THE HILBERT FUNCTION OF A REDUCED K-ALGEBRA

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Introduction

Let $A = \bigoplus_{i \geq 0} A_i$ be a graded k-algebra of finite type (where k is a field, $A_0 = k$ and A is generated as a k-algebra by A_1). Such algebras have been called *standard* G-algebras by Stanley [7]. They are those k-algebras which can be written in the form $A = k[X_0, \ldots, X_n]/I$, where the X_i are indeterminates of degree 1 and I is a homogeneous ideal. The Hilbert function $\{b_i\}$, $i \geq 0$, of A is defined by $b_i = \dim_k A_i$. Macaulay [4] characterized the Hilbert function of a standard G-algebra. In this paper we characterize the Hilbert function of a reduced standard G-algebra.

Our proof is based on properties of points in 'generic position' on an algebraic variety and on a partition of a differentiable 0-sequence into two differentiable pieces. These techniques are of some interest in their own right.

A preliminary version of some of the results in this paper can be found in the preprints [5, 6].

1. *Review of* 0-*sequences*

Let h and i be positive integers. Then h can be written uniquely in the form

$$h = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \ldots + \binom{m_j}{j} \tag{1.1}$$

where $m_i > m_{i-1} > \ldots > m_j \geq j \geq 1$ [3]. This expression for h is called the *i-binomial expansion* of h.

Also, define

$$h^{\langle i \rangle} = \binom{m_i+1}{i+1} + \binom{m_{i-1}+1}{i} + \ldots + \binom{m_j+1}{j+1} \tag{1.2}$$

and $0^{\langle i \rangle} = 0$.

DEFINITION 1.3. A sequence of non-negative integers $\{c_i\}$, $i \geq 0$, is called an 0-*sequence* if $c_0 = 1$ and $c_{i+1} \leq c_i^{\langle i \rangle}$ for all $i \geq 1$.

Macaulay's theorem (as expressed in modern form by Stanley [7, Theorem 2.2]) is as follows.

THEOREM 1.4. *The following are equivalent:*

(a) $\{c_i\}$, $i \geq 0$, *is an* 0-*sequence;*

(b) $\{c_i\}$, $i \geq 0$, *is the Hilbert function of a standard* G-*algebra.*

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A useful way to think about the function $h^{\langle i\rangle}$ is as follows. First write down Pascal's triangle

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & . & . & . \\ 1 & \mathbf{2} & 3 & 4 & 5 & . & . & . \\ 1 & 3 & \mathbf{6} & \mathbf{10} & 15 & . & . & . \\ 1 & 4 & 10 & 20 & 35 & . & . & . \\ 1 & 5 & 15 & 35 & 70 & . & . & . \\ & & & & \vdots & & & \end{array} \tag{1.5}$$

(best written in rectangular form).

The rows and columns are both numbered beginning with 0. The entry in the i-th row and j-th column is $\binom{i+j}{j}$. The term $\binom{m_i}{i}$ in the i-binomial expansion of h is the largest entry not exceeding h in the i-th column. Write $h = \binom{m_i}{i}+r$ with $r \geqslant 0$. If $r = 0$ we are done. Otherwise, repeat the process with r and column $i-1$. Ultimately the i-binomial expansion of h is achieved. The terms in the i-binomial expansion of h occur in consecutive columns $i, i-1, \ldots, j$, where $i \geqslant j \geqslant 1$. If l is chosen so that $j \leqslant l < i$ and the $l, l+1$ terms lie, respectively, in rows r_l, r_{l+1}, then $r_l \leqslant r_{l+1}$. Shifting each term to the right and adding yields $h^{\langle i\rangle}$.

For example, the bold numbers in the above array are the terms of the 3-binomial expansion of 18 and so $18^{\langle 3\rangle} = 15+10+3 = 28$.

Using the description above it is easy to verify the following properties of $h^{\langle i\rangle}$.

THEOREM 1.6. (a) *Let α, β, i be positive integers with $\alpha < \beta$. Then $\alpha^{\langle i\rangle} < \beta^{\langle i\rangle}$.*

(b) *Let $\{b_i\}$, $i \geqslant 0$, be an 0-sequence with $b_1 = m$. Then*

$$b_{i+1} \leqslant \binom{m+i}{i+1} \qquad \textit{for every } i \geqslant 0.$$

(c) *Let $h = \binom{(n-1)+i}{i} - 1$. Then*

$$h^{\langle i\rangle} = \binom{n+i}{i+1} - n.$$

Now suppose that A is a reduced standard G-algebra, with k infinite, and Hilbert function $\{b_i\}$, $i \geqslant 0$. Then A contains a non-zerodivisor x of degree 1 and A/xA has Hilbert function $\{1, b_1-1, b_2-b_1, \ldots\}$. This motivates the following.

DEFINITION 1.7. Let $\{b_i\}$, $i \geqslant 0$, be an 0-sequence. Then $\{b_i\}$ is *differentiable* if the difference sequence $\{c_i\}$, $c_i = b_i - b_{i-1}$, is again an 0-sequence. (We adopt the convention that $b_{-1} = 0$.)

Thus, a necessary condition that $\{b_i\}$ be the Hilbert function of a reduced standard G-algebra (for k infinite) is that $\{b_i\}$ be differentiable. We shall prove in §3 that this necessary condition is also sufficient.

For simplicity in the exposition we introduce the following definition.

DEFINITION 1.8. The 0-sequence $\{b_i\}$, $i \geqslant 0$, has *dimension* d if there is a polynomial $f(x)$, with rational coefficients, of degree d with the property that for all $n >> 0$, $f(n) = b_n$.

The fact that every 0-sequence has a dimension is an immediate consequence of Macaulay's theorem (1.4 above) and a theorem of Hilbert (see for example [8, p. 232, Theorem 41]).

2. *Points in generic position on an algebraic variety*

Let I be a homogeneous ideal in $k[x_0, \ldots, x_n] = R$ and let $A = R/I$. Then $X = \mathrm{Proj}(A) = V(I)$ is a closed subscheme of $\mathbb{P}^n_k$. The Hilbert function of A will sometimes be written $\{H_A(i)\}$ or $\{H_X(i)\}$. The map $I \to V(I)$ gives a one to one correspondence between homogeneous radical ideals in R and reduced closed subschemes of $\mathbb{P}^n_k$. Reduced closed subschemes of $\mathbb{P}^n_k$ will be referred to as algebraic varieties. If X is an algebraic variety, the corresponding ideal will be denoted by $I(X)$. A closed subset of $\mathbb{P}^n_k$ will be given its reduced subscheme structure unless stated otherwise.

Let I and J be two homogeneous ideals of R. Then we have an exact sequence

$$0 \longrightarrow R/I \cap J \longrightarrow (R/I) \oplus (R/J) \longrightarrow R/(I+J) \longrightarrow 0. \tag{2.1}$$

For $r, s \in R$, the maps are $\bar{r} \to (\bar{r}, \bar{r})$ and $(\bar{r}, \bar{s}) \to \bar{r} - \bar{s}$, the bars denoting the class in the appropriate quotient ring. Suppose that $R/I \cap J$, R/I, R/J and $R/(I+J)$ have Hilbert functions $\{b_i\}$, $\{c_i\}$, $\{d_i\}$ and $\{e_i\}$ respectively. Using the fact that if $e_s = 0$ then $e_i = 0$ for $i \geqslant s$, we obtain the following.

LEMMA 2.2. *With the above notation,*

$$c_i + d_i = b_i + e_i \qquad \text{for } i \geqslant 0.$$

If $e_s = 0$ then $b_i = c_i + d_i$ for $i \geqslant s$.

The assertion that $e_s = 0$ for some s is equivalent to $V(I) \cap V(J) = \varnothing$ which, in turn, is equivalent to $\sqrt{I+J} = (x_0, \ldots, x_n)$.

LEMMA 2.3. *Let $V \subset X$ be two subvarieties of $\mathbb{P}^n_k$.*

(a) *$H_V(i) \leqslant H_X(i)$ for $i \geqslant 0$.*

(b) *Let P be a k-rational point not in V. There is an integer d such that*

$$H_{V \cup \{P\}}(i) = H_V(i) \qquad \text{for } i < d$$

and

$$H_{V \cup \{P\}}(i) = H_V(i) + 1 \qquad \text{for } i \geqslant d.$$

(c) *Let k be algebraically closed and let e be the least integer such that $H_V(e) < H_X(e)$. There exists a k-rational point $P \in X - V$ for which $d = e$ (with d as in* (b) *above).*

Proof. Let $I = I(V)$ and $L = I(X)$. Then (a) holds because there is a surjection $R/L \to R/I$. Let $J = I(P)$. Then (b) is immediate from 2.2 once we observe that $R/J \cong k[t]$, $R/(I+J) \cong k[t]/(t^d)$ for some $d > 0$ and $I(V \cup \{P\}) = I \cap J$. To prove (c), let F be a (non-zero) form of degree e in the kernel of $R/L \to R/I$ and choose $P \in X$ such that $F(P) \neq 0$.

DEFINITION 2.4. The s k-rational points $P_i \in X$ $(1 \leqslant i \leqslant s)$ are in *generic position* on X if the Hilbert function of $V = \bigcup_{i=1}^{s} P_i$ satisfies

$$H_V(i) = \operatorname{Inf}\left(H_X(i), s\right) \qquad \text{for } i \geqslant 0\,.$$

THEOREM 2.5. *There exist s k-rational points in generic position on X in each of the following cases:*

(a) *k is algebraically closed and X contains more than s points;*
(b) *k is infinite and $X = \mathbb{P}_k^n$, $n \geqslant 1$;*
(c) *k is arbitrary and X is a union of $t \geqslant s$ k-rational points.*

Proof. We prove (a) by choosing s k-rational points one after the other as in Lemma 2.3(c). For (c) above note that Lemma 2.3(c) still holds because every point in X is k-rational. Generic position in $\mathbb{P}_k^n$ coincides with the concept of generic s-position introduced in [2], where (b) is proved.

EXAMPLES 2.6. (a) Let $X = \mathbb{P}_k^1$ or let X be a non-singular conic in $\mathbb{P}_k^2$. Then every finite set of k-rational points in X is in generic position on X.

(b) Let X be a rational non-singular quartic in $\mathbb{P}_k^3$ (not contained in a hyperplane). A hyperplane intersects X in four points, which are not in generic position on X.

Note that the curves X in (a), (b) are isomorphic as abstract varieties. Thus the notion of 'generic position on X' depends on the projective embedding of X (as was to be expected).

THEOREM 2.7. *Let X be a subvariety of $\mathbb{P}_k^n$ and let H be a hyperplane of $\mathbb{P}_k^n$ not containing any irreducible component of X. Let V be a subvariety of H such that*

$$H_V(i) = \binom{i+n-1}{n-1} \qquad \text{for } i \leqslant s$$

(that is, generic in H for $i \leqslant s$). Then

$$H_{X \cup V}(i) = H_V(i) + H_X(i-1) \qquad \text{for } i \leqslant s\,.$$

Proof. Let $J = I(X)$, $J' = I(X \cup V)$ and let L be a form of degree 1 defining H. Suppose that G is a form of degree $i \leqslant s$ vanishing on $X \cup V$. By the generic assumption on V, G vanishes on H and so contains L as a factor. Write $G = LF$. Since H does not contain any irreducible component of X, we must have $F \in J_{i-1}$. That is, $J'_i = (LJ)_i = LJ_{i-1}$. Theorem 2.7 now follows from the degree i portion of the exact sequence

$$0 \longrightarrow L/LJ \longrightarrow R/LJ \longrightarrow R/L \longrightarrow 0$$

and the fact that $(L/LJ)_i \cong (R/J)_{i-1}$.

COROLLARY 2.8. *If we maintain the hypotheses and notation of* 2.7 *and assume, moreover, that* $H_X(s-1) = H_X(s)$ (*that is if* k *is algebraically closed then* X *consists of* $H_X(s-1)$ *points*), *then*

$$H_{X \cup V}(i) = H_V(i) + H_X(i-1) \qquad \textit{for } i \leqslant s,$$

and

$$H_{X \cup V}(i) = H_V(i) + H_X(i) \qquad \textit{for } i \geqslant s.$$

Proof. The first conclusion is Theorem 2.7, while the second follows from 2.2 once we observe that

$$H_{X \cup V}(s) = H_V(s) + H_X(s-1) = H_V(s) + H_X(s).$$

3. *The Hilbert function of a reduced k-algebra*

Let $S = \{b_i\}$ be a differentiable 0-sequence, with $b_1 = n+1 \geqslant 2$. We construct, from S, two new sequences S_1 and S'_1.

First let $d_i = \binom{n+i-1}{i}$, $i \geqslant 0$ (that is the Hilbert function of a polynomial ring in n variables) and set $c_i = b_{i+1} - d_{i+1}$. There are two possibilities.

Case 1, *in which* $c_i \leqslant c_{i+1}$ *for all* $i \geqslant 0$. In this case let $S_1 = \{c_i\}$, $i \geqslant 0$.

Case 2, *in which* $1 = c_0 \leqslant c_1 \leqslant c_2 \leqslant \ldots \leqslant c_{h-1}$ *and* $c_{h-1} > c_h$. In this case let S_1 be the sequence $c_0, c_1, \ldots, c_{h-1} \rightarrow$ (where the arrow indicates that, if $i \geqslant h-1$, then the i-th term of S_1 is c_{h-1}).

We can maintain a uniform notation if in Case 1 we set $h = \infty$ and regard sets like $\{i \mid i \geqslant \infty\}$ to be empty.

With this convention in mind we now define $S'_1 = \{c'_i\}$, $i \geqslant 0$, where

$$c'_i = \begin{cases} d_i & \text{if } i \leqslant h, \\ b_i - c_{h-1} & \text{if } i \geqslant h. \end{cases}$$

(The two definitions agree if $i = h$.)

LEMMA 3.1. *Let $S = \{b_i\}$, $i \geqslant 0$, be a differentiable 0-sequence with $b_1 = n+1$ and let h be as in the above construction of S_1. Let $\{\alpha_i\}$ be the difference sequence of S (that is $\alpha_i = b_i - b_{i-1}$ for $i \geqslant 0$, $b_{-1} = 0$). Then*

(a) $\binom{r+n-1}{r+1} \leqslant \alpha_{r+1} \leqslant \binom{r+n}{r+1}$ *if* $1 \leqslant r \leqslant h-1$;

(b) $\alpha_{h+1} < \binom{n+h-1}{h+1}$;

(c) $\alpha_{r+1}^{\langle r+1\rangle} = \left[\alpha_{r+1} - \binom{r+n-1}{r+1}\right]^{\langle r\rangle} + \binom{n+r}{r+2}$, $1 \leqslant r \leqslant h-1$.

Proof. The right inequality of (a) is true for all r by 1.6(b). For all r we have $b_{r+1} = c_r + d_{r+1}$ and so $\alpha_{r+1} = \tilde{c}_r + \binom{n+r-1}{r+1}$ (where $\{\tilde{c}_i\}$ is the difference sequence of $\{c_i\}$). Part (b) and the remaining inequality of (a) now follow from the assumptions that $\tilde{c}_r \geqslant 0$ for $r \leqslant h-1$ and $c_{h-1} > c_h$ (that is $\tilde{c}_h < 0$).

Now (a) implies either that $\alpha_{r+1} = \binom{r+n}{r+1}$ in which case (c) can be checked directly, or that the first term in the $(r+1)$-binomial expansion of α_{r+1} is $\binom{r+n-1}{r+1}$. In the latter case (c) follows immediately from the definition of $\alpha_{r+1}^{\langle r+1\rangle}$.

THEOREM 3.2. *Let $S = \{b_i\}$, $i \geqslant 0$, with $b_1 = n+1$, be a differentiable 0-sequence. Then the sequences S_1 and S_1' constructed above are both differentiable 0-sequences.*

Proof. First consider S_1. Let $\{\beta_i\}$ be the difference sequence of S_1. It suffices to show that $\{\beta_i\}$ is an 0-sequence. Also, since $\beta_r = 0$ for $r \geqslant h$ it suffices to show that $\beta_{r+1} \leqslant \beta_r^{\langle r\rangle}$ for $1 \leqslant r \leqslant h-2$. Then $\beta_r^{\langle r\rangle} = (\tilde{c}_r)^{\langle r\rangle}$ (in the notation of 3.1) and

$$(\tilde{c}_r)^{\langle r\rangle} = \left(\alpha_{r+1} - \binom{n+r-1}{r+1}\right)^{\langle r\rangle} = \alpha_{r+1}^{\langle r+1\rangle} - \binom{n+r}{r+2}$$

by 3.1(c). But, since $\{\alpha_i\}$ is an 0-sequence, we have

$$\alpha_{r+1}^{\langle r+1\rangle} - \binom{n+r}{r+2} \geqslant \alpha_{r+2} - \binom{n+r}{r+2} = \beta_{r+1}$$

and so $\beta_{r+1} \leqslant \beta_r^{\langle r\rangle}$.

Now consider S_1'. Again it suffices to show that the first difference of S_1' is an 0-sequence and the only non-trivial verification there is that $c_{h+1}' - c_h' \leqslant (c_h' - c_{h-1}')^{\langle h\rangle}$. Writing this out one sees that this follows from 3.1(b).

THEOREM 3.3. *Let k be an infinite field and let $S = \{b_i\}$, $i \geqslant 0$, be a differentiable 0-sequence with $b_1 = n+1$. There is a radical ideal I in $k[x_0, \ldots, x_n]$ such that S is the Hilbert function of $k[x_0, \ldots, x_n]/I$.*

Proof. We prove the theorem by induction. Let $\mathbb{N} = \{1, 2, 3, \dots\}$. To any differentiable 0-sequence we associate an element $(a, b) \in \mathbb{N} \times \mathbb{N}$ where $a \geqslant 2$ and $b \geqslant 1$. We do this as follows.

To the sequence $T = \{t_i\}$, $t_i = \binom{t_1+i-1}{i}$, $i \geqslant 0$, we associate the element $(t_1+1, 1)$. Such an 0-sequence is the Hilbert function of a polynomial ring in t_1 variables. We call such an 0-sequence generic.

If $T = \{t_i\}$, $i \geqslant 0$, is a non-generic differentiable 0-sequence, we associate to T the element $(a, b) = (a(T), b(T))$, where $a = t_1$ and b is the smallest integer such that $t_b < \binom{t_1+b-1}{b}$.

The theorem is true for S associated to $(a, 1)$, $a \geqslant 1$ (such S are generic) and for S associated to $(2, l)$, $l \geqslant 2$ (such an S is the Hilbert function of l points on a line).

Now, let $S = \{b_i\}$, $i \geqslant 0$, be a non-generic differentiable 0-sequence and let S_1 and S_1' be the differentiable 0-sequences constructed above. It is clear from the construction that $a(S_1') = a(S)-1$ and either $a(S_1) < a(S)$ or $a(S_1) = a(S)$ and $b(S_1) = b(S)-1$.

Thus $(a(S_1'), b(S_1')) < (a(S), b(S))$ and $(a(S_1), b(S_1)) < (a(S), b(S))$ (in the usual lexicographic ordering). Thus, by induction, we may find a subvariety X of $\mathbb{P}_k^{b_1-1}$ with Hilbert function S_1. Let H be a hyperplane not containing any irreducible component of X (since k is infinite, such an H exists) and let V be a subvariety of H with Hilbert function S_1'. By Corollary 2.8, $X \cup V$ has Hilbert function S; this completes the proof.

If we carry out the induction all the way to the $(2, l)$, $l \geqslant 2$, and $(a, 1)$, $a \geqslant 1$, sequences, we obtain the following.

Corollary 3.4. *Let k be an infinite field and let $S = \{b_i\}$, $i \geqslant 0$, be a d-dimensional differentiable 0-sequence with $b_1 = n+1$. There is a d-dimensional variety $V \subset \mathbb{P}_k^n$, which is the union of a finite number of linear subvarieties of dimension $\leqslant d$, having Hilbert function S.*

Example 3.5. Let S be the zero-dimensional differentiable 0-sequence

$$S: 1 \ 3 \ 6 \ 10 \ 14 \ 17 \ 20 \ 21 \ 22 \rightarrow .$$

The procedure we have described gives S as the Hilbert function of 22 points in $\mathbb{P}^2$ where 9 are on a line L_1, 6 are on a line L_2, 5 are on a line L_3 and 2 are on a line L_4 where the L_i are any four distinct lines of $\mathbb{P}^2$ and no point chosen on L_i lies on L_j for $j < i$.

Example 3.6. If S is the 1-dimensional differentiable 0-sequence

$$S: 1 \ 4 \ 10 \ 19 \ 30 \ 40 \ 50 \rightarrow$$

(continuing with increments of 10) then our procedure gives S as the Hilbert function of the union of 10 lines lying in a hyperplane of $\mathbb{P}^3$, 20 points on another hyperplane and 5 points on a third hyperplane.

This S can also be shown to be the Hilbert function of the 10 lines of a double-five configuration on a non-singular cubic surface in $\mathbb{P}^3$. (See [1] for a description of the double-five.)

4. *The conductor of a finite set of points in* $\mathbb{P}^r$

Let $X = \{P_1, \ldots, P_s\}$ be any set of s k-rational points in $\mathbb{P}^r_k$, with homogeneous coordinate ring A. The integral closure of A is $\bar{A} = \prod_{i=1}^{s} k[t_i]$ and the conductor of A in $\bar{A}$ is of the form $\prod_{i=1}^{s} t_i^{n_i} k[t_i]$, for some integers $n_i \geqslant 1$. We shall refer to n_i as the *degree of the conductor of* P_i *in* X. Let $V = X - P_i$. It is clear that

$$H_V(i) = \begin{cases} H_X(i) & \text{for } 0 \leqslant i < n_i, \\ H_X(i) - 1 & \text{for } i \geqslant n_i. \end{cases}$$

Thus n_i is the 'd' of Lemma 2.3(b).

This motivates the following definition.

Definition 4.1. Let $S = \{b_i\}$, $i \geqslant 0$, be a zero-dimensional differentiable 0-sequence. We say that l is a *permissible value* for S if the sequence $\hat{S} = \{c_i\}$, where $c_i = b_i$ for $0 \leqslant i < l$ and $c_i = b_i - 1$ for $i \geqslant l$, is again a differentiable 0-sequence.

Hence, the possible n_i are a subset of the permissible values for the Hilbert function of A.

Example 4.2. Let S be the zero-dimensional differentiable 0-sequence

$$S: 1\ 3\ 6\ 7 \rightarrow .$$

The permissible values are 2 and 3. Seven points of $\mathbb{P}^2$ in uniform position [2] have Hilbert function S and each point has degree of conductor 3.

Thus, not every permissible value need occur as the degree of conductor for some point. However, we do have the following.

Theorem 4.3. *Let S be a zero-dimensional differentiable* 0*-sequence and let k be an infinite field. If X is a finite set of k-rational points with Hilbert function S* and *constructed by Corollary* 3.4 *then every permissible value for S is the degree of the conductor for some $P \in X$.*

Proof. If S is $1\ 2\ 3\ \ldots\ n \rightarrow$, then the only permissible value is $n-1$, which must be the degree of conductor of any point in X.

Otherwise, let S_1 and S'_1 be the sequences constructed from S in §3 and let l be a permissible value for S. It is straightforward to prove that either l is a permissible value for S'_1 or $l-1$ is a permissible value for S_1. We have $X = X_1 \cup X'_1$, where X_1 and X'_1 are constructed by applying the procedures of §3 (see Corollary 3.4). By induction on the number of points we conclude that either there is a point $P \in X_1$

with degree of conductor $l-1$ in X_1 or there is a point $P \in X'_1$ with degree of conductor l in X'_1. By Corollary 2.8 and Lemma 2.3(b) the point P has degree of conductor l in X.

5. *Some geometry associated to the Hilbert function of a set of points*

Let $S = \{b_i\}$, $i \geqslant 0$, be a d-dimensional differentiable 0-sequence. The variety X with Hilbert function S that is produced by Corollary 3.4 is obviously special if $d > 0$. Proposition 5.1 below suggests that X is, in some sense, extremal even if $d = 0$.

PROPOSITION 5.1. *Let $S = \{b_i\}$, $i \geqslant 0$, be a zero-dimensional differentiable 0-sequence with $b_1 = n+1$ and $b_m = s$ for $m >> 0$. Let d be the least integer for which $b_d = s$.*

(i) *If $Y = \{P_1, \ldots, P_s\}$ is any set of points in $\mathbb{P}^n$ with Hilbert function S then at most $d+1$ of the points of Y can be collinear.*

(ii) *If $X = \{Q_1, \ldots, Q_s\}$ is a set of points in $\mathbb{P}^n$ having Hilbert function S* and *constructed as in Corollary* 3.4, *then exactly $d+1$ points of X are collinear.*

Proof. Lemma 2.3(b) easily gives (i), and (ii) is obvious if $n = 1$. Hence to prove (ii) we may assume that $n \geqslant 2$.

Let $S'_1 = \{c'_i\}$ be produced from S as in §3. Then $c'_1 < b_1$ and d is also the degree at which S'_1 becomes constant. The proof now follows by induction on n.

In general there need not be $d+1$ collinear points. However, for some S, the X produced by Corollary 3.4 is the only one possible, as we now show.

PROPOSITION 5.2. *Let $S = \{b_i\}$, $i \geqslant 0$, $b_1 = n+1$, be a zero-dimensional differentiable 0-sequence for which $b_{d-2} = s-2$, $b_{d-1} = s-1$ and $b_i = s$ for $i \geqslant d$.*

(i) *If $X = \{P_1, \ldots, P_s\}$ is* any *set of s points in $\mathbb{P}^n$ having Hilbert function S then X contains a subset Y of $d+1$ collinear points.*

(ii) *If, moreover, $n = 2$ then $X \setminus Y$ has Hilbert function $\{c_i\}$ where*

$$c_i = \begin{cases} b_{i+1} - (i+2) & \text{for } 0 \leqslant i \leqslant d, \\ s-(d+1) & \text{for } i > d. \end{cases}$$

Proof. Note that (ii) follows from (i) by 2.8, and so it suffices to prove (i).

From Theorem 2.5(c) there are two points P_1, P_s in X such that $X \setminus \{P_1, P_s\}$ has Hilbert function

$$1 \;\; b_1 \;\; b_2 \;\; \ldots \;\; b_{d-2} = s-2 \;\; \rightarrow .$$

Thus, $X \setminus \{P_i\}$, $i = 1, s$, has Hilbert function

$$1 \;\; b_1 \;\; b_2 \;\; \ldots \;\; b_{d-2} = s-2 \;\; b_{d-1} = s-1 \;\; \rightarrow .$$

Let L be the line joining P_1 and P_s. The proof is by induction on the number of points of X that we have found on L (starting with $i = 2$, P_1 and P_s).

Suppose we have found i points $\{P_1, P_2, \ldots, P_{i-1}, P_s\}$ on L $(2 \leqslant i < d+1)$. The points $P_1, \ldots, P_{i-1}$ are collinear, and hence have Hilbert function

$$1\ 2\ 3\ \ldots\ (i-1) \rightarrow .$$

Now $\{P_1, P_2, \ldots, P_{i-1}\} \subset X \setminus \{P_s\}$ and $i-2 \leqslant d-2$, and so by Lemma 2.3(c) we can adjoin $s-i-1$ new points of $X \setminus \{P_s\}$ to $\{P_1, P_2, \ldots, P_{i-1}\}$ to obtain $R = \{P_1, \ldots, P_{i-1}, R_1, \ldots, R_{s-i-1}\}$, a set of $s-2$ points whose Hilbert function at $d-2$ has value exactly $s-2$. Let Q be the remaining point of $X \setminus \{P_s\}$ not in R. Let F be a form of degree $d-2$ vanishing at all points of R except P_1 and let G be any linear form vanishing at Q and P_s. Then FG is a form of degree $d-1$ vanishing at all points of X except possibly P_1. But $X \setminus \{P_1\}$ and X have the same Hilbert function in degree $d-1$; hence $FG(P_1) = 0$ and so $G(P_1) = 0$. Thus $Q \in L$, and this completes the induction and the proof.

EXAMPLE 5.3. Consider the zero-dimensional differentiable 0-sequence

$$S: 1\ 3\ 6\ 9\ 11\ 13\ 14\ 15 \rightarrow .$$

The procedures of §3 construct a set X in $\mathbb{P}^2$ having Hilbert function S and consisting of 8 points on one line, 5 points on another line and 2 points on a third line. Proposition 5.2 tells us that any set of 15 points in $\mathbb{P}^2$ with Hilbert function S must be so situated.

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