}_{\pi^{-1}(Y)}) \to H^0(\mathscr{E}_k \otimes \mathscr{I}_{\hat{Y}}) \to H^0(\mathscr{E}_k \otimes I_{\hat{Y},\pi^{-1}(Y)}) \to \cdots .$$

It is enough to prove $H^0(\mathscr{E}_k \otimes \mathscr{I}_{\hat{Y},\pi^{-1}(Y)}) = 0$.

One has:

$$\mathscr{E}_k|_{\pi^{-1}(Y)} = \mathscr{O}_X(1)|_{\pi^{-1}(Y)} \otimes \pi^* \mathscr{O}_{\mathbb{P}^2}(k)|_{\pi^{-1}(Y)} \cong \mathscr{O}_{\mathbb{P}(\Omega|_Y)}(1).$$

Moreover $\pi^{-1}(Y) = \mathbb{P}(\Omega|_Y)$ (e.g. see proof of Lemma 2.1 in [25] and [16], 9.7.9 and 9.7.6), while $\Omega|_Y = \mathscr{O}_Y^{\oplus 2}$; hence

$$H^0(\mathscr{E}_k \otimes \mathscr{I}_{\hat{Y},\pi^{-1}(Y)}) \cong H^0(\mathscr{O}_{\mathbb{P}(\Omega|_Y)}(1) \otimes \mathscr{I}_{\hat{Y},\mathbb{P}(\Omega|_Y)}),$$

and $\mathbb{P}(\Omega|_Y) \cong \mathbb{P}^1 \times Y$.

Let us consider now the projection $q : \mathbb{P}^1 \times Y \to \mathbb{P}^1$. One has: $\hat{Y} \cong q^{-1}(2\text{ pts})$, and $\mathscr{O}_{\mathbb{P}(\Omega|_Y)}(1) \cong q^* \mathscr{O}_{\mathbb{P}^1}(1)$.

So we only have to prove $H^0(q^* \mathscr{O}_{\mathbb{P}^1}(1) \otimes \mathscr{I}_{q^{-1}(2\text{pts})}) = 0$. This follows easily from the fact that: $T \times \mathbb{P}^1 = \text{Proj}(A[t_0, t_1])$, where $Y = \text{Spec}A$, and $\mathscr{O}_{\mathbb{P}^1_A}(1) = q^* \mathscr{O}_{\mathbb{P}^1_A}$. □

2.3. Notations. Following [1], we give the following definition:

In the algebra of formal functions $\kappa[[\mathbf{x}, y]]$, where $\mathbf{x} = (x_1, \ldots, x_{d-1})$, *a vertically graded* (with respect to y) ideal is an ideal of the form:

$$I = I_0 \oplus I_1 y \oplus \cdots \oplus I_{m-1} y^{m-1} \oplus (y^m)$$

where for $i = 0, \ldots, m-1$, $I_i \subset \kappa[[\mathbf{x}]]$ is an ideal.

Let Q be a smooth d-dimensional integral scheme, let K be a smooth irreducible divisor on Q. We say that $Z \subset Q$ is a *vertically graded subscheme* of Q with base K and support $z \in K$ if Z is a 0-dimensional scheme with support in the point z such that there is a regular system of parameters $(\mathbf{x}, y)$ at z such that $y = 0$ is a local equation for K and the ideal of Z in $\hat{\mathscr{O}}_{Q,z} \cong \kappa[[\mathbf{x}, y]]$ is vertically graded.

Let $Z_1, \ldots, Z_r \subset Q$ be vertically graded subschemes with base K and support z_i; let $p_i \geqslant 0$ be a fixed integer for $i = 1, \ldots, r$.

Let $Res_K^{p_i}(Z_i) \subset Q$ and $Tr_K^{p_i}(Z_i) \subset K$ be the closed subschemes defined, respectively, by the ideals:

$$\mathscr{I}_{Res_K^{p_i}(Z_i)} := \mathscr{I}_{Z_i} + (\mathscr{I}_{Z_i} : \mathscr{I}_K^{p_i+1}) \mathscr{I}_K^{p_i},$$

$$\mathscr{I}_{Tr_K^{p_i}(Z_i)} := (\mathscr{I}_{Z_i} : \mathscr{I}_K^{p_i}) \otimes \mathscr{O}_K.$$

In $Res_K^{p_i}(Z_i)$ we take away from Z_i the $(p_i + 1)$th “slice”, in $Tr_K^{p_i}(Z_i)$ we consider only the $(p_i + 1)$th “slice” (see Example 2.4 and picture, where the jth “row” of Z is the scheme corresponding to I_{j-1}).

We notice that for $p_i = 0$ we get the usual trace and residual schemes: $Tr_K(Z_i)$ and $Res_K(Z_i)$.

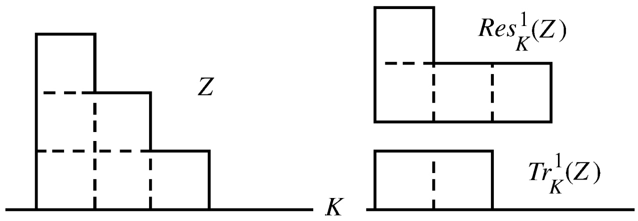
Finally, if $\mathbf{p} = (p_1, \ldots, p_r) \in \mathbb{N}^r$ and $Z = Z_1 \cup \cdots \cup Z_r$, we set:

$$Tr_K^{\mathbf{p}}(Z) := Tr_K^{p_1}(Z_1) \cup \cdots \cup Tr_K^{p_r}(Z_r), \quad Res_K^{\mathbf{p}}(Z) := Res_K^{p_1}(Z_1) \cup \ldots \cup Res_K^{p_r}(Z_r).$$

2.4. Example. Let (in $\mathbb{P}^2$), $I_{Z_i} = (x^3, x^2y, xy^2, y^3)$, $I_K = (y)$ and $p_i = 1$; then,

$$I_{Res_K^{p_i}(Z_i)} = (x^3, x^2y, xy^2, y^3) + ((x^3, x^2y, xy^2, y^3) : y^2)y = (x^3, xy, y^2),$$

$$I_{Tr_K^{p_i}(Z_i)} = ((x^3, x^2y, xy^2, y^3) : y) \otimes \frac{\kappa[[x, y]]}{(y)} = (x^2, xy, y^2) \otimes \frac{\kappa[[x, y]]}{(y)} = (\bar{x}^2).$$



2.5. Notations. Let C be a smooth conic in $\mathbb{P}^2$, and $H = \pi^{-1}(C) \subset X = \mathbb{P}(\Omega)$.

Let $W \subset X$ be a 0-dimensional closed subscheme and let TrW and $ResW$ be the trace and the residue of W with respect to H.

Let $S_1, \ldots, S_n, R_1, \ldots, R_n$ be 0-dimensional irreducible subschemes of $\mathbb{P}^2$ and $S = S_i \cup \cdots \cup S_n$ and $R = R_1 \cup \cdots \cup R_n$, such that:

(1) $S_i \cong R_i$, $i = 1, \ldots, n$;
(2) R_i has support on C and is vertically graded with base C;
(3) The supports of S and R are generic in their respective Hilbert schemes.

If $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (p_1, \ldots, p_n, p_1, \ldots, p_n)$ one has:

$$\widehat{Tr_C^{\mathbf{p}}(R)} = Tr_H^{\mathbf{q}}(\hat{R}); \quad \widehat{Res_C^{\mathbf{p}}(R)} = Res_H^{\mathbf{q}}(\hat{R}).$$

The main result in this section is:

2.6. Proposition. *With the previous notations, consider a scheme $W \cup \hat{S} \subset \mathbb{P}(\Omega)$; if*

(a) $H^0(\mathscr{I}_{TrW \cup Tr_H^{\mathbf{q}}(\hat{R}),H} \otimes \mathscr{E}_k|_H) = 0$ *and*
(b) $H^0(\mathscr{I}_{ResW \cup Res_H^{\mathbf{q}}(\hat{R})} \otimes \mathscr{E}_k(-H)) = 0$,

then

$$H^0(\mathscr{I}_{W \cup \hat{S}} \otimes \mathscr{E}_k) = 0.$$

2.7. Remark. This proposition would follow directly from Proposition 9.1 in [1] if the supports of $\hat{S}$ and $\hat{R}$ were generic in X, respectively in H; but S_i' and S_i'' lie on a fiber of π, as well as R_i' and R_i''. By semicontinuity, this is not a problem for the support of $\hat{R}$, since the vanishing of the cohomology in assumption (a) implies the generic vanishing required in [1, 9.1, assumption 1].

The idea of the proof of Proposition 2.6 is to mimic the proof of [1, 9.1]. If we were working in $\mathbb{P}^2$, we would move the support of each R_j along the germ of a curve C_j, transverse to the conic C, in order to get the thesis on S. In $\mathbb{P}(\Omega)$, for each j, we move "with the same speed" the points R'_j, R''_j along C'_j, C''_j, respectively; C'_j and C''_j are two "copies" of C_j in $\pi^{-1}(C)$, see Notations 2.1.

We need a technical lemma, which extends Proposition 8.2 in [1]. With this aim we premit some notations (the same as in [1]):

2.8. Notations. For $i=1,\dots,l$, with $l=2n$, let $B^{(i)}=\kappa[[\mathbf{x}_i, y_i]]$ be the algebra of formal functions in d variables where $\mathbf{x}_i=(x_{i,1},\dots,x_{i,d-1})$ and let

$$I^{(i)} = I_0^{(i)} \oplus I_1^{(i)} y_i \oplus \cdots \oplus I_{m_i-1}^{(i)} y_i^{m_i-1} \oplus (y_i^{m_i})$$

be a vertically graded ideal in $B^{(i)}$.

For $j=1,\dots,n$, let ψ_j be the κ-algebra isomorphisms defined as follows:

$$\psi_j : \kappa[[\mathbf{x}_j, y_j]] \to \kappa[[\mathbf{x}_{j+n}, y_{j+n}]], \quad \mathbf{x}_j \mapsto \mathbf{x}_{j+n}, \quad y_j \mapsto y_{j+n}.$$

We assume $I^{(j+n)} = \psi_j(I^{(j)})$.

Let

$$I = I^{(1)} \times \cdots \times I^{(l)} \subset B^{(1)} \times \cdots \times B^{(l)} = B.$$

Let $\kappa[[\mathbf{t}]] = \kappa[[t_1,\dots,t_l]]$ and let $I_{\mathbf{t}}$ in $B[[\mathbf{t}]]$ be the product of the ideals

$$I_{\mathbf{t}}^{(i)} = I_0^{(i)}[[\mathbf{t}]] \oplus I_1^{(i)}[[\mathbf{t}]](y_i - t_i) \oplus \cdots \oplus I_{m_i-1}^{(i)}[[\mathbf{t}]](y_i - t_i)^{m_i-1} \oplus ((y_i - t_i)^{m_i}).$$

Let $y=(y_1,\dots,y_l)$ and for any linear subspace $V \subset B$, let $V_{\mathrm{res}(y)} = \{v \in B \mid vy \in V\}$. Since y is not a zero-divisor, we get a residual exact sequence:

$$0 \to V_{\mathrm{res}(y)} \to V \to V/V \cap (y) \to 0.$$

2.9. Lemma. *Let $V \subset B$ be a κ-linear finite-dimensional subspace. Suppose that for $i=1,\dots,l$, with $l=2n$, there exist non-negative integers p_i, with $p_{j+n}=p_j$, $j=1,\dots,n$, such that the following two conditions are satisfied*:

1. *the canonical map*
$$V/V \cap (y) \to \kappa[[\mathbf{x}_1]]/I_{p_1}^{(1)} \times \cdots \times \kappa[[\mathbf{x}_l]]/I_{p_1}^{(l)}$$
is injective.
2. *the canonical map*
$$V_{\mathrm{res}(y)} \to B/J$$
is injective, where $J = J^{(1)} \times \cdots \times J^{(l)}$ and
$$J^{(i)} = I_0^{(i)} \oplus I_1^{(i)} y_i \oplus \cdots \oplus I_{p_i-1}^{(i)} y_i^{p_i-1} \oplus I_{p_i+1}^{(i)} y_i^{p_i} \oplus \cdots \oplus I_{m_i-1}^{(i)} y_i^{m_i-2} \oplus (y_i^{m_i-1}).$$

Then the canonical map $\varphi_{\mathbf{t}}: V\otimes\kappa[[\mathbf{t}]] \to B[[\mathbf{t}]]/I_{\mathbf{t}}$ in injective for $\mathbf{t}$ in an open subset with non-empty intersection with $D=\{(z_1,\dots,z_l)\in\mathbb{A}^l \mid z_{j+n}=z_j,\ j=1,\dots,n\}\cong\mathbb{A}^n$.

Proof. This proof is a slight modification of the proof of [1, Proposition 8.2]; the statement there is that $\varphi_{\mathbf{t}}$ is generically injective, hence the only difference in our statement is that we point out that in our case the open set where injectivity holds intersects D.

Notice that, as it has been done in [1], here too it is possible to assume that the p_i's are positive, so we set $\mathrm{lcm}(p_1,\dots,p_l)=r_i p_i$, hence $r_j=r_{j+n}$.

The key tool in the proof is the semicontinuity of $\varphi_{\mathbf{t}}$ and the fact that the canonical map $\varphi_t: V_t \to B[[t]]/I_t$ obtained by formal base change $\psi:\kappa[[t_1,\dots,t_l]]\to\kappa[[t]]$, $t_i\mapsto t^{r_i}$, is injective, where V_t is the image of $V\otimes\kappa[[t_1,\dots,t_l]]$ in $V\otimes\kappa[[t]]$, and I_t is the image of $I_{\mathbf{t}}$ in $B[[t]]$.

Since φ_t is injective (see [1]), there is an $u\in\kappa$ such that φ_u is 1-1. Since $\kappa=\bar{\kappa}$, we can find $u_1,\dots,u_n\in\kappa$ such that $u_1^{r_1}=\cdots=u_n^{r_n}=u$. We set $u_{j+n}=u_j,\ j=1,\dots,n$ and $\mathbf{u}=(u_1,\dots,u_n,u_1,\dots,u_n)$, so that $\varphi_{\mathbf{u}}$ is injective.

Let $D=\{(t_1,\dots,t_l)\in\mathbb{A}^l \mid t_{j+n}=t_j,\ j=1,\dots,n\}\cong\mathbb{A}^n$; then $\mathbf{u}\in D$, and by semicontinuity there is an open subset U of D such that $\varphi_{\mathbf{t}}$ is injective for $\mathbf{t}\in U$. □

Proof of Proposition 2.6. Here $d=3$. For $j=1,\dots,n$ let x_j, y_j be local coordinates in $\mathbb{P}^2$ at R_j ($y_j=0$ is the local equation for C). Let $(x_{j,1},x_{j,2},y_j)$ and $(x_{j+n,1},x_{j+n,2},y_{j+n})$ be local coordinates at R'_j, R''_j, respectively; we can assume $x_{j,1}=x_{j+n,1}=x_j,\ y_j=y_{j+n}$ and that $x_{j,2},x_{j+n,2}$ are local coordinates along the fiber.

Let C_j be a curve which meets C transversally at R_j; let t_j be a local parameter for S_j on C_j. Above, in $\mathbb{P}(\Omega)$, we can assume that t_j is a local parameter for both the points S'_j, S''_j (on C'_j and, respectively, on C''_j).

We can assume $W\cap\hat{R}=\emptyset$, hence $H^0(\mathscr{I}_W\otimes\mathscr{E}_k)\hookrightarrow(\mathscr{I}_W\otimes\mathscr{E}_k)_x\cong\mathscr{O}_{X,x}\hookrightarrow\hat{\mathscr{O}}_{X,x}$ $\forall x\in\mathrm{Supp}\,\hat{R}$. We can view $\hat{R}$ as an embedding of $Spec(B/I)$, see notations 2.8, into X. This allows us to choose $V=H^0(\mathscr{I}_W\otimes\mathscr{E}_k)$ in Lemma 2.9, and hypotheses (a) and (b) of our statement give hypotheses (1) and (2) of 2.9. Then the result follows by noting that $\ker\varphi_{\mathbf{t}}\cong H^0(\mathscr{I}_{W\cup\hat{S}}\otimes\mathscr{E}_k)$. We refer to [1, Section 9] for details. □

3. Proof of the main result

3.1. Notations. We introduce now some notations that will allow us to express ourselves as if we were working in $\mathbb{P}^2$, while our environment is actually $\mathbb{P}(\Omega)$.

Let Y be a 0-dimensional scheme of $\mathbb{P}^2$ with support at a point P. We have defined $\hat{Y}$ in 2.1; if Y is vertically graded with base a smooth conic C with local equation $y=0$, then Y' and Y'' are vertically graded with base $H=\pi^{-1}(C)$; let x,y be local coordinates at P.

If $I_Y=(x^h,y)$, we will denote $\hat{Y}$ by (h). In the general case we will denote $\hat{Y}$ by $\begin{pmatrix} a_s \\ \vdots \\ a_0 \end{pmatrix}$, where $Tr_H^j\hat{Y}=(a_j)$.

For example:

$\hat{Y}$	I_Y	$\hat{Y}$	I_Y
$\begin{pmatrix}1\\2\end{pmatrix}$	$(x,y)^2$	$\begin{pmatrix}2\\2\end{pmatrix}$	(x^2,y^2)
$\begin{pmatrix}1\\3\end{pmatrix}$	(x^3,xy,y^2)	$\begin{pmatrix}1\\1\end{pmatrix}$	(x,y^2)
$\begin{pmatrix}2\\3\end{pmatrix}$	(x^3,x^2y,y^2)	$\begin{pmatrix}1\\2\\3\end{pmatrix}$	$(x,y)^3$

If $\hat{Y}$ is a $\binom{1}{3}$, we will say that $\hat{Y}$ is "a $\binom{1}{3}$ scheme over C". Moreover, if $h,l,\in\mathbb{N}$, we will use, for example, the notation "$h\binom{1}{2}+l\binom{1}{3}$" to denote the union of h schemes of type $\binom{1}{2}$ and l schemes of type $\binom{1}{3}$. Finally we write, for example, "$h\binom{1}{2}$ schemes general over $\mathbb{P}^2$" to mean a union of h schemes of type $\binom{1}{2}$, whose projection in $\mathbb{P}^2$ is general.

Moreover, if h, a_i and b_i, $i=1,\dots,m$ are positive integers such that $h=\sum_{i=1}^m a_ib_i$, we will not distinguish between (h) and $\sum_{i=1}^m a_i(b_i)$. We are allowed to do that since the role of these schemes boils down only to the vanishing of the global sections of a twist of Ω restricted to the conic C, hence it is only their length over C which matters.

3.2. Definition. Let b,c,d,e,f,r,k be integers $\geqslant 0$; with $Z(b,c,d,e,f,r,k)$ we denote a 0-dimensional subscheme of X, union of:

$$b\begin{pmatrix}1\\2\end{pmatrix}+c(1)+d\begin{pmatrix}1\\3\end{pmatrix}+e\begin{pmatrix}2\\3\end{pmatrix}\ \text{over } C,\quad \text{and}$$

$$f\begin{pmatrix}1\\2\\3\end{pmatrix}+T_r\ \text{general over }\mathbb{P}^2,$$

with following assumptions:

$(0)_k$ $2b+c+3d+3e\leqslant 2k,\ 0\leqslant d+e\leqslant 1,$
$(1)_k$ $2(3b+c+4d+5e+6f)+r=k(k+2)$ (i.e., $length(Z(b,c,d,e,f,r,k))=h^0(\mathscr{E}_{k+1})$)
$(2)_k$ $r=0$ or $r=8$ if k is even; $r=3$ or $r=11$ if k is odd (i.e., r is one of the possible remainders in 1.2).

Notice that condition $(0)_k$ means "not too much stuff over C", since $h^0(\mathscr{E}_{k+1}|_H)=h^0(\Omega(k+1)|_C)=4k$, see [26]. Moreover, condition $(1)_k$ implies that $((k(k+2)-r)/2)-$

$(3b+c+4d+5e)\equiv 0\,(\mathrm{mod}\,6)$, and $f=\frac{1}{6}(((k(k+2)-r)/2)-(3b+c+4d+5e))$, that is, f is known if b,c,d,e,r,k are (suitably) given, so we can write $f=f(b,c,d,e,r,k)$.

Also recall that the scheme T_r has been defined in 1.3: it is uniquely defined for $r=0,3,11$, while it can be a $\binom{2}{2}$ or a $\binom{1}{3}$ scheme for $r=8$.

The set

$$\mathscr{A}(k):=\{(b,c,d,e,f,r,k)\in\mathbb{N}^7\mid(0)_k,(1)_k,(2)_k\text{ hold for }(b,c,d,e,f,r,k)\}$$

is called the set of admissible values for a given k; hence, a scheme $Z(b,c,d,e,f,r,k)$ is defined if and only if the 7-ple (b,c,d,e,f,r,k) is admissible, that is, if $(b,c,d,e,f,r,k)\in\mathscr{A}(k)$.

Soon we shall also need the following integers $\geqslant 0$:

$$g=g(b,c,d,e,r,k),\quad h=h(b,c,d,e,r,k),$$

$$i=i(b,c,d,e,r,k),\text{ defined by the relations :}$$

$$(3)_k\quad 3g+2h+i=2k-2b-c-3d-3e,\quad 0\leqslant h+i\leqslant 1,$$

i.e. g,h and i measure, modulo 3, what is missing over C in order to annihilate all the sections of $H^0(\mathscr{E}_{k+1}|_H)$.

3.3. Proposition. *For each* $Z=Z(b,c,d,e,f,r,k)$, $k\geqslant 8$, *and* $(b,c,d,e,f,r,k)\neq(0,0,0,0,6,8,8)$ *there exists a scheme* $Z'=Z(b',c',d',e',f',r,k-2)$ *such that if* $H^0(\mathscr{E}_{k-1}\otimes\mathscr{I}_{Z'})=0$, *then* $H^0(\mathscr{E}_{k+1}\otimes\mathscr{I}_Z)=0$.

More precisely,

$$b'=g,\quad c'=b+d+2e,\quad d'=h,\quad e'=i,\quad f'=f-g-h-i;$$

we will say that "Z' comes from Z in one step", or that "Z' comes from one step before".

Proof. The idea of the proof is to specialize Z to a scheme $W\cup\hat{S}$ (where part of the $f\left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right)$ points of Z have been specialized over C) and then apply Proposition 2.6 in order to get $h^0(\mathscr{E}_{k+1}\otimes\mathscr{I}_{W\cup\hat{S}})=0$, from which $H^0(\mathscr{E}_{k+1}\otimes\mathscr{I}_Z)=0$ follows by semicontinuity.

More precisely, since by forthcoming Lemma 3.4 $f\geqslant g+h+i$, we specialize g schemes of type $\left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right)$ over C, and we apply Proposition 2.6 (recall also Notations 2.5) with:

W union of $b\binom{1}{2}+c(1)+d\binom{1}{3}+e\binom{2}{3}+g\left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right)$ over C, and of $(f-g-h-i)\left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right)+T_r$ general over $\mathbb{P}^2$; and with

$R=R_1=\left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right)$ and $p_1=1$, if $h=1$ and $i=0$ $\left(\text{in this case }\mathbf{q}=(1,1),Tr_H^{\mathbf{q}}(\hat{R})=(2),\text{ and }Res_H^{\mathbf{q}}(\hat{R})=\binom{1}{3}\right)$; $R=R_1=\left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right)$ and $p_1=2$, if $h=0$ and $i=1$ $\left(\text{in this case }\mathbf{q}=(2,2),Tr_H^{\mathbf{q}}(\hat{R})=(1),\text{ and }Res_H^{\mathbf{q}}(\hat{R})=\binom{2}{3}\right)$;

$R=\emptyset$, if $h=0$ and $i=0$.

We say that we have done an HD (Horace différentiel) step, with the HD trace and the HD residue defined by:

$$Tr^{HD}Z := Tr_H W \cup Tr_H^{\mathbf{q}}(\hat{R}) = b(2) + c(1) + d(3) + e(3) + g(3) + h(2) + i(1) \text{ over } C;$$

$$Res^{HD}Z := Res_H W \cup Res_H^{\mathbf{q}}(\hat{R}),$$

where $Res_H W$ is $b(1)+d(1)+e(2)+g\begin{pmatrix}1\\2\end{pmatrix}$ over C union with $(f-g-h-i)\begin{pmatrix}1\\2\\3\end{pmatrix}+Tr$ general over $\mathbb{P}^2$, and $Res_H^{\mathbf{q}}(\hat{R}) = h\begin{pmatrix}1\\3\end{pmatrix} + i\begin{pmatrix}2\\3\end{pmatrix}$ (over C).

For sake of brevity, we will express all this also with the following table (in the first column, "c.n.o.C" means "conditions needed over C"):

c.n.o.C	What we have over C	What we "add" over C	Residue not over C
$2k$	$b\begin{pmatrix}1\\ [2]\end{pmatrix} + c([1])$	$g\begin{pmatrix}1\\2\\ [3]\end{pmatrix}$	$(f-g-h-i)\begin{pmatrix}1\\2\\3\end{pmatrix} + T_r$
	$+d\begin{pmatrix}1\\ [3]\end{pmatrix} + e\begin{pmatrix}2\\ [3]\end{pmatrix}$	$+h\begin{pmatrix}1\\ [2]\\3\end{pmatrix} + i\begin{pmatrix}[1]\\2\\3\end{pmatrix}$	

The $Tr^{HD}Z$ is given by the numbers in square brackets in the 2nd and 3rd column, while the $Res^{HD}Z$ is obtained by the 2nd, 3rd and 4th columns eliminating the part in the square brackets.

Since condition $(3)_k$ holds for Z, that is, $2b+c+3d+3e+3g+2h+i=2k$, by Lemma 2.2 and by [26, (1.2)iv] one has $H^0(\mathscr{I}_{Tr^{HD}Z,H} \otimes \mathscr{E}_{k+1}|_H) = 0$; hence Proposition 2.6. (together with semicontinuity) gives $H^0(\mathscr{I}_Z \otimes \mathscr{E}_{k+1}) = 0$, provided that $H^0(\mathscr{I}_{Res^{HD}Z} \otimes \mathscr{E}_{k-1}) = 0$.

So set $b'=g$, $c'=b+d+2e$, $d'=h$, $e'=i$, $f'=f-g-h-i$, and $Z'=Res^{HD}Z$; Z' is a scheme $Z(b',c',d',e',f',r,k-2)$ provided that conditions $(0)_{k-2},(1)_{k-2},(2)_{k-2}$ of Definition 3.1 hold for these integers; hence we will be done if those conditions hold.

Condition $(2)_{k-2}$ for Z' follows immediately from condition $(2)_k$ for Z.

Condition $(1)_{k-2}$ for Z' also holds, since $(1)_k$ for Z gives $length(Z)=k(k+2)$; we have seen that $2b+c+3d+3e+3g+2h+i=2k$, that is, $length(Tr^{HD}Z)=2(2k)$; so we get $length(Res^{HD}Z) = length(Z) - length(Tr^{HD}Z) = k(k+2) - 4k = k(k-2)$.

Condition $(0)_{k-2}$ for Z' is $0 \leqslant d'+e' \leqslant 1$, that is, $0 \leqslant h+i \leqslant 1$, which is true by relations $(3)_k$ for Z, together with $2b'+c'+3d'+3e' \leqslant 2k-4$, that is,

$$3(2g+b+d+2e+3h+3i) \leqslant 3(2k-4). \tag{$*$}$$

Since $3g = 2k-2b-c-3d-3e-2h-i$ (by $(3)_k$ for Z), ($*$) is equivalent to

$$-b-2c-3d+5h+7i \leqslant 2k-12, \tag{$**$}$$

but $-b-2c-3d \leqslant 0$, and $5h+7i \leqslant 7$, hence $(**)$ is true if $2k-12 \geqslant 7$, that is, for all $k \geqslant 10$.

If $k=9$, $(**)$ becomes $-b-2c-3d+5h+7i \leqslant 6$, and this may fail only for $b=c=d=h=0$, $i=1$; relation $(3)_k$ in this case gives $3g+1=18-3e$, which is impossible (just read it mod 3).

If $k=8$, $(**)$ becomes $-b-2c-3d+5h+7i \leqslant 4$. If $h+i=0$ this is always true. If $h+i=1$ and $b=c=d=0$, $(**)$ is false; in this case, $r=0$ or $r=8$, and $e=0$ or $e=1$; condition $(1)_k$ for Z (i.e. $(k(k+2)-r)/2-(3b+c+4d+5e) \equiv 0 \pmod 6$) allows us to exclude $(r,e)=(0,0),(0,1),(8,1)$. The remaining case $r=8$, $e=0$ is possible, but it gives $f=6$, i.e. we are in the exceptional case $(0,0,0,0,6,8,8)$ that the statement excludes.

Now assume $h+i=1$, $-b-2c-3d<0$; $(**)$ is always true if $h=1$, $i=0$, and it is also true if $h=0$, $i=1$ and $b \geqslant 3$, or $c \geqslant 2$, or $d>0$, or $b>0$ and $c>0$.

So the remaining cases are: $(b,c,d,h,i) \in \{(0,1,0,0,1),(1,0,0,0,1),(2,0,0,0,1)\}$; in all three cases, relation $(3)_k$ for Z gives: $3g+1=16-2b-c-3e$ which is impossible (read it mod 3). □

3.4. Lemma. *If* $k \geqslant 8$, $Z(b,c,d,e,f,r,k)$ *satisfies the condition*:

$$f \geqslant g+h+i \qquad (*)$$

Proof. Condition $(1)_k$ and $(3)_k$ give:

$$f=\frac{1}{6}\left(\frac{k(k+2)-r}{2}-(3b+c+4d+5e)\right),$$

$$g=\frac{2k-2b-c-3d-3e-2h-i}{3}$$

hence $(*)$ becomes:

$$k^2-6k \geqslant r-2b-2c-4d-2e+4h+8i.$$

Now $-2b-2c-4d-2e \leqslant 0$, and $4h+8i \leqslant 8$, while $r \leqslant 8$ if k is even, and $r \leqslant 11$ if k is odd, so in order to have $(*)$ it is enough to have
$k^2-6k-16 \geqslant 0$ if k is even, which is true for $k \geqslant 8$, and
$k^2-6k-19 \geqslant 0$ if k is odd, which is true for $k \geqslant 9$. □

3.5. Lemma. *If* $Z=Z(b,c,d,e,f,r,k)$ *comes from one step before, the following relation holds*:

$$(4)_k \quad 3b+2c+2d+e \leqslant 2k+5.$$

Proof. If $Z=Z(b,c,d,e,f,r,k)$ comes from $Z''=Z(b'',c'',d'',e'',f'',r,k+2)$, by Proposition 3.3 one has:

$$b=g'', \quad c=b''+d''+2e'', \quad d=h'', \quad e=i'', \quad f=f''-g''-h''-i''. \qquad (\circ)$$

Relation $(3)_{k+2}$: $3g''+2h''+i''=2k+4-(2b''+2d''+4e'')-c''-d''+e''$ hence gives:

$$3b+2c+2d+e=2k+4-c''-d''+e''\leqslant 2k+5. \qquad (\circ\circ)$$

□

3.6. Lemma. *If $Z=Z(b,c,d,e,f,r,k)$ comes from two steps before, the following relation holds*:

$$(5)_k \quad 6b+c+4d+2e\geqslant 2k-4; \quad \text{if } c=0, \quad \text{then } 6b+4d+2e\geqslant 2k-1.$$

Proof. Assume $Z=Z(b,c,d,e,f,r,k)$ comes from $Z''=Z(b'',c'',d'',e'',f'',r,k+2)$, which at its turn comes from one step before.

We have (see $(\circ)$ and $(\circ\circ)$ in proof of Lemma 3.5) $b''=c-d''-2e''$, $c''=2k+4-3b-2c-2d-e-d''+e''$. Substituting in $(4)_{k+2}$: $3b''+2c''+2d''+e''\leqslant 2(k+2)+5$, we get:

$$6b+c+4d+2e\geqslant 2k-1-3d''-3e''\geqslant 2k-4;$$

moreover, $d''+e''=1$ implies $c\geqslant 1$, hence if $c=0$, then $2k-1-3d''-3e''=2k-1$. □

3.7. Lemma. *If $Z=Z(b,c,d,e,f,r,k)$ comes from three steps before, the following relation holds*:

$$(6)_k \quad 3b+2d+e\leqslant 4c+4.$$

Proof. Assume $Z=Z(b,c,d,e,f,r,k)$ comes from $Z''=Z(b'',c'',d'',e'',f'',r,k+2)$, which at its turn comes from two steps before.

We have (see $(\circ)$ and $(\circ\circ)$ in proof of Lemma 3.5) $b''=c-d''-2e''$, $c''=2k+4-3b-2c-2d-e-d''+e''$; substituting in $(5)_{k+2}$: $6b''+c''+4d''+2e''\geqslant 2k$, we get: $3b+2d+e\leqslant 4+4c-3d''-9e''\leqslant 4c+4$. □

3.8. Remark. Let $k\geqslant 5$; if $(b,c,d,e,f,r,k)\in\mathscr{A}(k)$ satisfies $(5)_k$ and $(6)_k$, then $c\neq 0$. In fact, if $c=0$, $(5)_k$ and $(6)_k$ give: $2k-1\leqslant 2(3b+2d+e)\leqslant 8$, that is, $2k\leqslant 9$.

3.9. Proposition. *For all $k\geqslant 10$, $\mathbf{B}(k)$ is true.*

Proof. First of all, notice that the schemes $T(k)$ of Section 1 are the schemes $Z(0, 0, 0, 0, q(k), r(k), k)$. In the following we denote by $B(b,c,d,e,f,r,k)$ the statement:

$$(b,c,d,e,f,r,k)\in\mathscr{A}(k), \quad \text{and} \quad H^0(\mathscr{E}_{k+1}\otimes\mathscr{I}_{Z(b,c,d,e,f,r,k)})=0,$$

hence $\mathbf{B}(k)$ is $B(0,0,0,0,q(k),r(k),k)$.

In Sections 4 and 5 (see 4.4, 5.1 and 5.2) we will prove that, if $k=6$ or $k=7$, $B(b,c,d,e,f,r,k)$ is true when (b,c,d,e,f,r,k) satisfies $(4)_k$, $(5)_k$ and $(6)_k$, or when $Z(b,c,d,e,f,r,k)$ comes in two steps from one of the following schemes: $T(10), T(11)$.

The last fact, together with Proposition 3.3, says that **B**(10) and **B**(11) are true; the first means (Lemmata 3.5, 3.6, and 3.7) that, if $k=6$ or $k=7$, $B(b,c,d,e,f,r,k)$ is true for all $Z(b,c,d,e,f,r,k)$ coming from (at least) three steps before (and maybe for some others).

Now we prove $B(b,c,d,e,f,r,k)$ for $k \geqslant 12$ by induction on k. Assume that the statement is true for k; then, by Proposition 3.3, it is true for $k+2$, hence it is enough to prove the initial cases $B(b,c,d,e,f,r,12)$ and $B(b,c,d,e,f,r,13)$.

For $B(b,c,d,e,f,r,13)$ it is enough to apply 3 times Proposition 3.3, and to use the fact that $B(b,c,d,e,f,r,7)$ is true for all $Z(b,c,d,e,f,r,7)$ coming from three steps before.

For $B(b,c,d,e,f,r,12)$ we do the same. This is possible since the only exception, $(0,0,0,0,6,8,8)$, does not come from two steps before by Remark 3.6. □

4. Initial cases for k even

We set $\mathscr{S}(6) := \{(b,c,d,e,f,r,6) \in \mathscr{A}(6) \mid (b,c,d,e,f,r,6)$ satisfies conditions $(4)_6$, $(5)_6$ and $(6)_6\}$. In this section we wish to prove that $H^0(\mathscr{E}_7 \otimes \mathscr{I}_{Z(b,c,d,e,f,r,6)}) = 0$ for $(b,c,d,e,f,r,6) \in \mathscr{S}(6)$, and also for $Z(b,c,d,e,f,r,6)$ coming from $T(10)$ in two steps.

We recall that $\mathscr{S}(6)$ is the set of $(b,c,d,e,f,r,6)$ such that $c \neq 0$ (see Remark 3.8) and such that the following hold:

$(0)_6$ $\quad 2b+c+3d+3e \leqslant 12,\ 0 \leqslant d+e \leqslant 1,$

$(1)_6$ $\quad 2(3b+c+4d+5e+6f)+r=48,$

$(2)_6$ $\quad r=0$ or $r=8,$

$(4)_6$ $\quad 3b+2c+2d+e \leqslant 17,$

$(5)_6$ $\quad 6b+c+4d+2e \geqslant 8,$

$(6)_6$ $\quad 3b-4c+2d+e \leqslant 4.$

Then it is elementary, but tedious, to check that

$$\begin{aligned}\mathscr{S}(6) = \{&(0,7,0,1,2,0,6),(1,3,0,0,3,0,6),(1,4,0,1,2,0,6),(1,5,1,0,2,0,6),\\ &(2,1,0,1,2,0,6),(2,2,1,0,2,0,6),(3,3,0,0,2,0,6),(0,4,1,0,2,8,6),\\ &(0,8,0,0,2,8,6),(1,1,1,0,2,8,6),(1,5,0,0,2,8,6),(1,6,0,1,1,8,6),\\ &(2,2,0,0,2,8,6),(2,3,0,1,1,8,6),(2,4,1,0,1,8,6),(4,2,0,0,1,8,6)\}.\end{aligned}$$

On the other hand, it is immediate to see, using Proposition 3.3, that if $Z(b,c,d,e,f,r,6)$ comes from $T(10)=Z(0,0,0,0,10,0,10)$ in two steps, then $(b,c,d,e,f,r,6)=(0,7,0,1,2,0,6)$, which already belongs to $\mathscr{S}(6)$.

So let $(b,c,d,e,f,r,6) \in \mathscr{S}(6)$; in the following we prove, by techniques and notations analogous to the ones in the proof of Proposition 3.3, that $H^0(\mathscr{E}_7 \otimes \mathscr{I}_{Z(b,c,d,e,f,r,6)}) = 0$.

4.1. Proposition. *Let* $(b,c,d,e,f,r,6) \in \mathscr{S}(6)$, *let* $g,h,i \geqslant 0$ *be defined as before by relation* $(3)_6$, *and assume*:

$(*)$ $f = g + h + i$,
$(**)$ $1 \leqslant f + i \leqslant 3$, *if* $r = 8, f + i = 4$, *if* $r = 0$.

Then, $H^0(\mathscr{E}_7 \otimes \mathscr{I}_{Z(b,c,d,e,f,r,6)}) = 0$.

Proof. We do the same HD step as in the general case (see proof of Proposition 3.3), described by the following table:

c.n.o.C	What we have over C	What we "add" over C
12	$b\begin{pmatrix}1\\ [2]\end{pmatrix} + c([1]) + d\begin{pmatrix}1\\ [3]\end{pmatrix} + e\begin{pmatrix}2\\ [3]\end{pmatrix}$	$g\begin{pmatrix}1\\ 2\\ [3]\end{pmatrix} + h\begin{pmatrix}1\\ [2]\\ 3\end{pmatrix} + i\begin{pmatrix}[1]\\ 2\\ 3\end{pmatrix}$

Condition $(*)$ assures that this can be done. Hence, exactly as in the general case, the proposition is true provided that the HD residue Z', consisting of $b(1) + d(1) + e(2) + g\binom{1}{2} + h\binom{1}{3} + i\binom{2}{3}$ over C, and of T_r general over $\mathbb{P}^2$, satisfies $H^0(\mathscr{E}_5 \otimes \mathscr{I}_{Z'}) = 0$.

We set $a := \frac{1}{2}\mathit{length}(Z' \cap H) = b + d + 2e + 2g + 3h + 3i$. As in proof of Proposition 3.3, Condition $(1)_4$ is automatically satisfied by the integers of Z', that is, we have:

$$a + (g + h + 2i) = b + d + 2e + 3g + 4h + 5i = 12 - \frac{r}{2},$$

hence, since by $(*)$ $f + i = g + h + 2i$, by $(**)$ we get:

if $r = 0, a = 8$ and $g + h + 2i = 4$,
if $r = 8, 5 \leqslant a \leqslant 7$ and $g + h + 2i = 8 - a$.

Recall that if $r = 0, T_r = \emptyset$, and if $r = 8$, we have the choice between $T_r = \binom{1}{3}$ or $T_r = \binom{2}{2}$.

Now we do another HD step, "adding" on C respectively:

$$\emptyset \quad \text{if } r = 0,\ a = 8 \quad \text{and} \quad g + h + 2i = 4,$$

$$\begin{pmatrix}[1]\\ 3\end{pmatrix} \quad \text{if } r = 8,\ a = 7 \quad \text{and} \quad g + h + 2i = 1,$$

$$\begin{pmatrix} 2 \\ [2] \end{pmatrix} \quad \text{if } r=8,\ a=6 \quad \text{and} \quad g+h+2i=2,$$

$$\begin{pmatrix} 1 \\ [3] \end{pmatrix} \quad \text{if } r=8,\ a=5 \quad \text{and} \quad g+h+2i=3,$$

Since a plus the length of the Tr^{HD} of the scheme we have just added is 8, then by Lemma 2.2 and by [26] (1.2)iv) one has $H^0(\mathscr{I}_{Tr^{HD}Z',H} \otimes \mathscr{E}_{5|H}) = 0$.

Moreover, since $g+h+2i$ plus the length of the Res^{HD} of the scheme we have added is 4, then by the remark below $H^0(\mathscr{I}_{Res^{HD}Z'} \otimes \mathscr{E}_3) = 0$, hence Proposition 2.6 gives $H^0(\mathscr{I}_{Z'} \otimes \mathscr{E}_5) = 0$. □

4.2. Remark. If M is a 0-dimensional curvilinear scheme of length 4 contained in a smooth conic $C \subset \mathbb{P}^2$, then $H^0(\mathscr{E}_3 \otimes \mathscr{I}_{\hat{M}})=0$. This follows by Lemma 2.2 and [26] (see proof of $A(4)$ in 2.2 there, for M reduced; the proof holds also for M non reduced).

4.3. Corollary. *Let* $(b,c,d,e,f,r,6) \in \mathscr{S}(6)$ *with* $r=8$ *or* $(b,c,d,e,f,r,6) = (1,3,0,0,3,0,6)$; *then*, $H^0(\mathscr{E}_7 \otimes \mathscr{I}_{Z(b,c,d,e,f,r,6)}) = 0$.

Proof. Just check that $(*)$ and $(**)$ of Proposition 4.1 hold.

4.4. Propostion. *Let* $(b,c,d,e,f,r,6) \in \mathscr{S}(6)$; *then*, $H^0(\mathscr{E}_7 \otimes \mathscr{I}_{Z(b,c,d,e,f,r,6)}) = 0$.

Proof. Since Corollary 4.3 holds, we are only left to prove the remaining cases:

$$(0,7,0,1,2,0,6),(1,4,0,1,2,0,6),(1,5,1,0,2,0,6),(2,1,0,1,2,0,6),$$
$$(2,2,1,0,2,0,6),(3,3,0,0,2,0,6).$$

We prove each case separately, and the proofs are described by tables, according to the conventions used in the proof of Proposition 3.3. Each table consists of two lines, that is, of two Horace steps; since in each case, denoting by $\hat{M}$ the HD residue of the last step, M consists of a 0-dimensional curvilinear scheme of $\mathbb{P}^2$ of length 4 contained in a smooth conic, we are done since $H^0(\mathscr{E}_3 \otimes \mathscr{I}_{\hat{M}}) = 0$ (see Remark 4.2).

In every proof we use either the conic C or another conic C' meeting C transversally at four points. In the latter case, we will always set $A := C \cap C'$, and the notations $()_A$ mean that the support of the scheme is over A.

Notice that a scheme $\begin{pmatrix} 1 \\ 3 \end{pmatrix}_A$ over C will appear as a scheme $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}_A$ when viewed over C', and analogously a scheme $\begin{pmatrix} 2 \\ 3 \end{pmatrix}_A$ over C will appear as a scheme $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}_A$ over C'.

Case $(0,7,0,1,2,0,6)$:

c.n.o.C	What we have over C	What we "add" over C	Residue not over C
12	$7([1]) + \begin{pmatrix} 2 \\ [3] \end{pmatrix}$	$2\begin{pmatrix} [1] \\ 2 \\ 3 \end{pmatrix}$	
8	$([2]) + 2\begin{pmatrix} 2 \\ [3] \end{pmatrix}$		

Case $(1,4,0,1,2,0,6)$:

c.n.o.C'	What we have over C'	What we "add" over C'	Residue not over C'
12		$\begin{pmatrix} 1 \\ [2] \end{pmatrix}_A + \begin{pmatrix} 1 \\ 2 \\ [2] \end{pmatrix}_A$	2(1) over C
		$+2([1])_A + 2\begin{pmatrix} 1 \\ 2 \\ [3] \end{pmatrix}$	
8	$([1])_A + \begin{pmatrix} 1 \\ [2] \end{pmatrix}_A + 2\begin{pmatrix} 1 \\ [2] \end{pmatrix}$	$([1])_A$	(1) over C

Case $(1,5,1,0,2,0,6)$:

c.n.o.C	What we have over C	What we "add" over C	Residue not over C
12	$\begin{pmatrix} 1 \\ [2] \end{pmatrix} + 5([1]) + \begin{pmatrix} 1 \\ [3] \end{pmatrix}$	$2\begin{pmatrix} [1] \\ 2 \\ 3 \end{pmatrix}$	
8	$([1]) + ([1]) + 2\begin{pmatrix} 2 \\ [3] \end{pmatrix}$		

Case $(2,1,0,1,2,0,6)$:

c.n.o.C'	What we have over C'	What we "add" over C'	Residue not over C'
12		$2\begin{pmatrix}1\\2\\ [3]\end{pmatrix}+2\begin{pmatrix}1\\ [2]\end{pmatrix}_A+\begin{pmatrix}1\\2\\ [2]\end{pmatrix}_A$	(1) over C
8	$2\begin{pmatrix}1\\ [2]\end{pmatrix}+2([1])_A+\begin{pmatrix}1\\ [2]\end{pmatrix}_A$		(1) over C

Case $(2,2,1,0,2,0,6)$:

c.n.o.C'	What we have over C'	What we "add" over C'	Residue not over C'
12		$\begin{pmatrix}1\\1\\ [2]\end{pmatrix}_A+\begin{pmatrix}1\\ [2]\end{pmatrix}_A+2([1])_A+2\begin{pmatrix}1\\2\\ [3]\end{pmatrix}$	$\begin{pmatrix}1\\2\end{pmatrix}$ over C
8	$\begin{pmatrix}1\\ [1]\end{pmatrix}_A+([1])_A+2\begin{pmatrix}1\\ [2]\end{pmatrix}$	$\begin{pmatrix}1\\ [2]\end{pmatrix}_A$	

Case $(3,3,0,0,2,0,6)$:

c.n.o.C'	What we have over C'	What we "add" over C'	Residue not over C'
12		$2\begin{pmatrix}1\\ [2]\end{pmatrix}_A+2([1])_A+2\begin{pmatrix}1\\2\\ [3]\end{pmatrix}$	$\begin{pmatrix}1\\2\end{pmatrix}+(1)$ over C
8	$2([1])_A+2\begin{pmatrix}1\\ [2]\end{pmatrix}$	$\begin{pmatrix}1\\ [2]\end{pmatrix}_A$	(1) over C

□

5. Initial cases for k odd

In order to complete the induction for k odd, we should prove all the initial cases with $k=7$, i.e. that $H^0(\mathcal{I}_Z\otimes\mathcal{E}_8)=0$ for all the admissible schemes $Z=Z(b,c,d,e,f,r,7)$.

Recall that "Z is admissible" means:

$(0)_7 \quad 2b + c + 3d + 3e \leqslant 14, \quad d + e \leqslant 1,$

$(1)_7 \quad 12f = 63 - r - 36b - 6c - 24d - 30e,$

$(2)_7 \quad r = 3 \quad \text{or} \quad r = 11.$

On the other hand, Lemmata 3.5, 3.6, 3.7 and Remark 3.8 show that if Z is needed for the induction in order to prove $H^0(\mathscr{I}_{Z''} \otimes \mathscr{E}_{k+1}) = 0$, for $k \geqslant 13$ and some admissible scheme Z'', then we have also:

$(4)_7 \quad 3b + 2c + 2d + e \leqslant 19,$

$(5)_7 \quad 6b + c + 4d + 2e \geqslant 10,$

$(6)_7 \quad 3b + 2d + e \leqslant 4c + 4.$

Again, it is elementary, but tedious, to check that the $(b, c, d, e, f, r, 7)$'s satisfying $(0)_7, (1)_7, (2)_7, (4)_7, (5)_7, (6)_7$ are the elements of the following set:

$$\begin{aligned}\mathscr{S}(7) = \{&(0,8,1,0,3,3,7), (1,4,0,1,3,3,7), (1,5,1,0,3,3,7), (2,1,0,1,3,3,7),\\ &(2,2,1,0,3,3,7), (2,6,0,0,3,3,7), (3,3,0,0,3,3,7), (3,4,0,1,2,3,7),\\ &(1,1,1,0,3,11,7), (0,9,0,1,2,11,7), (1,5,0,0,3,11,7), (1,6,0,1,2,11,7),\\ &(1,7,1,0,2,11,7), (2,2,0,0,3,11,7), (2,3,0,1,2,11,7), (2,4,1,0,2,11,7),\\ &(3,5,0,0,2,11,7), (4,2,0,0,2,11,7), (4,3,0,1,1,11,7)\}.\end{aligned}$$

Moreover, a straightforward computation (see Proposition 3.3) shows that if $Z(b, c, d, e, f, r, 7)$ comes from $T(11)$ in two steps, then $(b, c, d, e, f, r, 7) = (0, 9, 0, 1, 2, 11, 7)$, which already belongs to $\mathscr{S}(7)$.

In the following Lemmata 5.1, 5.2 we prove that $H^0(\mathscr{E}_8 \otimes \mathscr{I}_{Z(b,c,d,e,f,r,7)}) = 0$ for all $(b, c, d, e, f, r, 7) \in \mathscr{S}(7)$.

5.1. Lemma. *Let* $(b, c, d, e, f, r, 7) \in \mathscr{S}(7)$ *with* $r = 3$; *then*, $H^0(\mathscr{E}_8 \otimes \mathscr{I}_{Z(b,c,d,e,f,r,7)}) = 0$.

Proof. The following table, in the manner of what we did in Section 3, illustrates a procedure that gives the required proof in almost all cases with $r = 3$.

In the second step we set $w = f - g - h - i - g' - h' - i'$.

c.n.o.C	What we have over C	What we “add” over C	Residue not over C
14	$b\begin{pmatrix}1\\ [2]\end{pmatrix} + c + d\begin{pmatrix}1\\ [3]\end{pmatrix} + e\begin{pmatrix}2\\ [3]\end{pmatrix}$	$g\begin{pmatrix}1\\ 2\\ [3]\end{pmatrix} + h\begin{pmatrix}1\\ [2]\\ 3\end{pmatrix} + i\begin{pmatrix}[1]\\ 2\\ 3\end{pmatrix}$	$(f-g-h-i)\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} + T_3$
10	$b + d + 2e + g\begin{pmatrix}1\\ [2]\end{pmatrix} + h\begin{pmatrix}1\\ [3]\end{pmatrix} + i\begin{pmatrix}2\\ [3]\end{pmatrix}$	$g'\begin{pmatrix}1\\ 2\\ [3]\end{pmatrix} + h'\begin{pmatrix}1\\ [2]\\ 3\end{pmatrix} + i'\begin{pmatrix}[1]\\ 2\\ 3\end{pmatrix}$	$w\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} + T_3$
6	$g + h + 2i + g'\begin{pmatrix}1\\ [2]\end{pmatrix} + h'\begin{pmatrix}1\\ [3]\end{pmatrix} + i'\begin{pmatrix}2\\ [3]\end{pmatrix}$	T_3	$w\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$

This procedure is possible whenever:

(†) in the first step: $f \geqslant g + h + i$,
(#) in the second step: $b + d + 2e + 2g + 3h + 3i \leqslant 10$;
and we will set

$$10 - (b + d + 2e + 2g + 3h + 3i) = 3g' + 2h' + i',$$

where $g', h', i' \geqslant 0$, with $h' + i' \leqslant 1$.

Then we require that there are no more 3-fat points left in the last step, i.e. $w = 0$, in other words:

$$(\#\#) \quad g' + h' + i' = f - g - h - i.$$

Notice that (##), with the requirement $g', h', i' \geqslant 0$, implies (†).

In the third step, we need:

$$(\ddagger)\quad g+h+2i+2g'+3h'+3i'=5$$

so that, specializing T_3 over C (i.e. specializing the support of the projection $\pi(T_3)$ on C) in such a way that η_2 (see Definition 1.3) is not tangent to $H=\pi^{-1}(C)$, we get six conditions over C and the residue R is given by $g'+h'+2i'$ points over C plus the residual of T_3, i.e. a point $T\in H=\pi^{-1}(C)$. Since at each step, relative to $\mathscr{E}_{q+1}$, the length of the residual scheme is $h^0(\mathscr{E}_{q-1})$, we know that the length of this residual scheme R is $3=h^0(\mathscr{E}_2)$, so $R=T\cup\hat{P}$, where $P\in C\subset\mathbb{P}^2$.

Hence the last step is proving that $H^0(\mathscr{E}_2\otimes\mathscr{I}_R)=0$; this is statement $A(1)$ in [26, Lemma 2.2].

So this proof works whenever (#), (##) and (‡) hold; it is immediate to check that these conditions are satisfied except for $(b,c,d,e,f,3,7)=(0,8,1,0,3,3,7)$.

In this case we proceed with an *ad hoc* construction as follows (the specialization of T_3 in the last step and the conclusion are as in the general case):

Case $(0,8,1,0,3,3,7)$:

c.n.o.C	What we have over C	What we "add" over C	Residue not over C
14	$8([1])+\begin{pmatrix}1\\ [3]\end{pmatrix}$	$\begin{pmatrix}[1]\\ 2\\ 3\end{pmatrix}+\begin{pmatrix}1\\ [2]\\ 3\end{pmatrix}$	$\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}+T_3$
10	$([1])+\begin{pmatrix}1\\ [3]\end{pmatrix}+\begin{pmatrix}2\\ [3]\end{pmatrix}$	$\begin{pmatrix}1\\ 2\\ [3]\end{pmatrix}$	T_3
6	$([1])+([2])+\begin{pmatrix}1\\ [2]\end{pmatrix}$	T_3	

□

5.2. Lemma. *Let* $(b,c,d,e,f,r,7)\in\mathscr{S}(7)$ *with* $r=11$; *then*, $H^0(\mathscr{E}_8\otimes\mathscr{I}_{Z(b,c,d,e,f,r,7)})=0$.

Proof. We are unfortunately obliged to give specific procedures for each case in $\mathscr{S}(7)$ with $r=11$.

We will always write $\begin{pmatrix}1/0\\ 2\\ 3\end{pmatrix}$ for T_{11}, with obvious meaning, see its description in Section 1. In every proof we can use either the conic C or C and another conic C'; in the latter case we will always set $A=C\cap C'$, as in 5.1. In all cases the last step is as in 5.1, general case, and the last residue will always be of type $(1/0)+(1)$, i.e. $T\cup\hat{P}$, where $T\in H=\pi^{-1}(C)$ and $P\in C\subset\mathbb{P}^2$, as required.

Case $(0,9,0,1,2,11,7)$:

Needed over which conic	Already on that conic	What we "add" over that conic
14 over C	$9([1]) + \begin{pmatrix} 2 \\ [3] \end{pmatrix}$	$\begin{pmatrix} 1 \\ [2] \\ 3 \end{pmatrix}$
10 over C	$([2]) + \begin{pmatrix} 1 \\ [3] \end{pmatrix}$	$\begin{pmatrix} 1/0 \\ 2 \\ [3] \end{pmatrix} + \begin{pmatrix} 1 \\ [2] \\ 3 \end{pmatrix}$
6 over C	$([1]) + \begin{pmatrix} 1 \\ [3] \end{pmatrix} + \begin{pmatrix} 1/0 \\ [2] \end{pmatrix}$	

Case $(1,5,0,0,3,11,7)$:

Needed over which conic	Already on that conic	What we "add" over that conic
14 over C	$\begin{pmatrix} 1 \\ [2] \end{pmatrix} + 5([1])$	$2\begin{pmatrix} 1 \\ 2 \\ [3] \end{pmatrix} + \begin{pmatrix} [1] \\ 2 \\ 3 \end{pmatrix}$
10 over C'		$([1])_A + 2\begin{pmatrix} 1 \\ [2] \end{pmatrix}_A + \begin{pmatrix} 1 \\ 2 \\ [2] \end{pmatrix}_A + \begin{pmatrix} 1/0 \\ 2 \\ [3] \end{pmatrix}$
6 over C'	$2([1])_A + \begin{pmatrix} 1 \\ [2] \end{pmatrix}_A + \begin{pmatrix} 1/0 \\ [2] \end{pmatrix}$	

Case $(1,6,0,1,2,11,7)$:

Needed over which conic	Already on that conic	What we "add" over that conic
14 over C	$\begin{pmatrix} 1 \\ [2] \end{pmatrix} + 6([1]) + \begin{pmatrix} 2 \\ [3] \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \\ [3] \end{pmatrix}$
10 over C	$([1]) + ([2]) + \begin{pmatrix} 1 \\ [2] \end{pmatrix}$	$\begin{pmatrix} 1/0 \\ 2 \\ [3] \end{pmatrix} + \begin{pmatrix} 1 \\ [2] \\ 3 \end{pmatrix}$
6 over C	$([1]) + \begin{pmatrix} 1 \\ [3] \end{pmatrix} + \begin{pmatrix} 1/0 \\ [2] \end{pmatrix}$	

Case $(1,7,1,0,2,11,7)$:

Needed over which conic	Already on that conic	What we “add” over that conic
14 over C	$\begin{pmatrix}1\\ {[2]}\end{pmatrix}+7([1])+\begin{pmatrix}1\\ {[3]}\end{pmatrix}$	$\begin{pmatrix}1\\ {[2]}\\ 3\end{pmatrix}$
10 over C	$([1])+([1])+\begin{pmatrix}1\\ {[3]}\end{pmatrix}$	$\begin{pmatrix}1/0\\ 2\\ {[3]}\end{pmatrix}+\begin{pmatrix}1\\ {[2]}\\ 3\end{pmatrix}$
6 over C	$([1])+\begin{pmatrix}1\\ {[3]}\end{pmatrix}+\begin{pmatrix}1/0\\ {[2]}\end{pmatrix}$	

Case $(2,2,0,0,3,11,7)$:

Needed over which conic	Already on that conic	What we “add” over that conic
14 over C'		$2\begin{pmatrix}1\\ {[2]}\end{pmatrix}_A+([1])_A+3\begin{pmatrix}1\\ 2\\ {[3]}\end{pmatrix}$
10 over C	$([1])+2([1])_A$	$2\begin{pmatrix}1\\ {[2]}\end{pmatrix}_A+\begin{pmatrix}1/0\\ 2\\ {[3]}\end{pmatrix}$
6 over C	$2([1])_A+\begin{pmatrix}1/0\\ {[2]}\end{pmatrix}$	$\begin{pmatrix}1\\ {[2]}\end{pmatrix}_A$

Case $(2,3,0,1,2,11,7)$:

Needed over which conic	Already on that conic	What we “add” over that conic
14 over C'		$2\begin{pmatrix}1\\ {[2]}\end{pmatrix}_A+([1])_A+2\begin{pmatrix}1\\ 2\\ {[3]}\end{pmatrix}+\begin{pmatrix}1/0\\ 2\\ {[3]}\end{pmatrix}$
10 over C'	$2([1])_A+2\begin{pmatrix}1\\ {[2]}\end{pmatrix}+\begin{pmatrix}1/0\\ {[2]}\end{pmatrix}$	$\begin{pmatrix}1\\ 2\\ {[2]}\end{pmatrix}_A$
6 over C	$2([1])+\begin{pmatrix}1\\ {[2]}\end{pmatrix}_A$	$2([1])_A$

Notice that for the last step it remains a (1/0) over C' and a (1) over C.

Case $(2,4,1,0,2,11,7)$:

Needed over which conic	Already on that conic	What we "add" over that conic
14 over C	$2\begin{pmatrix}1\\ [2]\end{pmatrix}+4([1])+\begin{pmatrix}1\\ [3]\end{pmatrix}$	$\begin{pmatrix}1\\ 2\\ [3]\end{pmatrix}$
10 over C	$2([1])+([1])+\begin{pmatrix}1\\ [2]\end{pmatrix}$	$\begin{pmatrix}1\\ [2]\\ 3\end{pmatrix}+\begin{pmatrix}1/0\\ 2\\ [3]\end{pmatrix}$
6 over C	$([1])+\begin{pmatrix}1\\ [3]\end{pmatrix}+\begin{pmatrix}1/0\\ [2]\end{pmatrix}$	

Case $(3,5,0,0,2,11,7)$:

Needed over which conic	Already on that conic	What we "add" over that conic
14 over C	$3\begin{pmatrix}1\\ [2]\end{pmatrix}+5([1])$	$\begin{pmatrix}1\\ 2\\ [3]\end{pmatrix}$
10 over C	$3([1])+\begin{pmatrix}1\\ [2]\end{pmatrix}$	$\begin{pmatrix}1\\ [2]\\ 3\end{pmatrix}+\begin{pmatrix}1/0\\ 2\\ [3]\end{pmatrix}$
6 over C	$([1])+\begin{pmatrix}1/0\\ [2]\end{pmatrix}+\begin{pmatrix}1\\ [3]\end{pmatrix}$	

Case $(4,2,0,0,2,11,7)$:

Needed over which conic	Already on that conic	What we "add" over that conic
14 over C'		$4\begin{pmatrix}1\\ [2]\end{pmatrix}_A+\begin{pmatrix}1\\ 2\\ [3]\end{pmatrix}+\begin{pmatrix}1/0\\ 2\\ [3]\end{pmatrix}$
10 over C	$3([1])_A+\begin{pmatrix}1\\ 1\\ [2]\end{pmatrix}_A+2([1])$	$\begin{pmatrix}1\\ 2\\ [3]\end{pmatrix}$
6 over C'	$([2])_A+\begin{pmatrix}1/0\\ [2]\end{pmatrix}$	$\begin{pmatrix}1\\ [2]\end{pmatrix}_A$

Here in the second step we made a collision of a $\binom{1}{2}$ over C' on a $(1)_A$, so we got a $\binom{1}{3}_A$ over C', which is a $\begin{pmatrix}1\\1\\2\end{pmatrix}_A$ over C. The residue of such a scheme is a $\binom{1}{1}_A$ over C, which in turn gives a $(2)_A$ over C' in the last step.

Case $(4,3,0,1,1,11,7)$:

Needed over which conic	Already on that conic	What we "add" over that conic
14 over C	$4\begin{pmatrix}1\\ [2]\end{pmatrix}+3([1])+\begin{pmatrix}2\\ [3]\end{pmatrix}$	
10 over C'	$4([1])_A$	$\begin{pmatrix}1\\2\\ [3]\end{pmatrix}+\begin{pmatrix}1/0\\2\\ [3]\end{pmatrix}$
6 over Γ	$\begin{pmatrix}1\\ [2]\end{pmatrix}+\begin{pmatrix}1/0\\ [2]\end{pmatrix}$	$([2])_A$

In the last step we specialize C' to a conic Γ which is tangent to C in one point, so that A is of type $2(1)+(2)$ over both C and Γ, and we can specialize the residue (2) over C to $(2)_A$ over Γ.

Case $(1,1,1,0,3,11,7)$:

Needed over which conic	Already on that conic	What we "add" over that conic
14 over C	$\begin{pmatrix}1\\ [2]\end{pmatrix}+([1])+\begin{pmatrix}1\\ [3]\end{pmatrix}$	$2\begin{pmatrix}1\\2\\ [3]\end{pmatrix}+\begin{pmatrix}1\\ [2]\\3\end{pmatrix}$
10 over C'		$([1])_A+\begin{pmatrix}1\\1\\ [2]\end{pmatrix}_A+2\begin{pmatrix}1\\ [2]\end{pmatrix}_A+\begin{pmatrix}1/0\\2\\ [3]\end{pmatrix}$
6 over C'	$2([1])_A+\begin{pmatrix}1\\ [1]\end{pmatrix}_A+\begin{pmatrix}1/0\\ [2]\end{pmatrix}$	$([1])_A$

□

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