M953, Homework 6, due Friday, March 29, 2013

Instructions: Do any three problems.

(1) Let k be a field and let $\varnothing \neq S \subseteq \mathbf{A}^m(k)$. Recall that $\text{affspan}(S) = V(\text{Lin}(S))$, where $\text{Lin}(S) = \{a_1x_1 + \cdots + a_mx_m + a_0 \in k[x_1, \ldots, x_m] = k[\mathbf{A}^m] : S \subseteq V(a_1x_1 + \cdots + a_mx_m + a_0)\}$. For $p \in \text{affspan}(S)$, let $W_p(S) = \{v - p : v \in \text{affspan}(S)\}$. Show that $W_p(S)$ is a vector subspace of $k^m = \mathbf{A}^m(k)$ and that this subspace is independent of p (i.e., $W_p(S) = W_q(S)$ for all $q \in \text{affspan}(S)$).

(2) Let k be a field and let $\varnothing \neq S \subseteq \mathbf{A}^m(k)$. We define $\dim \text{affspan}(S)$ to be the vector space dimension of $W_p(S)$ for any $p \in \text{affspan}(S)$ (this is well-defined by Problem (1)). Let $d = \dim \text{affspan}(S)$. Show that there exist $d+1$ points $p_0, \ldots, p_d \in S$ such that $\text{affspan}(\{p_0, \ldots, p_d\}) = \text{affspan}(S)$.

(3) Let k be a field and let $\phi = (\phi_1, \ldots, \phi_m) : \mathbf{A}^1(k) \to \mathbf{A}^m(k)$ be a morphism. If $\deg(\phi_i) \leq d$ for all i, show that $\dim \text{affspan}(\text{Im}(\phi)) \leq d$.

(4) Let k be a field. Let $S \subseteq \mathbf{A}^m(k)$ and let V be the Zariski closure of S. Show that $\text{affspan}(S) = \text{affspan}(V)$.

(5) Let $\phi = (\phi_1, \ldots, \phi_m) : \mathbf{A}^1(k) \to \mathbf{A}^m(k)$ be a morphism such that $m > 1$ and $\deg \phi_i \leq 2$ for all i, not all constant. Let C be the closure of $\phi(\mathbf{A}^1(k))$. Show that $\dim \text{affspan}(C) \leq 2$. Conclude that $\text{Sec}_2(C)$ is contained in a 2-dimensional plane. [Aside: This shows that $\dim \text{Sec}_2(C) \leq 2$, and thus that rational curves of degree at most 2 in $\mathbf{A}^m(k)$ with $m > 2$ are always defective. One can also show that any curve of degree at most 2 is rational, hence curves of degree at most 2 are always defective.]

(6) Let $L_1 \subset \mathbf{A}^3(k)$ be the line $V(x, y+1)$ and let $L_2 \subset \mathbf{A}^3(k)$ be the line $V(z, y-1)$, where $k[\mathbf{A}^3] = k[x, y, z]$. Let P_1 be the plane $V(y+1)$ and let P_2 be the plane $V(y-1)$. Let $V = L_1 \cup L_2$. Let $\sigma_2 : V^2 \times \Delta_2 \to \mathbf{A}^3(k)$, so $\text{Sec}_2(V)$ is the closure of the image of σ_2. Show that $\text{Im}(\sigma_2) = (\mathbf{A}^3(k) \setminus (P_1 \cup P_2)) \cup (L_1 \cup L_2)$. [Hint: If $q \in \mathbf{A}^3(k) \setminus (P_1 \cup P_2)$, consider the intersections of the planes Q_1 and Q_2 where Q_i contains q and L_i.]

(7) Let k be a field of characteristic not equal to 2 and let V be the image of $\nu : \mathbf{A}^2(k) \to \mathbf{A}^5(k)$ defined by $\nu((a,b)) = (a^2, ab, b^2, a, b)$ (so the component functions of ν are the nontrivial monomials of degree at most 2); V is known as the Veronese variety.
  (a) Show that V is closed and that ν is an isomorphism to its image.
  (b) Show that the set of points $(p', q') \in V \times V$ such that $p' \neq q'$ and such that the line $\ell_{pq} \subset \mathbf{A}^2(k)$ through p and q is neither vertical nor horizontal and does not go through the origin, where $\nu(p) = p'$ and $\nu(q) = q'$, is a nonempty open subset $U' \subseteq V \times V$.
  (c) Recall that the secant variety $\text{Sec}_2(V)$ is the closure of the image of $\sigma_2 : V^2 \times \Delta_2 \to \mathbf{A}^5(k)$, and hence the closure of the image of $f : \mathbf{A}^5(k) \to \mathbf{A}^5(k)$ defined as $f(a,b,c,d,e) = e(a^2, ab, b^2, a, b) + (1-e)(c^2, cd, d^2, c, d)$. Show in fact that $\text{Sec}_2(V)$ is contained in the closure of the image of $g : \mathbf{A}^4(k) \to \mathbf{A}^5(k)$ defined by $g(a,b,c,d) = c\nu((a,0)) + d\nu((0,b)) + (1-c-d)\nu((a/2, b/2))$. [Hint: for points $p \neq q \in \mathbf{A}^2(k)$, let $p' = \nu(p)$ and $q' = \nu(q)$. Show that the secant line $L_{pq} \subset \mathbf{A}^5(k)$ through p' and q' lies in the affine span of the image $\nu(\ell_{pq})$ of the line $\ell_{pq} \subset \mathbf{A}^2(k)$ through p and q. Now use Problem (5).] (Aside: This implies that $\dim \text{Sec}_2(V) \leq 4$ and hence that $\text{Sec}_2(V)$ is defective. In fact, $\dim \text{Sec}_2(V) = 4$, so $\text{Sec}_2(V)$ is defined by a single polynomial equation on $\mathbf{A}^5(k)$.)

(8) Let k be a field of characteristic not 2 or 3, and let C be the image of $\tau : \mathbf{A}^1(k) \to \mathbf{A}^3(k)$, defined as $\tau(t) = (t, t^2, t^3)$. For any $a \in k$, the line L_p through $p = \tau(a)$ with direction vector $(1, 2a, 3a^2)$ is called the tangent line to C at p. Let $k[\mathbf{A}^1] = k[t]$ and $k[\mathbf{A}^3] = k[x, y, z]$. Let $f \in k[x, y, z]$ be a nonzero polynomial of degree 1.
  (a) Show that C is closed and that τ is an isomorphism to its image. (In fact, the image of $\mathbf{A}^1(k)$ under a morphism is always closed, but this is harder to show.)
  (b) Show that $\tau^*(f)$ has at most 3 roots, counted with multiplicity.
  (c) Let $\tau(a) = p \in C$. If $L_p \subset V(f)$, show that $t = a$ is a root of multiplicity at least 2 (i.e., show that $(t-a)^2$ divides $\tau^*(f)$). [Aside: if $t = a$ is a root of multiplicity 3, we say $V(f)$ is an osculating plane for C at p.]
  (d) Show that if q is on a tangent line but not on C, then q is on no secant line of C. Conclude for any $q \neq p \in C$, if $q \in L_p$, then q is not in the image of $\sigma_2 : C^2 \times \Delta_2 \to \mathbf{A}^3(k)$.